ON OSCILLATION OF SOLUTIONS OF DIFFERENTIAL INEQUALITIES WITH RETARDED ARGUMENT

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We consider the following differential inequality

$$\{[r(t)\, y^{(n-1)}(t)]' + f(t, y(t), y[h(t)]\} \operatorname{sgn} y[h(t)] \leqq 0\,, \quad n \geqq 2\,, \tag{1}$$

where

$$r : [0, \infty) \to (0, \infty)\,; \quad h : [0, \infty) \to R\,; \tag{2}$$

$$f : [0, \infty) \times R^2 \to R \quad \text{are continuous functions}\,,$$

$$h(t) \leqq t\,, \quad \lim h(t) = \infty \quad \text{for} \quad t \to \infty\,, \tag{3}$$

$$y f(t, x, y) > 0 \quad \text{for} \quad (t, x, y) \in [0, \infty) \times R^2\,, \quad xy > 0\,; \tag{4}$$

$$|f(t, x_1, y_1)| \leqq |f(t, x_2, y_2)| \quad \text{for} \quad |x_1| \leqq |x_2|\,, \quad |y_1| \leqq |y_2|\,, \quad x_1 x_2 > 0\,,$$

$$y_1 y_2 > 0\,, \quad x_1 y_1 > 0\,.$$

Denote by W the set of all solutions $y(t)$ of the differential inequality (1), which exist on a ray $[t_0, \infty] \subset [0, \infty)$ and satisfy

$$\sup \{|y(s)| : s \geqq t\} > 0$$

for every $t \in [t_0, \infty)$.

A solution $y(t) \in W$ is said to be *oscillatory* if the set of zeros of $y(t)$ is not bounded from the right. Otherwise the solution $y(t) \in W$ is said to be *nonoscillatory*.

Definition 1. We shall say the that *the inequality* (1) has *the property A* if every solution $y(t) \in W$ is oscillatory for n even, while for n odd is either oscillatory or $y^{(i)}(t)$ $(i = 0, 1, \ldots, n - 2)$ and $r(t)\, y^{(n-1)}(t)$ tend monotonically to zero as $t \to \infty$.

Definition 2. Let $m \in \{0, 1, \ldots, n - 1\}$. We shall say that *the inequality* (1) has *the property* A_m if every solution $y(t) \in W$ is either oscillatory or $y^{(i)}(t)$ $(i = m, m + 1, \ldots, n - 2)$ and $r(t)\, y^{(n-1)}(t)$ tend monotonically to zero as $t \to \infty$.

The oscillatory properties of solutions of differential equations of the n-th order with the term $[r(t)\, y^{(n-1)}(t)]'$ $(n = 2,\ n \geqq 2,\ r(t) > 0)$ are studied, for example, in [1, 2, 4, 7, 9 ÷ 12]. In this paper we shall prove sufficient conditions for the inequality (1) to have either the property A or A_0.

Finally, with the help of the inequality (1) we shall prove a sufficient condition for the equation (r) to have the property A_m, $m \in \{0, 1, \ldots, n-1\}$. Our results generalize some of those in the papers [1–3, 6, 9, 12].

Let us denote

$$\bar{r}(t) = \max\{r(s) : t/2 \leqq s \leqq t\},$$

$$b(t) = \frac{r(t)}{\bar{r}(t)}, \quad b_0 = \inf\{b(t) : t \geqq t_0\},$$

$$R_k(t) = \int_T^t \frac{x^k}{r(x)}\, dx, \quad k = 0, 1, \ldots, n-2, \quad T \in [0, \infty),$$

$$R_k(t, u) = \int_u^t \frac{(x-u)^k}{r(x)}\, dx, \quad k = 0, 1, \ldots, n-2, \quad u \leqq t,$$

$$\varrho(t) = \frac{r[h(t)]}{\min\{r(s) : h(t) \leqq s \leqq t\}}, \quad H(t) = \frac{t}{h(t)} \quad \text{for} \quad h(t) > 0.$$

Let $m \in \{0, 1, \ldots, n-1\}$, $t_0 \in [0, \infty)$. Put

$$D_{t_0}^{(m)} = \{(t, x_1, y_1, \ldots, x_n, y_n) \in [0, \infty) \times R^{2n} : t_0 \leqq h(t),$$

$$\frac{(n-m-1)!}{(n-j)!}\left(\frac{b_0}{2}\, t\right)^{m-j+1} \leqq \frac{x_j}{x_{m+1}}, \quad \frac{(n-m+1)!}{(n-j)!}\left(\frac{b_0}{2}\, h(t)\right)^{m-j+1} \leqq$$

$$\leqq \frac{y_j}{y_{m+1}}, \quad (j = 1, 2, \ldots, m+1), \quad x_{m+1} y_{m+1} > 0, \quad x_i, y_i \in R,$$

$$(i = m+2, \ldots, n)\}.$$

Lemma 1. *Let $y(t), \ldots, y^{(n-1)}(t)$ be continuous functions of constant sign in the interval $[t_0, \infty) \subset [0, \infty)$. If*

(5) $$y(t)\,[r(t)\, y^{(n-1)}(t)]' \leqq 0, \quad y(t) \neq 0 \quad for \quad t \geqq t_0;$$

(5') $$y(t)\, y^{(n-1)}(t) \geqq 0 \quad for \quad t \geqq t_0,$$

where the function r satisfies (2), then there exists an integer $k \in \{0, 1, \ldots, n-1\}$, $n + k$ odd, such that

(6) $$y^{(i)}(t)\, y(t) \geqq 0 \quad (i = 0, 1, \ldots, k), \quad t \geqq t_0,$$

(7) $$(-1)^{k+i}\, y^{(i)}(t)\, y(t) \geqq 0 \quad (i = k+1, \ldots, n-1), \quad t \geqq t_0,$$

$$(8)\qquad |y^{(i)}(t)| \geqq L_i\, b(t)\, t^{n-i-1} |y^{(n-1)}(t)|\,, \quad where \quad k \in \{1, 2, \ldots, n-1\}\,,$$

$$L_i = \frac{2^{-n^2}}{(n-i-1)!} \quad (i = 0, 1, \ldots, k-1)\,, \quad t \geqq 2^{n-k} t_0\,,$$

$$(9)\qquad |y^{(k)}(t)| \geqq t^{n-k-1}\, b(2^{n-k-1} t)\, |y^{(n-1)}(2^{n-k-1} t)|\,, \quad t \geqq t_0\,,$$

$$(10)\qquad i! \left(\frac{b_0}{2}\right)^{j-i} t^{j-i} |y^{(k-i)}(t)| \leqq j!\, |y^{(k-j)}(t)| \quad (j = 0, 1, \ldots, k\,, \quad i = 0, 1, \ldots, j)\,,$$

$$t \geqq 2t_0\,.$$

Proof. Under the assumption (5′), assertions (6) and (7) follow from Kiguradze's lemma 14.2 in [5]. Further, we may suppose, without loss of generality, that $y(t) > 0$ for $t \geqq t_0$.

(a) Let $k = n - 1$. Then (6) implies

$$y^{(i)}(t) \geqq 0\,, \quad i = 1, 2, \ldots, n-1\,, \quad t \geqq t_0\,.$$

Using Taylor's theorem, the last inequality, and the monotonicity of $[r(t)\, y^{(n-1)}(t)]$, we get

$$(11)\qquad y^{(i)}(t) = \sum_{j=0}^{n-i-2} \frac{y^{(i+j)}(t/2)}{j!} \left(\frac{t}{2}\right)^j + \int_{t/2}^{t} y^{(n-1)}(s) \frac{(t-s)^{n-i-2}}{(n-i-2)!}\, ds \geqq$$

$$\geqq \int_{t/2}^{t} y^{(n-1)}(s) \frac{(t-s)^{n-i-2}}{(n-i-2)!}\, ds \geqq \frac{b(t)\, y^{(n-1)}(t)}{(n-i-1)!} \left(\frac{t}{2}\right)^{n-i-1}$$

$$(i = 0, 1, \ldots, n-2)\,, \quad t \geqq 2t_0\,.$$

From (11), we obtain (8) for $k = n - 1$.

The inequality (9) for $k = n - 1$ is evident.

(b) Let $k \in \{0, 1, \ldots, n-3\}$ and let $n + k$ be an odd integer. Then, in view of Lemma 1 in [8], we get

$$(12)\qquad y^{(i)}(t) \geqq \bar{L}_i t^{n-i-3}\, y^{(n-3)}(t)\,, \quad t \geqq 2^{n-k-2} t_0\,,$$

$$\bar{L}_i = \frac{2^{-(n-2)^2}}{(n-i-3)!}\,, \quad (i = 0, 1, \ldots, k-1)\,,$$

and

$$(13)\qquad y^{(k)}(t) \geqq t^{n-k-3}\, y^{(n-3)}(2^{n-k-3} t)\,, \quad t \geqq t_0\,.$$

With the help of (7) and (5), we get

$$(14)\qquad -y^{(n-2)}(t/2) \geqq \int_{t/2}^{t} y^{(n-1)}(s)\, ds \geqq \frac{t}{2}\, b(t)\, y^{(n-1)}(t)\,, \quad t \geqq 2t_0\,.$$

For $t \geqq 4t_0$, using (6), (7) and (14), we obtain

$$(15) \qquad y^{(n-3)}\left(\frac{t}{4}\right) \geqq y^{(n-3)}\left(\frac{t}{4}\right) - y^{(n-3)}\left(\frac{t}{2}\right) \geqq -\frac{t}{4}\, y^{(n-2)}\left(\frac{t}{2}\right) \geqq \frac{t^2}{8}\, b(t)\, y^{(n-1)}(t)\,.$$

The inequalities (15), (12) and (13) imply (8) and (9).

If $k \in \{0, 1, \ldots, n-3\}$, then the inequality (10) follows from Kiguradze's lemma in [5]. It remains to prove (10) for $k = n - 1$.

Let $k = n - 1$. Using (6) we can show that

$$y^{(n-2-i)}(t) \geqq \frac{t}{2}\, y^{(n-i-1)}\left(\frac{t}{2}\right), \quad (i = 1, 2, \ldots, n-2)\,, \quad t \geqq 2t_0\,.$$

Utilizing the last inequality and (6), we can easily verify the correctness of the following relation

$$(16) \qquad (1 + i)\, y^{(n-2-i)}(t) - \int_{t/2}^{t} \left[i\, y^{(n-i-1)}(s) - \frac{b_0}{2}\, s\, y^{(n-i)}(s) \right] \mathrm{d}s \geqq$$

$$\geqq (1 - b_0)\, y^{(n-2-i)}(t) + \frac{b_0}{2}\, t\, y^{(n-i-1)}(t) + \frac{b_0}{2}\left[y^{(n-i-2)}(t) - \frac{t}{2}\, y^{(n-i-1)}\left(\frac{t}{2}\right)\right] \geqq$$

$$\geqq \frac{b_0}{2}\, t\, y^{(n-i-1)}(t)\,, \quad (i = 1, 2, \ldots, n-2)\,, \quad t \geqq 2t_0\,.$$

For $i = n - 2$,

$$(17) \qquad y^{(n-2)}(t) \geqq \frac{b(t)}{2}\, t\, y^{(n-1)}(t) \geqq \frac{b_0}{2}\, t\, y^{(n-1)}(t)\,, \quad t \geqq 2t_0$$

follows from (11).

Further, (16) and (17) imply

$$(1 + i)\, y^{(n-2-i)}(t) \geqq \frac{b_0}{2}\, t\, y^{(n-i-1)}(t)\,, \quad t \geqq 2t_0\,, \quad (i = 0, 1, \ldots, n-2)\,.$$

For $k = n - 1$, (10) follows from the last inequality.

This completes the proof of Lemma 1.

Lemma 1 is an extension of Lemma 2 in [9].

Lemma 2. *Let* (2)–(4) *hold.*

(a) *If*

$$(18) \qquad \int^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty\,,$$

then conditions (5) *and* (5′) *are satisfied for every nonoscillatory solution* $y(t) \in W$ *of* (1).

(b) *If*

$$\int^{\infty}\left(\frac{1}{r(t)}\int_T^t |f(s, c, c)|\,\mathrm{d}s\right)\mathrm{d}t = \infty \tag{19}$$

for every $c \neq 0$ *and* $T \geqq 0$, *then conditions* (5) *and* (5′) *hold for every nonoscillatory solution* $y(t) \in W$ *of* (1) *such that* $\lim_{t\to\infty} y(t) \neq 0$.

Proof. We assume, without loss of generality, that $y(t) > 0$ for $t \geqq t_0$. Then, in view of (3), there exists $t_1 \geqq t_0$ such that $y[h(t)] > 0$ for $t \geqq t_1$. From (1), with regard to (4), we obtain

$$[r(t)\, y^{(n-1)}(t)]' \leqq -f(t, y(t), y[h(t)]) < 0 \quad \text{for} \quad t \geqq t_1\,. \tag{20}$$

(a) If (18) holds, then using the same method as in Lemma 1 in [9], we get $y^{(n-1)}(t) > 0$ for $t \geqq t_1$.

(*b*) Via contradiction we prove that $y^{(n-1)}(t) > 0$ for $t \geqq t_1$. We suppose that for some $t_2 \geqq t_1$ we have $y^{(n-1)}(t_2) \leqq 0$. Then (20) implies $y^{(n-1)}(t) < 0$ for $t \geqq t_2$. If $\lim_{t\to\infty} y(t) > 0$, then there exist $\varepsilon > 0$ and $t_3 \geqq t_2$ such that $y(t) \geqq \varepsilon$ and $y[h(t)] \geqq \geqq \varepsilon$ hold for every $t \geqq t_3$. Thus (20), under the assumption (4), yields

$$[r(t)\, y^{(n-1)}(t)]' \leqq -f(t, \varepsilon, \varepsilon) < 0 \quad \text{for} \quad t \geqq t_3\,.$$

Integrating the last inequality from T ($T \geqq t_3$) to t and using $y^{(n-1)}(t) \leqq 0$ for $t \geqq t_2$ we have

$$y^{(n-1)}(t) \leqq \frac{1}{r(t)}\int_T^t f(s, \varepsilon, \varepsilon)\,\mathrm{d}s\,.$$

Integrating the last relation from T to t, with regard to (19) we get $\lim_{t\to\infty} y^{(n-2)}(t) = -\infty$ which contradicts the positivity of $y(t)$ for $t \geqq t_0$.

The proof of Lemma 2 is complete.

Theorem 1. *Let* r, h, f *be functions satisfying conditions* (2), (3), (4). *Let* K, α, δ *be constants* ($K > 0$, $0 \leqq \alpha < 1$, $\delta > 0$) *and* $g : [0, \infty) \to [K, \infty)$ *a continuous function such that*

$$|f(t, g(t)\, x, g(t)\, y)| = [g(t)]^{\alpha}\, |f(t, x, y)| \tag{21}$$

holds for every $t \geqq 0$ *and* $|y| \geqq \delta$, $|x| \geqq \delta$.

(a) *If* (18) *and*

$$\int^{\infty} |f(t, \pm\bar{r}^{-1}(t)\, t^{n-1}, \pm\bar{r}^{-1}[h(t)]\,(h(t))^{n-1})|\,\mathrm{d}t = \infty \tag{22}$$

hold, then the inequality (1) *has the property A.*

(b) *When* (19) *and* (22) *hold, then the inequality* (1) *has the property* A_0.

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (1) such that $\lim_{t\to\infty} y(t) \neq 0$. We assume, without loss of generality, that

$$\lim_{t\to\infty} y(t) > 0 . \tag{23}$$

Then, in view of (3), we can choose $\bar{t}_0$ such that $y[h(t)] > 0$ for every $t \geqq \bar{t}_0$. Then (1), with regard to (4), implies $[r(t)\, y^{(n-1)}(t)]' < 0$ for $t \geqq \bar{t}_0$. If any of the conditions (18) and (19) is satisfied, Lemma 2 implies $y^{(n-1)}(t) > 0$ for $t \geqq \bar{t}_0$. Then by Lemma 1, there exists $t_0 \geqq \bar{t}_0$ such that the inequalities (6)–(9) hold for $t \geqq t_0$.

Integrating (1) from t ($t \geqq t_0$) to ∞, we get

$$\infty > r(t)\, y^{(n-1)}(t) \geqq \int_t^\infty f(s, y(s), y[h(s)])\, ds \quad \text{for} \quad t \geqq t_0 , \tag{24}$$

and then, in view of the monotonicity of $r(t)\, y^{(n-1)}(t)$, we have

$$r[h(t)]\, y^{(n-1)}[h(t)] \geqq \int_t^\infty f(s, y(s), y[h(s)])\, ds \quad \text{for} \quad t \geqq t_1 \geqq t_0 . \tag{24'}$$

I. Let $k \in \{1, 2, \ldots, n-1\}$. Then we obtain by (8) for $i = 0$

$$y(t) \geqq L_0\, b(t)\, t^{n-1}\, y^{(n-1)}(t) , \quad t \geqq 2^{n-k} t_0 = t_2 , \tag{25}$$

$$y[h(t)] \geqq L_0\, b[h(t)]\, (h(t))^{n-1}\, y^{(n-1)}[h(t)] \quad \text{for} \quad t \geqq t_3 , \tag{25'}$$

where $L_0 = 2^{-n^2}/(n-1)!$ and t_3 is chosen such that $h(t) \geqq \max\{t_2, t_1\}$ for $t \geqq t_3$.

Let us denote

$$\Phi(t) = \int_t^\infty f(s, y(s), y[h(s)])\, ds .$$

From (25) or (25′), with regard to (24) or (24′), we get, respectively,

$$y(t) \geqq L_0\, \bar{r}^{-1}(t)\, t^{n-1}\, \Phi(t) \quad \text{for} \quad t \geqq t_3 , \tag{26}$$

or

$$y[h(t)] \geqq L_0\, \bar{r}^{-1}[h(t)]\, (h(t))^{n-1}\, \Phi(t) \quad \text{for} \quad t \geqq t_3 . \tag{26'}$$

Because $k \geqq 1$, there exists $\delta > 0$ such that $y(t) \geqq y[h(t)] \geqq \delta$ for $t \geqq t_3$. Then, in view of the monotonicity of the function f, (26), (26′) and (21) we have

$$\begin{aligned} & f(t, \bar{r}^{-1}(t)\, t^{n-1}, \bar{r}^{-1}[h(t)]\, (h(t))^{n-1}) \leqq \\ & \leqq f(t, y(t)\, \{L_0\, \Phi(t)\}^{-1}, y[h(t)]\, \{L_0\, \Phi(t)\}^{-1}) = \\ & = \{L_0\, \Phi(t)\}^{-\alpha} f(t, y(t), y[h(t)]) \quad \text{for} \quad t \geqq t_3 . \end{aligned} \tag{27}$$

By integrating (27) from t_3 to t_4 ($t_3 < t_4$) we have

(28)

$$\int_{t_3}^{t_4} f(t, \bar{r}^{-1}(t)\, t^{n-1},\ \bar{r}^{-1}[h(t)]\,(h(t))^{n-1})\, dt \leqq \frac{L_0^{-\alpha}}{1-\alpha}\left[\left(\int_t^\infty f(s, y(s), y[h(s)])\, ds\right)^{1-\alpha}\right]_{t_4}^{t_3}.$$

From (28), in view of (24), we obtain

$$\int_{t_3}^{\infty} f(t, \bar{r}^{-1}(t)\, t^{n-1},\ \bar{r}^{-1}[h(t)]\,(h(t))^{n-1})\, dt < \infty\,,$$

which contradicts (22).

II. Let $k = 0$ (n is an odd integer). Then (9) with $k = 0$ implies in view of (23)

$$y(t) \geqq M_0\, b(t)\, t^{n-1}\, y^{(n-1)}(t) \quad \text{for} \quad t \geqq 2^n t_0\,, \tag{29}$$

where

$$M_0 = \inf_{t \geqq t_0} \left\{\frac{y(t)}{y(2^{1-n}t)}\right\} 2^{-(n-1)^2} > 0\,.$$

Further, using an analogous method as in the case I, we get a contradiction with (22).

If (18) holds and $k \in \{1, 2, \ldots, n-1\}$, then, with regard to (6), (36) is fulfilled. In all other cases (i.e. either (18) holds and $k = 0$ or (19) holds and $k \in \{0, 1, \ldots$ $\ldots, n-1\}$) we have to assume that (23) holds. But, as shown above, this leads to a contradiction with (22). Then $\lim_{t\to\infty} y(t) = 0$ for every nonoscillatory solution $y(t) \in$ $\in W$. Hence it follows that $\lim_{t\to\infty} y^{(i)}(t) = 0$ ($i = 0, 1, \ldots, n-2$) and $\lim_{t\to\infty} r(t)\, y^{(n-1)}(t) =$ $= 0$.

The proof of Theorem 1 is complete.

Lemma 3. *Let the assumptions of Lemma 1 be fulfilled. Let $b_0 > 0$ and let $h : [0, \infty) \to R$ be a function such that (3) holds. Then there exists $T \geqq 2t_0$ such that, for $t \geqq T$, we have*

$$|y(t)| \leqq C\, \varrho(t)\, (H(t))^{n-1}\, |y[h(t)]|\,, \quad \textit{where} \quad C \geqq (2/b_0)^{n-1}\,. \tag{30}$$

Proof. The case $h(t) = t$ for $t > 2t_0$ is trivial. Consider t such that $t > h(t) \geqq 2t_0$. Without loss of generality, we assume that $y(t) > 0$ for $t \geqq t_0$. Then, with regard to (3), (5)–(7), there exists $t_1 \geqq t_0$ such that for $t \geqq t_1$ we have $h(t) \geqq t_0$, and either

(a) $y^{(i)}[h(t)] \geqq 0$ ($i = 0, 1, \ldots, n-1$), $(r[h(t)]\, y^{(n-1)}\,[h(t)])' \leqq 0$ or

(b) $y^{(i)}[h(t)] \geqq 0$ ($i = 0, 1, \ldots, k$, $k \in \{0, 1, \ldots, n-3\}$, $n + k$ is odd) and $y^{(k+1)}[h(t)] \leqq 0$.

Consider the case (a). Applying Taylor's theorem and (5) we get

$$(31)\quad y(t) \leqq \sum_{i=0}^{n-2} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i + \frac{r[h(t)]\, y^{(n-1)}[h(t)]}{(n-2)!} \int_{h(t)}^{t} \frac{(t-s)^{n-2}}{r(s)}\, ds \leqq$$

$$\leqq \varrho(t) \sum_{i=0}^{n-1} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i .$$

Because of the assumptions of Lemma 1, (10) implies

$$(32)\qquad (b_0/2)^i (h(t))^i y^{(i)}[h(t)] \leqq k(k-1)\dots(k-i+1)\, y[h(t)]$$
$$(i = 0, 1, \dots, k), \quad h(t) \geqq 2t_0 .$$

Using (31) and (32) we get

$$(b_0/2)^{n-1} y(t) \leqq \varrho(t)\, y[h(t)] \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{t - h(t)}{h(t)}\right)^i =$$

$$= \varrho(t)\, y[h(t)] \left(\frac{t}{h(t)}\right)^{n-1} \quad \text{for} \quad h(t) \geqq 2t_0 .$$

From the last inequality we get

$$y(t) \leqq C\, \varrho(t)\, y[h(t)]\, (H(t))^{n-1} \quad \text{for} \quad t \geqq T \geqq 2t_0 ,$$

where $C \geqq (2/b_0)^{n-1}$ and T is chosen so that $h(t) \geqq 2t_0$ for $t \geqq T$.

(b) Applying Taylor's theorem and the fact that $y^{(k+1)}[h(t)] \leqq 0$ for $h(t) \geqq t_0$ we have

$$y(t) \leqq \sum_{i=0}^{k-i} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i .$$

Next, using the same method as in the case (a) we get

$$y(t) \leqq y[h(t)]\, (H(t))^k \leqq C\, \varrho(t)\, y[h(t)]\, (H(t))^{n-1} \quad \text{for} \quad h(t) \geqq 2t_0 .$$

This completes the proof

Lemma 3 is an extension of Lemma 4 obtained by Grimmer in [3].

Theorem 2. *Suppose that* (2)–(4) *are satisfied and, in addition, suppose that*

(i) $r(t) \geqq r_0 > 0$ *for* $t \geqq 0$ *and* $b_0 > 0$;

(ii) *there exist a positive continuous function* $\varphi_1(t)$ *and positive nondecreasing continuous functions* $\varphi(t)$, $\varphi_2(t)$, $\psi(t)$ *for* $t \geqq a$ *such that* $\varphi(t) = \varphi_1(t)\, \varphi_2(t)$,

$$(33)\qquad \int_a^\infty \frac{dt}{\varphi(t)} < \infty ;$$

(iii) *for* $x \geqq y \geqq a$, $t \geqq b > 0$, *and for every constants* α, β, γ (*where* $0 < \alpha \leqq 1$, $\beta > 1$, $\gamma > 0$) *we have*

$$\liminf_{y\to\infty} \frac{\psi(\alpha x) f(t, x, y)}{\varphi_1(x)\, \varphi_2(\beta\, \varrho(t)\, (H(t))^{n-1}\, y)} \geqq d \frac{f(t, \gamma, \gamma)}{\varphi_2(\varrho(t)\, (H(t))^{n-1})} > 0 . \tag{34}$$

(a) *If* (18) *holds and*

$$\int^{\infty} \frac{R_{n-2}(t) f(t, \gamma, \gamma)}{\psi(t^{n-1})\, \varphi_2(\varrho(t)\, (H(t))^{n-1})}\, dt = \infty , \tag{35}$$

then inequality (1) *has the property A.*

(b) *If* (19) *and* (35) *hold, then inequality* (1) *has the property* A_0.

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (1) such that $\lim_{t\to\infty} y(t) \neq 0$. We assume, without loss of generality, that

$$\lim_{y\to\infty} y(t) > 0 . \tag{36}$$

Further, exactly as in the proof of Theorem 2 we prove that the conditions of Lemma 1 and Lemma 2 are satisfied and the inequalities (5)–(9) and (24) hold.

I. Let $k \in \{1, 2, \ldots, n-1\}$. By virtue of (5)–(7) and the assumption $r(t) \geqq \geqq r_0 > 0$, it is easy to show that there exist constants $\bar{\alpha}, \bar{\gamma}$ ($0 < \bar{\alpha} \leqq 1, \bar{\gamma} > 0$) and $t_1 \geqq \geqq t_0$ such that

$$\bar{\alpha}\, y(t) \leqq t^{n-1} , \quad y[h(t)] \geqq \bar{\gamma} \quad \text{for} \quad t \geqq t_0 . \tag{37}$$

In view of Lemma 3, the monotonicity of the function ψ, (4), (34) and (37) we get

$$\frac{f(t, y(t), y[h(t)])}{\varphi(y(t))} \geqq \frac{\psi(\bar{\alpha}\, y(t))}{\psi(t^{n-1})\, \varphi_1(y(t))} , \tag{38}$$

$$\frac{f(t, y(t), y[h(t)])}{\varphi_2(C\, \varrho(t)\, (H(t))^{n-1}\, y[h(t)])} \geqq d \frac{f(t, \bar{\gamma}, \bar{\gamma})}{\psi(t^{n-1})\, \varphi_2(\varrho(t)\, (H(t))^{n-1})}$$

for $t \geqq T_1 \geqq t_1$.

I_a. If $k \in \{2, 3, \ldots, n-1\}$, then (8) and the fact that $b(t) \geqq b_0 > 0$ imply

$$\dot{y}(t) \geqq L_1\, b_0 t^{n-2}\, y^{(n-1)}(t) \quad \text{for} \quad t \geqq 2^n t_0 . \tag{39}$$

Let $k = 1$ and $\lim_{t\to\infty} \dot{y}(t) \neq 0$. Then (9), with regard to $b(t) \geqq b_0 > 0$, yields

$$\dot{y}(t) \geqq \bar{L}_1 b_0 t^{n-2}\, y^{(n-1)}(t) \quad \text{for} \quad t \geqq t_1 , \tag{40}$$

where

$$\bar{L}_1 = \inf_{t \geqq t_0} \left\{ \frac{\dot{y}(t)}{\dot{y}(2^{2-n} t)} \right\} 2^{-n^2} > 0 .$$

Put $B = \min\{L_1 b_0, \bar{L}_1 b_0\}$. Using (24), (39) and (40) we get

$$\dot{y}(t) \geqq B \frac{t^{n-2}}{r(t)} \int_t^\infty f(s, y(s), y[h(s)]) \, ds .$$

With regard to (38) and the monotonicity of y and φ, after multiplying the last inequality by $\{\varphi(y(t))\}^{-1}$, we obtain

$$(41) \qquad \frac{\dot{y}(t)}{\varphi(y(t))} \geqq B \frac{t^{n-2}}{r(t)} \int_t^\infty \frac{f(s, y(s), y[h(s)])}{\varphi(y(s))} \, ds \geqq$$
$$\geqq dB \frac{t^{n-2}}{r(t)} \int_t^\infty \frac{f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \, \varphi_2(\varrho(s) (H(s))^{n-1})} \, ds$$
$$\text{for} \quad t \geqq T \geqq \max\{T_1, 2^n t_0\} .$$

In view of (33), after integrating (41) from T to t $(t > T)$ we get

$$\infty > \int_T^\infty \frac{\dot{y}(t)}{\varphi(y(t))} \, dt \geqq dB \int_T^t \frac{R_{n-2}(s, T) f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \, \varphi_2(\varrho(s) (H(s))^{n-1})} \, ds ,$$

which contradicts (35).

I$_b$. Let $k = 1$ and $\lim_{t \to \infty} \dot{y}(t) = 0$. Integrating (24) from t $(t \geqq t_0)$ to ∞ we obtain

$$-y^{(n-2)}(t) \geqq \int_t^\infty R_0(s, t) f(s, y(s), y[h(s)]) \, ds .$$

Repeating this procedure $n - 3$ times, we get

$$(42) \qquad (-1)^n \dot{y}(t) \geqq \int_t^\infty \frac{R_{n-3}(s, t)}{(n-3)!} f(s, y(s), y[h(s)]) \, ds \quad \text{for} \quad t \geqq t_0 .$$

Multiplying (42) by $\{\varphi(y(t))\}^{-1}$, using the monotonicity of the functions y, φ, (38), and the fact that n is even ($n + k$ is odd), we obtain

$$(43) \qquad \frac{\dot{y}(t)}{\varphi(y(t))} \geqq d \int_t^\infty \frac{R_{n-3}(s, t) f(s, y(s), y[h(s)])}{(n-3)! \, \varphi(y(s))} \, ds \geqq$$
$$\geqq \frac{d}{(n-3)!} \int_t^\infty \frac{R_{n-3}(s, t) f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \, \varphi_2(\varrho(s) (H(s))^{n-1})} \, ds \quad \text{for} \quad t \geqq T .$$

Integrating (43) from T to t $(t \geqq T)$ and using (33) we get a contradiction with (35).

II. Let $k = 0$ (n is an odd number). In view of (36), (3) and (7), there exist constants σ, ε $(0 < \sigma \leqq 1, \varepsilon > 0)$ and $t_3 \geqq t_0$ such that

$$\sigma \, y(t) \leqq t^{n-1} , \quad y[h(t)] \geqq y(t) \geqq \varepsilon \quad \text{for} \quad t \geqq t_3 .$$

By virtue of the monotonicity of ψ, φ_2, f, the last inequality and (30) we have

$$(44) \qquad f(t, y(t), y[h(t)]) \geqq \frac{\psi(\sigma\, y(t))\, \varphi_2(y(t)/C\, y[h(t)])\, f(t, y(t), y[h(t)])}{\psi(t^{n-1})\, \varphi_2(\varrho(t)\, (H(t))^{n-1})} \geqq$$

$$\geqq \frac{K f(t, \varepsilon, \varepsilon)}{\psi(t^{n-1})\, \varphi_2(\varrho(t)\, (H(t))^{n-1})} \quad \text{for} \quad t \geqq t_3\,,$$

where

$$K = \psi(\varepsilon\sigma)\, \varphi_2(C_0)\,, \quad C_0 = \frac{1}{C} \inf_{t \geqq t_3} \left\{\frac{y(t)}{y[h(t)]}\right\} > 0\,.$$

It is obvious that (42) holds also for $k = 0$. Then (42) with n odd, in view of (44), implies

$$-\dot{y}(t) \geqq \frac{K}{(n-3)!} \int_t^\infty \frac{R_{n-3}(s, t)\, f(s, \varepsilon, \varepsilon)}{\psi(s^{n-1})\, \varphi_2(\varrho(s)\, (H(s))^{n-1})}\, ds \quad \text{for} \quad t \geqq t_3\,.$$

Integrating the last inequality from $T(\geqq t_3)$ to ∞ we get

$$y(T) > y(T) - y(\infty) \geqq \frac{K}{(n-2)!} \int_T^\infty \frac{R_{n-2}(s, T)\, f(s, \varepsilon, \varepsilon)}{\psi(s^{n-1})\, \varphi_2(\varrho(s)\, (H(s))^{n-1})}\, ds\,,$$

which contradicts (35).

If (18) holds and $k \in \{1, 2, \ldots, n-1\}$, then, with regard to (6), (36) is fulfilled. In all other cases (i.e. either (18) holds and $k = 0$ or (19) holds and $k \in \{0, 1, \ldots, n-1\}$) we have to assume that (36) holds. But, as shown above, this leads to a contradiction with (35). Then $\lim_{t\to\infty} y(t) = 0$ for every nonoscillatory solution $y(t) \in \boldsymbol{W}$. Hence it follows that $\lim_{t\to\infty} y^{(i)}(t) = 0$ $(i = 0, 1, \ldots, n-2)$ and $\lim_{t\to\infty} r(t)\, y^{(n-1)}(t) = 0$.

The proof of Theorem 2 is complete.

Remark. If $\psi(t) \equiv 1$, it is evident from the proof that Theorem 2 holds without the assumption $r(t) \geqq r_0 > 0$.

In the case that $n = 2$, $r(t) \equiv 1$, we get Theorem 2.9 in [6].

Further, consider the following equation

$$(\mathrm{r}) \qquad \{r(t)\, y^{(n-1)}(t)\}' + F(t, y(t), y[h_0(t)], \ldots, y^{(n-1)}(t), y^{(n-1)}[h_{n-1}(t)]) = 0\,, \quad n \geqq 2\,,$$

where

$$(45) \qquad r : [0, \infty) \to (0, \infty)\,, \quad h_i : [0, \infty) \to R \quad (i = 0, 1, \ldots, n-1)\,,$$

$$F : D(\equiv [0, \infty) \times R^{2n}) \to R \quad \text{are continuous functions}\,;$$

(46) $t \geqq h_i(t)$ for $t \geqq 0$ and $\lim_{t\to\infty} h_i(t) = \infty$ $(i = 0, 1, \ldots, n-1)$;

(47) $y_1 F(t, x_1, y_1, \ldots, x_n, y_n) > 0$ for $(t, x_1, y_1, \ldots, x_n, y_n) \in D$ and $x_1 y_1 > 0$.

The next theorem follows directly from Theorem 1 and Theorem 2.

Theorem 3. *Let equation* (r) *fulfil conditions* (45)–(47), *and in addition, let there exist a function f which satisfies* (2), (4) *and*

$$|F(t, x_1, y_1, \ldots, x_n, y_n)| \geqq |f(t, x_1, y_1)|$$

for every point $(t, x_1, y_1, \ldots, x_n, y_n) \in D_{t_0}^{(0)}$. *If inequality* (1) *has either the property A or A_0 then equation* (r) *has the same property.*

Corollary. *Let the function h satisfy conditions* (2), (3). *Let p be a continuous function and* ν, σ *real numbers such that* $p : [0, \infty) \to (0, \infty)$, $\nu \geqq 0$, $\sigma > 1$. *If*

$$\int^{\infty} t^{(n-1)(1-\sigma)} [h(t)]^{(n-1)\sigma} p(t)\, dt = \infty,$$

then the equation

$$y^{(n)}(t) + p(t) |y(t)|^{\nu} |y[h(t)]|^{\sigma} \operatorname{sgn} y[h(t)] = 0, \quad n \geqq 2$$

has the property A.

Proof. If we put $F(t, x_1, y_1, \ldots, x_n, y_n) = p(t) |x_1|^{\nu} |y_1|^{\sigma} \operatorname{sgn} y_1$, $\psi(x) \equiv 1$, $\varphi_1(x) = |x|^{\nu}$, $\varphi_2(y) = |y|^{\sigma}$, then the assertion follows from Theorem 3 and Theorem 2.

Theorem 4. *Let* $m \in \{1, 2, \ldots, n-1\}$ *and let the conditions* (18), (45)–(47), $b_0 > 0$ *be fulfilled. Further, we suppose:*

(a) $h_m(t) \leqq \{\min [h_0(t), h_1(t), \ldots, h_{m-1}(t)];\ t \geqq 0\}$;

(b) *there exists a function f which satisfies* (2), (4), *and*

(48) $$|F(t, x_1, y_1, \ldots, x_n, y_n)| \geqq |f(t, x_{m+1}, y_{m+1})|$$

for every point $(t, x_1, y_1, \ldots, x_n, y_n) \in D_{t_0}^{(m)}$;

(c) *the following inequality*

(49) $$\{[r(t)\, y^{(n-m-1)}(t)]' + f(t, y(t), y[h_m(t)])\} \operatorname{sgn} y[h_m(t)] \leqq 0$$

has the property A.

Then equation (r) *has the property* A_m.

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (r) such that

$$\liminf_{t\to\infty} y^{(m)}(t) = C > 0$$

(the case $\limsup_{t\to\infty} y^{(m)}(t) = C < 0$ is treated similarly).

From (49), in view of (46), we get

$$(50) \qquad y^{(i)}(t) > 0\,, \quad y^{(i)}[h_i(t)] > 0 \quad (i = 0, 1, \ldots, m) \quad \text{for} \quad t \geqq t_0 > 0\,.$$

Thus, with regard to (50), (47) and (18), it is obvious that the assumptions of Lemma 1 and Lemma 2 are fulfilled and therefore (5)–(10) hold, where $m \leqq k \in \{0, 1, \ldots, n-1\}$, $n + k$ is odd. By (10) and the assumption (*a*), it is easy to prove that the following inequalities

$$(51) \qquad \frac{(n-m-1)!}{(n-j)!}\left(\frac{b_0 t}{2}\right)^{m-j+1} \leqq \frac{y^{(j-1)}(t)}{y^{(m)}(t)}\,,$$

$$\frac{(n-m-1)!}{(n-j)!}\left(\frac{b_0}{2}\,h_m(t)\right)^{m-j+1} \leqq \frac{y^{(j-1)}[h_{j-1}(t)]}{y^{(m)}[h_m(t)]} \quad \text{for} \quad t \geqq t_1 \geqq 2t_0$$

hold.

Evidently, $u(t) = y^{(m)}(t)$ satisfies

$$(52) \qquad \liminf_{t\to\infty} u(t) = C > 0$$

and, for $t \geqq t_1$, $u(t)$ is a solution of the following equation

$$(53) \qquad [r(t)\, u^{(n-m-1)}(t)]' + G(t, u(t), u[h_m(t)], \ldots, u^{(n-m-1)}(t), u^{(n-m-1)}[h_{n-1}(t)]) = 0\,,$$

where

$$G(t, x_1, y_1, \ldots, x_{n-m}, y_{n-m}) = F\left(t, \frac{y(t)}{y^{(m)}(t)}\,x_1, \frac{y[h_0(t)]}{y^{(m)}[h_m(t)]}\,y_1, \ldots \right.$$
$$\left. \ldots, \frac{y^{(m-1)}(t)}{y^{(m)}(t)}\,x_1, \frac{y^{(m-1)}[h_{m-1}(t)]}{y^{(m)}[h_m(t)]}\,y_1, x_1, y_1, \ldots, x_{n-m}, y_{n-m}\right).$$

In view of the last relation, (47), (48) and (51) we get

$$(54) \qquad y_1\, G(t, x_1, y_1, \ldots, x_{n-m}, y_{n-m}) > 0\,,$$

$$(55) \qquad |G(t, x_1, y_1, \ldots, x_{n-m}, y_{n-m})| \geqq |f(t, x_1, y_1)| \quad \text{for} \quad x_1 y_1 > 0\,,$$
$$x_i, y_i \in R \quad (i = 1, 2, \ldots, n-m)\,.$$