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ON THE EXISTENCE OF PERIODIC BOUNDARY CONDITIONS FOR CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATIONS

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In [5], B. MEHRI uses a special case of a theorem about contractions given in [3] (which, due to the finiteness of distance functions considered in [5], is in fact the usual theorem about contractions; see, e.g. [6]), and a result reported by ĎURIKOVIČ [1], to establish the existence and uniqueness of solution of the nonlinear differential equation $x'' + Kx = f(t, x, x')$, satisfying the periodic boundary conditions $x(0) - x(\omega) = x'(0) - x'(\omega) = 0$. Although Mehri's Theorem 1 covers both cases $K > 0$ and $K < 0$, his Theorems 2 and 3 are restricted only to the case $K > 0$.

In this note, we first extend all the results in [5] to a system of nonlinear second order differential equations. Then we establish two theorems whose scalar cases give analogues of Theorems 2 and 3 of [5] for the case $K < 0$.

Consider the vector boundary value problem

$$x'' + Ax = f(t, x, x')\,, \tag{1}$$

$$x(0) - x(\omega) = x'(0) - x'(\omega) = 0\,, \tag{2}$$

where $x = (x_1, \ldots, x_n)$ is an n-dimensional vector; A is a constant diagonal $n \times n$ matrix; and $f(t, x, y) = (f_1(t, x_1, \ldots, x_n, y_1, \ldots, y_n), \ldots, f_n(t, x_1, \ldots, x_n, y_1, \ldots, y_n))$ is a vector valued function, defined for $(t, x, y) \in E = [0, \omega] \times R^n \times R^n$.

Throughout this paper, we take $\|x\| = \operatorname{Max}_i |x_i|$ and $\|A\| = \operatorname{Max}_{i,k} |a_{ik}|$ respectively as the norm of $x = (x_1, \ldots, x_n)$ and of $A = (a_{ik})$.

Theorem 1. *Suppose that the matrix $A = (a_i\delta_{ik})_1^n$ (δ_{ik} is the Kronecker delta) is such that all the a_i are nonzero and have the same sign. Suppose further that the vector function $f(t, x, y)$ is continuous, bounded in E and satisfies the inequality*

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqq C\{\|x_1 - x_2\| + 1/b\|y_1 - y_2\|\}\,, \tag{3}$$

where $b = \min_i \sqrt{|a_i|}$, $C > 0$ *is a constant such that*

$$\frac{2C}{b^2} < 1 . \tag{4}$$

Then in $[0, \omega] \subseteq [0, \pi/a]$, *where* $a = \max_i \sqrt{a_i}$ *if*

$$a_i > 0 , \quad i = 1, \ldots, n , \tag{5}$$

and in $[0, \omega] \subseteq [0, +\infty)$, *if*

$$a_i < 0 , \quad i = 1, \ldots, n , \tag{6}$$

the problem (1) (2) *has a unique solution. Moreover, Picard's sequence of successive approximations defined by*

$$x_n(t) = \int_0^\omega G(t, s) f(s, x_{n-1}(s), x'_{n-1}(s)) \, ds , \quad n = 1, 2, \ldots \tag{7}$$

(*where* $G(t, s)$ *is Green's matrix for the problem* (1), (2)) *for any vector function* $x_0(t)$ *specified below, converges in distance to this unique solution.*

Proof. If (5) holds, then problem (1), (2) is equivalent to the integral equation

$$x(t) = \int_0^\omega G(t, s) f(t, x(s), x'(s)) \, ds , \tag{8}$$

where $G(t, s)$ is Green's matrix for the problem (1), (2),

$$G(t, s) = \begin{cases} 2^{-1}(\sqrt{A})^{-1}\left[\operatorname{Sin}\sqrt{(A)}\,\frac{\omega}{2}\right]^{-1} \operatorname{Cos}\sqrt{(A)}\left(\frac{\omega}{2} + s - t\right) & \text{for} \\ 0 \leqq s \leqq t \leqq \omega & \\ 2^{-1}(\sqrt{A})^{-1}\left[\operatorname{Sin}\sqrt{(A)}\,\frac{\omega}{2}\right]^{-1} \operatorname{Cos}\sqrt{(A)}\left(\frac{\omega}{2} + t - s\right) & \text{for} \\ 0 \leqq t \leqq s \leqq \omega , & \end{cases} \tag{9}$$

and the matrix functions $\operatorname{Sin}\sqrt{(A)}\,t$ and $\operatorname{Cos}\sqrt{(A)}\,t$ are defined by the matrix series ([2], p. 118),

$$\operatorname{Sin}\sqrt{(A)}\,t = \sum_{p=0}^{\infty} (-1)^p \frac{(\sqrt{A})^{2p+1}}{(2p + 1)!} t^{2p+1} ,$$

$$\operatorname{Cos}\sqrt{(A)}\,t = \sum_{p=0}^{\infty} (-1)^p \frac{(\sqrt{A})^{2p}}{(2p)!} t^{2p} .$$

If (6) holds, then problem (1), (2) is equivalent to (8) where

(10)

$$G(t,s)=\begin{cases}2^{-1}(\sqrt{|A|})^{-1}\,[E-\exp\sqrt{|A|}\,\omega]^{-1}\,\{\exp[-\sqrt{|A|}\,(t-s)]\exp(\sqrt{|A|}\,\omega)\\ \qquad\qquad+\exp[\sqrt{|A|}\,(t-s)]\}\quad\text{for}\quad s\leqq t\\ 2^{-1}(\sqrt{|A|})^{-1}\,[E-\exp\sqrt{|A|}\,\omega]^{-1}\,\{\exp[-\sqrt{|A|}\,(s-t)]\exp(\sqrt{|A|}\,\omega)\\ \qquad\qquad+\exp[\sqrt{|A|}\,(s-t)]\}\quad\text{for}\quad t\leqq s\,,\end{cases}$$

and the matrix functions $\exp[\sqrt{|A|}\,t]$ and $\exp[-\sqrt{|A|}\,t]$ are defined by the matrix series

$$\exp[\sqrt{|A|}\,t]=\sum_{p=0}^{\infty}\frac{(\sqrt{|A|})^p}{p!}\,t^p\,,\quad \exp[-\sqrt{|A|}\,t]=\sum_{p=0}^{\infty}(-1)^p\,\frac{(\sqrt{|A|})^p}{p!}\,t^p\,.$$

Let S be the set of all continuous vector functions $x(t)=(x_1(t),\ldots,x_n(t))$ with continuous first derivatives $x'(t)=(x'_1(t),\ldots,x'_n(t))$ on $[0,\omega]$, and define the distance

(11) $$d(x_1,x_2)=\operatorname*{Max}_{t\in[0,\omega]}\left\{\|x_1(t)-x_2(t)\|+\frac{1}{b}\,\|x'_1(t)-x'_2(t)\|\right\},$$

for an arbitrary pair of elements $x_1(t)$, $x_2(t)$ of S. Then $X=(S,d)$ is a complete metric space. We define an operator U on X by

(12) $$Ux(t)=\int_0^{\omega}G(t,s)\,f(s,x(s),x'(s))\,\mathrm{d}s\,.$$

The operator U maps the space X into itself.

Let $x_1(t)$, $x_2(t)$ be any two elements from X, then

$$\|Ux_1(t)-Ux_2(t)\|\leqq C\,d(x_1,x_2)\operatorname*{Max}_i\frac{1}{|a_i|}\leqq\frac{C}{b^2}\,d(x_1,x_2)\,,$$

and

$$\frac{1}{b}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\,Ux_1(t)-\frac{\mathrm{d}}{\mathrm{d}t}\,Ux_2(t)\right\|\leqq\frac{C}{b}\,d(x_1,x_2)\operatorname*{Max}_i\frac{1}{\sqrt{|a_i|}}\leqq\frac{C}{b^2}\,d(x_1,x_2)\,.$$

Hence

$$d(Ux_1,Ux_2)\leqq\frac{2C}{b^2}\,d(x_1,x_2)\,.$$

Now (4) and the fact that any two elements of X have a finite distance, complete the proof of the theorem.

In the following two theorems we shall assume that (5) holds. Since $\omega \in [0, \pi/a]$, it follows that $\sqrt{(a_i)}\,(\omega/2) \in [0, \pi/2]$ for each i, and hence $\operatorname{Sin} \sqrt{(a_i)}\,(\omega/2) \geqq$ $\geqq (2/\pi) \sqrt{(a_i)}\,(\omega/2)$ for each i involving

$$\|G(t, s)\| \leqq \frac{\pi}{2b^2\omega}, \quad \|G_t(t, s)\| \leqq \frac{\pi}{2b\omega}.$$

Let S and U be as before, then $US \subseteq S$. Let (S^*, d) be the completion of (US, d) where d is given by (11).

Theorem 2. *Let $f(t, x, y)$ be a vector function defined and continuous on E, and satisfying the following conditions*

$$\|f(t, x, y)\| \leqq \frac{b^2}{2\pi} t^p, \quad p \geqq 0, \quad (t, x, y) \in E, \tag{13}$$

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqq \frac{b^2}{\pi t^r} \left\{ \|x_1 - x_2\|^q + \left[\frac{1}{b} \|y_1 - y_2\|\right]^q \right\} \tag{14}$$

for $(t, x_i, y_i) \in E$, $i = 1, 2$, where $q \geqq 1$, $0 < r < 1$, $r = p(q - 1)$ and

$$\frac{1}{(1 - r)} \left(\frac{1}{p + 1}\right)^{q-1} < 1.$$

Then problem (1), (2) has a unique solution $x(t) \in S^*$, and the successive approximations defined by (7) for any $x_0(t) \in S$, converge in distance to this unique solution.

Proof. The space $X = (S^*, d)$ is a complete metric space, and U, defined by (12), maps X into itself. Let $z_1(t)$, $z_2(t)$ be any two elements of X, then from (12) and (13)

$$\|z_1(t) - z_2(t)\| \leqq \frac{b^2}{\pi} \int_0^\omega \|G(t, s)\| \, s^p \, ds \leqq \frac{1}{2(p + 1)} \omega^p$$

and

$$\frac{1}{b} \|z_1'(t) - z_2'(t)\| \leqq \frac{b^2}{\pi b} \int_0^\omega \|G_t(t, s)\| \, s^p \, ds \leqq \frac{1}{2(p + 1)} \omega^p.$$

From (14) and (11) we obtain

$$\|Uz_1(t) - Uz_2(t)\| \leqq \frac{b^2}{\pi} \left(\frac{\omega^p}{p + 1}\right)^{q-1} \cdot \frac{\pi}{2b^2\omega} \cdot \frac{d(z_1, z_2)}{(1 - r)} \omega^{1-r} \leqq$$
$$\leqq \frac{1}{2} \cdot \frac{1}{(p + 1)^{q-1}} \cdot \frac{d(z_1, z_2)}{(1 - r)}$$

and

$$\frac{1}{b} \left\| \frac{d}{dt} Uz_1(t) - \frac{d}{dt} Uz_2(t) \right\| \leqq \frac{1}{2} \cdot \frac{1}{(p + 1)^{q-1}} \cdot \frac{d(z_1, z_2)}{(1 - r)}.$$

From the last two inequalities, it follows that

$$d(Uz_1, Uz_2) \leqq \frac{1}{(p+1)^{q-1}} \cdot \frac{1}{1-r} d(z_1, z_2)$$

which completes the proof.

Remark. In Theorem 2 it is assumed that $f(t, x, y)$ is bounded on E. The following theorem (whose proof is similar to that of Theorem 2) shows that this assumption is not necessary.

Theorem 3. *Let $f(t, x, y)$ be continuous on E and satisfy the following conditions*

(15) $$\|f(t, x, y)\| \leqq \frac{b^2}{2\pi} t^{-p}, \quad 0 < p < 1, \quad (t, x, y) \in E,$$

(16) $$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqq \frac{b^2}{\pi} t^{p(q-1)} \left\{\|x_1 - x_2\|^q + \left[\frac{1}{b}\|y_1 - y_2\|\right]^q\right\},$$

where $q \geqq 1$ and

$$\left(\frac{1}{1-p}\right)^{q-1} \cdot \frac{1}{p(q-1)+1} < 1.$$

Then problem (1), (2) has a unique solution, and the successive approximations defined by (7) for any $x_0(t) \in S$, converge in distance to this unique solution.

In the following two theorems we shall assume that (6) holds. Then we have

$$\|G(t, s)\| \leqq \frac{2 + b\omega}{2b^2\omega}, \quad \|G_t(t, s)\| \leqq \frac{2 + b\omega}{2b\omega}.$$

Theorem 4. *Let $f(t, x, y)$ be continuous on E, and let $C > 0$ be a constant such that*

(17) $$\|f(t, x, y)\| \leqq \frac{b^2C}{2} t^p, \quad p \geqq 0, \quad (t, x, y) \in E,$$

(18) $$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leqq \frac{b^2C}{t^r} \left\{\|x_1 - x_2\|^q + \left[\frac{1}{b}\|y_1 - y_2\|\right]^q\right\},$$

where $q \geqq 1$, $0 < r < 1$, $r = p(q-1)$ and

(19) $$2C\left(\frac{1}{1-r}\right)^{1/q}\left(\frac{1}{p+1}\right)^{q-1/q} < 1.$$

Then there exists an $\omega_0 > 0$ such that for every ω, $0 < \omega \leqq \omega_0$, (1), (2) has a unique solution $x(t) \in S^*$, and the successive approximations defined by (7) for any $x_0(t) \in S$, converge in distance to this unique solution.

Proof. Let $X = (S^*, d)$, and let $z_1(t)$, $z_2(t)$ be any two elements of X, then from (12) and (17)

$$\|z_1(t) - z_2(t)\| \leqq b^2 C \int_0^\omega \|G(t, s)\| \, s^p \, \mathrm{d}s \leqq \frac{C(2 + b\omega)\, \omega^p}{2(p + 1)}$$

and

$$\frac{1}{b} \|z_1'(t) - z_2'(t)\| \leqq \frac{b^2 C}{b} \int_0^\omega \|G_t(t, s)\| \, s^p \, \mathrm{d}s \leqq \frac{C(2 + b\omega)\, \omega^p}{2(p + 1)} .$$

From (18) and (11), it follows that

$$\|Uz_1(t) - Uz_2(t)\| \leqq b^2 C \,. \left(\frac{C(2 + b\omega)\, \omega^p}{p + 1}\right)^{q-1} . \frac{2 + b\omega}{2b^2\omega} . \frac{d(z_1, z_2)}{1 - r} \, \omega^{1-r} \leqq$$

$$\leqq \frac{1}{2} . \frac{(C(2 + b\omega))^q}{(p + 1)^{q-1}} . \frac{1}{1 - r} \, d(z_1, z_2)$$

and

$$\frac{1}{b} \left\|\frac{\mathrm{d}}{\mathrm{d}t} Uz_1(t) - \frac{\mathrm{d}}{\mathrm{d}t} Uz_2(t)\right\| \leqq \frac{1}{2} . \frac{(C(2 + b\omega))^q}{(p + 1)^{q-1}} . \frac{1}{1 - r} \, d(z_1, z_2) .$$

From the last two inequalities we obtain

$$d(Uz_1, Uz_2) \leqq \frac{(C(2 + b\omega))^q}{(p + 1)^{q-1}} . \frac{1}{1 - r} \, d(z_1, z_2) .$$

U is a contraction map provided that

$$\frac{(C(2 + b\omega))^q}{(p + 1)^{q-1}} . \frac{1}{1 - r} < 1 . \tag{20}$$

Clearly (20) is satisfied if

$$\omega < \frac{1}{b} \left\{\frac{1}{C} (p + 1)^{q-1/q} (1 - r)^{1/q} - 2 . \right. \tag{21}$$

Therefore, if $\omega > 0$ is chosen so that (21) is satisfied, then problem (1), (2) has a unique solution with the desired property.

Theorem 5. *Let $f(t, x, y)$ be continuous on E, and let $C > 0$ be a constant such that*

$$\|f(t, x, y)\| \leqq \frac{b^2 C}{2} t^{-p} , \quad 0 < p < 1 , \quad (t, x, y) \in E , \tag{22}$$

$$\|f(t, x_1, y_1) - f(t. x_2, y_2)\| \leqq b^2 C t^{p(q-1)} \left\{\{\|x_1 - x_2\|^q + \left[\frac{1}{b} \|y_1 - y_2\|\right]^q\right\} \tag{23}$$