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# LOCAL DETERMINACY OF SYMMETRIC PSEUDOPROCESSES

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A number of various physical systems can be described by means of relations in the cartesian product $P \times T$, where $P$ is the set of all possible states of the system concerned and $T$ is a set of time instants. This approach to the study of the behaviour of systems was used in [1] to [5], where a wide class of such relations is investigated in detail. The present paper is a direct continuation of [5] so that as far as the notation and the terminology is concerned, the reader is refered to [5]. To make the text of the paper as self-contained as possible, the basic notions and notation from [5] will be recalled in the first point of the next section.

## 1. SYMMETRIC PSEUDOPROCESSES

**1.1. Notation.** In what follows, $P$ denotes an arbitrary set, $R$ the set of all reals, $R^* = R \cup \{+\infty, -\infty\}$ the extended real line with the ordering extended from $R$ to $R^*$ in the natural way, $T$ a subset of $R$.

If $X$, $Y$ are sets, then any subset of the cartesian product $X \times Y$ (in this order) is called a relation between $X$ and $Y$. If $X = Y$, then a relation $\mathsf{r} \subset X \times X$ is called a relation in $X$. The relation inverse to a relation $\mathsf{r}$ is denoted by $\mathsf{r}^{-1}$. The identity relation in $X$ is denoted by $1_X$. If $\mathsf{r} \subset X \times Y$, $\mathsf{s} \subset Y \times Z$, then the composition of the relations $\mathsf{r}$ and $\mathsf{s}$ (in this order) is denoted by $\mathsf{r} \circ \mathsf{s}$. If a pair $(x, y) \in X \times Y$ belongs to a relation $\mathsf{r} \subset X \times Y$, then we write either $(x, y) \in \mathsf{r}$ or $x\mathsf{r}y$. Given $\mathsf{r} \subset X \times Y$, we set

$$D_\mathsf{r} = \{y \in Y \mid x\mathsf{r}y \text{ for some } x \in X\}\,, \tag{1.1.1}$$

$$I_\mathsf{r} = \{x \in X \mid x\mathsf{r}x\} \text{ if } X = Y\,, \tag{1.1.2}$$

$$\mathsf{r}y = \{x \in X \mid (x, y) \in D_\mathsf{r}\}\,, \tag{1.1.3}$$

$$\mathsf{r}A = \{x \in X \mid (x, y) \in D_\mathsf{r} \text{ for some } y \in A\}\,, \tag{1.1.4}$$

$$(1.1.5) \qquad x\mathsf{r} = \{y \in Y \mid (x, y) \in D_{\mathsf{r}}\},$$

$$(1.1.6) \qquad B\mathsf{r} = \{y \in Y \mid (x, y) \in D_{\mathsf{r}} \text{ for some } x \in B\},$$

$$(1.1.7) \qquad \mathsf{r}|_A = \mathsf{r} \cap (X \times A),$$

for each $y \in Y$, $A \subset Y$, $x \in X$, $B \subset X$.

In the present paper we shall be concerned mainly with relations $\mathsf{p}$, $\mathsf{q}$ in $P \times T$, i.e. with subsets of $(P \times T) \times (P \times T)$. Each such relation $\mathsf{q} \subset (P \times T) \times (P \times T)$ can be uniquely described by the two-parametric system of relations ${}_v\mathsf{q}_u$ in $P$ with $u, v \in T$ as follows:

$$(1.1.8). \qquad (y, v)\, \mathsf{q}(x, u) \quad \text{iff} \quad y_v\mathsf{q}_u x\,, \quad x, y \in P\,, \quad u, v \in T\,.$$

A relation $\mathsf{p}$ in $P \times T$ such that

$$\text{(I)} \qquad {}_u\mathsf{p}_u \subset 1_P \quad \text{for each} \quad u \in T$$

and

$$\text{(R)} \qquad {}_v\mathsf{p}_u \neq \emptyset \quad \text{implies} \quad u \leqq v \quad \text{for all} \quad u, v \in T$$

is called a *right pseudoprocess in $P$ over $T$*. The set of all right pseudoprocesses in $P$ over $T$ i sdenoted by $\mathsf{Ps}(P, T)$. A right pseudoprocess $\mathsf{p} \in \mathsf{Ps}(P, T)$ is said to be a *compositive right pseudoprocess*, a *transitive right pseudoprocess* or a *right process in $P$ over $T$* iff the condition

$$\text{(RC)} \qquad {}_v\mathsf{p}_u \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T\,,$$

$$\text{(RT)} \qquad {}_v\mathsf{p}_u \supset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T$$

or

$$\text{(RP)} \qquad {}_v\mathsf{p}_u = {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T\,,$$

is satisfied, respectively. The set of all compositive right pseudoprocesses, transitive right pseudoprocesses and right processes in $P$ over $T$ will be denoted by $\mathsf{Psc}(P, T)$, $\mathsf{Pst}(P, T)$ and $\mathsf{P}(P, T)$. A more detailed explanation of the theory of right pseudoprocesses may be found in [5].

**1.2. Definition.** Let $P$ be an arbitrary set, $T \subset R$, $\mathsf{q}$ a relation in $P \times T$. The relation $\mathsf{q}$ is called a *symmetric pseudoprocess in $P$ over $T$* iff it satisfies the conditions

$$\text{(I)} \qquad {}_u\mathsf{q}_u \subset 1_P \quad \text{for all} \quad u \in T\,,$$

$$\text{(S)} \qquad {}_v\mathsf{q}_u = ({}_u\mathsf{q}_v)^{-1} \quad \text{for all} \quad u, v \in T\,.$$

The set of all symmetric pseudoprocesses in $P$ over $T$ will be denoted by $\mathsf{Ss}(P, T)$.

**1.3. Remark.** The property (S) in 1.2 may be reformulated as

$$y_v\mathsf{q}_u x \quad \text{iff} \quad x_u\mathsf{q}_v y \quad \text{for all} \quad x, y \in P, \quad u, v \in T. \tag{1.3.1}$$

Hence, this property is equivalent with

$$\mathsf{q} = \mathsf{q}^{-1}. \tag{1.3.2}$$

Thus

$${}_v\mathsf{q}_u = {}_u\mathsf{q}_v \quad \text{for all} \quad u, v \in T. \tag{1.3.3}$$

For a symmetric pseudoprocess $\mathsf{q}$, the sets $D_\mathsf{q}$ and $I_\mathsf{q}$ from (1.1.1) and (1.1.2) may be characterized as follows:

$$D_\mathsf{q} = \{(x, u) \in P \times T \mid {}_v\mathsf{q}_u x \neq \emptyset \text{ for some } v \in T\}$$

and

$$I_\mathsf{q} = \{(x, u) \in D_\mathsf{q} \mid x_u\mathsf{q}_u x\}.$$

**1.4. Construction.** Let $\mathsf{p} \in \mathsf{Ps}(P, T)$. It is not difficult to verify that the relation $\mathsf{p} \cup \mathsf{p}^{-1}$ in $P \times T$ fulfils the conditions of Definition 1.2 so that it is a symmetric pseudoprocess in $P$ over $T$. The symmetric pseudoprocess $\mathsf{q}$ in $P$ over $T$ defined by

$$\mathsf{q} = \mathsf{p} \cup \mathsf{p}^{-1} \tag{1.4.1}$$

is said to be *induced by the right pseudoprocess* $\mathsf{p}$ in $P$ over $T$.

Let us show that, given a symmetric pseudoprocess $\mathsf{q} \in \mathsf{Ss}(P, T)$, there exists a right pseudoprocess $\mathsf{p} \in \mathsf{Ps}(P, T)$ such that (1.4.1) holds.

**1.5. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$. The right pseudoprocess $\mathsf{q}^+$ is said to be *positively induced by the symmetric pseudoprocess* $\mathsf{q}$ iff it satisfies the condition

$${}_v\mathsf{q}^+{}_u = {}_v\mathsf{q}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad T. \tag{1.5.1}$$

**1.6. Remark.** Since $\mathsf{q}^+$ is a right pseudoprocess, it holds

$${}_v\mathsf{q}^+{}_u = \emptyset \quad \text{for all} \quad u > v \quad \text{in} \quad T \tag{1.6.1}$$

so that we obtain from 1.5 and (1.3.1) that

$$D_{\mathsf{q}^+} \subset D_\mathsf{q}. \tag{1.6.2}$$

The inclusion in (1.6.2) cannot be in general replaced by the equality. However, if $\mathsf{q} \in \mathsf{Ss}(P, T)$ is such that for each $(x, u) \in D_\mathsf{q}$ there exists $t \in T$ fulfilling the conditions $t \geqq u$ and ${}_t\mathsf{q}_u x \neq \emptyset$, then equality

$$D_{\mathsf{q}^+} = D_\mathsf{q} \tag{1.6.3}$$

holds.

**1.7. Lemma.** *Let* $\mathsf{q}, {}^{\sim}\mathsf{q} \in \mathsf{Ss}(P, T)$. *Then the following assertions hold*:

(i) $\mathsf{q} = \mathsf{q}^+ \cup (\mathsf{q}^+)^{-1}$.

(ii) $\mathsf{q} \subset {}^{\sim}\mathsf{q}$ iff $\mathsf{q}^+ \subset {}^{\sim}\mathsf{q}^+$.

**1.8. Definition.** Let $\mathsf{p} \in \mathsf{Ps}(P, T)$, $\mathsf{q} \in \mathsf{Ss}(P, T)$, $\mathsf{p}' \in \mathsf{Ps}(P, -T)$, $\mathsf{q}' \in \mathsf{Ss}(P, -T)$, where $-T = \{t \in R \mid -t \in T\}$.

The right pseudoprocess $\mathsf{p}'$ is said to be *orientation-change produced from the right pseudoprocess* $\mathsf{p}$ iff

$$(1.8.1) \qquad {}_v\mathsf{p}'_u = ({}_{-u}\mathsf{p}_{-v})^{-1} \quad \text{for all} \quad u \leqq v \quad \text{in} \quad -T\,.$$

The symmetric pseudoprocess $\mathsf{q}'$ is said to be *orientation-change produced from the symmetric pseudoprocess* $\mathsf{q}$ iff

$$(1.8.2) \qquad {}_v\mathsf{q}'_u = ({}_{-u}\mathsf{q}_{-v})^{-1} \quad \text{for all} \quad u, v \quad \text{in} \quad -T\,.$$

**1.9. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, let $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ be positively induced by $\mathsf{q}$, and let $\mathsf{q}^- \in \mathsf{Ps}(P, -T)$.

The right pseudoprocess $\mathsf{q}^-$ is said to be *negatively induced* by the symmetric pseudoprocess $\mathsf{q}$ iff $\mathsf{q}^-$ is orientation-change produced from $\mathsf{q}^+$.

**1.10. Remark.** A right pseudoprocess $\mathsf{q}^-$ from the preceding definition can be described directly by the relations ${}_v\mathsf{q}_u$ as follows.

The equality

$$(1.10.1) \qquad {}_v\mathsf{q}^-{}_u = ({}_{-u}\mathsf{q}^+{}_{-v})^{-1} = ({}_{-u}\mathsf{q}_{-v})^{-1} = {}_{-v}\mathsf{q}_{-u}$$

holds for all $u \leqq v$ in $-T$, hence

$$(1.10.2) \qquad {}_v\mathsf{q}_u = {}_{-v}\mathsf{q}^-{}_{-u} \quad \text{for all} \quad v \leqq u \quad \text{in} \quad T\,.$$

Clearly

$$(1.10.3) \qquad D_{\mathsf{q}^-} \subset D_{\mathsf{q}'} = \{(x, u) \in P \times (-T) \mid (x, -u) \in D_{\mathsf{q}}\},$$

where the symmetric pseudoprocess $\mathsf{q}'$ is orientation-change produced from the symmetric pseudoprocess $\mathsf{q}$. If for each $(x, u) \in D_{\mathsf{q}}$ there exists $t \leqq u$ in $T$ such that ${}_t\mathsf{q}_u x \neq \emptyset$, then the inclusion in (1.10.3) can be replaced by the equality.

**1.11. Lemma.** *Let* $\mathsf{q}, {}^{\sim}\mathsf{q} \in \mathsf{Ss}(P, T)$ *and let* $\mathsf{q}' \in \mathsf{Ss}(P, -T)$ *be orientation-change produced from* $\mathsf{q}$. *Then the following assertions hold*:

(i) $\mathsf{q}' = \mathsf{q}^- \cup (\mathsf{q}^-)^{-1}$.

(ii) $\mathsf{q} \subset {}^{\sim}\mathsf{q}$ *iff* $\mathsf{q}^- \subset {}^{\sim}\mathsf{q}^-$.

Proof. The assertions follow from (1.8.2), 1.2 (ii), (1.10.1), (1.10.2) and from Lemmas 1.2 and 1.3 in [5].

**1.12. Corollary.** *Let* $\mathsf{q}$, $\tilde{\mathsf{q}} \in \mathsf{Ss}(P, T)$. *Then the following three inclusions are equivalent*:

(i) $\mathsf{q} \subset \tilde{\mathsf{q}}$;

(ii) $\mathsf{q}^+ \subset \tilde{\mathsf{q}}^+$;

(iii) $\mathsf{q}^- \subset \tilde{\mathsf{q}}^-$.

**1.13. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, $s \subset P \times T$. The relation $s$ is called a *solution* of the symmetric pseudoprocess $\mathsf{q}$ iff the following three conditions are satisfied:

(i) the domain $D_s$ of $s$ is an interval in $T$;

(ii) $s$ is a map of $D_s$ into $P$;

(iii) $s(v)\,{}_v\mathsf{q}_u\,s(u)$ holds for all $u, v \in D_s$.

The set of all solutions of $\mathsf{q}$ will be denoted by $S_\mathsf{q}$.

**1.14. Theorem.** *Let* $\mathsf{q}$, $\tilde{\mathsf{q}} \in \mathsf{Ss}(P, T)$, *let* $\mathsf{q}' \in \mathsf{Ss}(P, -T)$ *be orientation-change produced from* $\mathsf{q}$ *and let* $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ *and* $\mathsf{q}^- \in \mathsf{Ps}(P, -T)$ *be the right pseudoprocesses positively and negatively induced by* $\mathsf{q}$, *respectively. Let* $s : T \to P$ *and* $s' : -T \to P$ *be maps such that* $D_s$ *is an interval in* $T$; $D_{s'} = \{t \mid -t \in D_s\}$ *and* $s'(-t) = s(t)$ *for all* $t \in D_s$. *Then the following assertions hold*:

(i) $s \in S_\mathsf{q}$ *iff* $s \times s \subset \mathsf{q}$.

(ii) $s \in S_\mathsf{q}$ *iff* $s' \in S_{\mathsf{q}'}$.

(iii) $S_\mathsf{q} = S_{\mathsf{q}^+}$.

(iv) $S_{\mathsf{q}'} = S_{\mathsf{q}^-}$.

(v) $S_{\mathsf{q} \cap \tilde{\mathsf{q}}} = S_\mathsf{q} \cap S_{\tilde{\mathsf{q}}}$.

**1.16. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$. The maps

(1.16.1) $$e^+ : D_\mathsf{q} \to R^\#, \quad e^- : D_\mathsf{q} \to R^\#$$

defined by

(1.16.2) $$e^+(x, u) = \sup \{t \in T \mid {}_t\mathsf{q}_u x \neq \emptyset\},$$

(1.16.3) $$e^-(x, u) = \inf \{t \in T \mid {}_t\mathsf{q}_u x \neq \emptyset\}$$

are called the *positive* and the *negative extent of existence* of $\mathsf{q}$, respectively.

**1.17. Remark.** Let us recall that if $\mathsf{q}^+$ or $\mathsf{q}^-$ is a right pseudoprocess positively or negatively induced by a symmetric pseudoprocess $\mathsf{q}$, then the extents of existence $e$ or $e'$ of these pseudoprocesses are defined, according to Definition 2.3 in [5], by

(1.17.1) $$e(x, u) = \sup \{t \in T \mid {}_t\mathsf{q}^+{}_u x \neq \emptyset\}, \quad (x, u) \in D_{\mathsf{q}^+}$$

or

$$(1.17.2) \qquad e'(x, u) = \sup \{t \in T \mid {}_t\mathsf{q}^-{}_u x \neq \emptyset\}, \quad (x, u) \in D_{\mathsf{q}^-}.$$

If $(x, u) \in D_{\mathsf{q}^+}$, then $(x, u) \in D_{\mathsf{q}}$ so that $e^+(x, u)$ as well as $e(x, u)$ are defined and it is evident that the equality

$$(1.17.3) \qquad e^+(x, u) = e(x, u), \quad (x, u) \in D_{\mathsf{q}^+}$$

holds. Analogously, if $(x, -u) \in D_{\mathsf{q}^-}$, then $(x, u) \in D_{\mathsf{q}}$ so that both $e'(x, -u)$ and $e^-(x, u)$ are defined and the equality

$$(1.17.4) \qquad e^-(x, u) = -e'(x, -u)$$

takes place. From (1.16.2) and (1.16.3) we obtain immediately the inequality

$$(1.17.5) \qquad e^-(x, u) \leqq e^+(x, u) \quad \text{for all} \quad (x, u) \in D_{\mathsf{q}}.$$

**1.18. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *local existence at a point* $(x, u) \in D_{\mathsf{q}}$ iff $e^+(x, u) > u$ $(e^-(x, u) < u,\ e^-(x, u) < u < e^+(x, u))$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *local existence* iff it has positive (negative, bilateral) local existence at each point $(x, u) \in D_{\mathsf{q}}$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *global existence at a point* $(x, u) \in D_{\mathsf{q}}$ iff $e^+(x, u) = \sup T$ $(e^-(x, u) = \inf T$, $e^+(x, u) = \sup T$ and $e^-(x, u) = \inf T)$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *global existence* iff it has positive (negative, bilateral) global existence at each point $(x, u) \in D_{\mathsf{q}}$.

If $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ is a right pseudoprocess positively induced by a symmetric pseudoprocess $\mathsf{q}$, then $\mathsf{q}^+$ is said to have some of the properties described above iff $\mathsf{q}$ has the property.

**1.19. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$. A point $(x, u) \in I_{\mathsf{q}}$ is called a *start point* or an *end point* of the symmetric pseudoprocess $\mathsf{q}$ iff ${}_t\mathsf{q}_u x = \emptyset$ holds for all $t \in T$ such that $t < u$ or $u < t$, respectively.

**1.20. Remark.** Let a (right or symmetric) pseudoprocess $\mathsf{r}$ in $P$ over $T$ be given. In accordance with the notation introduced in [5], item 5.1, the symbol $\mathscr{S}_{\mathsf{r}}$ or $\mathscr{E}_{\mathsf{r}}$ will stand for the set of all start or end points of the pseudoprocess $\mathsf{r}$, respectively.

Let $\mathsf{q} \in \mathsf{Ss}(P, T)$, $(x, u) \in D_{\mathsf{q}}$. Then $(x, u) \in \mathscr{S}_{\mathsf{q}}$ or $(x, u) \in \mathscr{E}_{\mathsf{q}}$ iff $e^-(x, u) = u$ or $e^+(x, u) = u$, respectively. If $(x, u) \in \mathscr{S}_{\mathsf{q}}$ or $(x, u) \in \mathscr{E}_{\mathsf{q}}$ and $s \in S_{\mathsf{q}}$ is such that $s(u) = x$, then $u = \min D_s$ or $u = \max D_s$, respectively. The converse of this assertion is not valid.

**1.21. Lemma.** *Let* $\mathsf{q} \in \mathrm{Ss}(P, T)$. *Let* $\mathsf{q}^+ \in \mathrm{Ps}(P, T)$ *or* $\mathsf{q}^- \in \mathrm{Ps}(P, -T)$ *be positively or negatively induced by* $\mathsf{q}$, *respectively. Let* $(x, u) \in I_\mathsf{q}$. *Then the following assertions hold.*

(i) $\mathscr{S}_\mathsf{q} = \mathscr{S}_{\mathsf{q}^+}$, $\mathscr{E}_\mathsf{q} = \mathscr{E}_{\mathsf{q}^+}$.

(ii) $(x, u) \in \mathscr{S}_\mathsf{q}$ *iff* $(x, -u) \in \mathscr{E}_{\mathsf{q}^-}$, $(x, u) \in \mathscr{E}_\mathsf{q}$ *iff* $(x, -u) \in \mathscr{S}_{\mathsf{q}^-}$.

Proof follows from 1.2 (ii), (1.5.1) and (1.10.2).

**1.22. Definition.** Let $\mathsf{q} \in \mathrm{Ss}(P, T)$. The maps

(1.22.1) $$d^+ : D_\mathsf{q} \to R^\# \quad \text{and} \quad d^- : D_\mathsf{q} \to R^\#$$

defined by

(1.22.2) $$d^+(x, u) = \sup \{w \in R \mid \operatorname{card} ({}_t\mathsf{q}_u x) \leqq 1 \text{ for all } t \in T \cap \langle u, w\rangle\},$$

(1.22.3) $$d^-(x, u) = \inf \{w \in R \mid \operatorname{card} ({}_t\mathsf{q}_u x) \leqq 1 \text{ for all } t \in T \cap \langle w, u\rangle\}$$

are called the *positive* and the *negative extent of unicity* of the symmetric pseudoprocess $\mathsf{q}$.

**1.23. Remark.** Notice that $+\infty$ may belong to the range of the function $d^+$ and that $d^+$ can assume this value also in the case of a bounded $T$. Similarly for $d^-$ and $-\infty$. In general, it holds

(1.23.1) $$-\infty \leqq d^-(x, u) \leqq u \leqq d^+(x, u) \leqq +\infty .$$

Let $\mathsf{q}^+$ and $\mathsf{q}^-$ be the right pseudoprocesses positively and negatively induced by a symmetric pseudoprocess $\mathsf{q}$, respectively. Then, according to Definition 2.5 in [5] the extents of unicity $d$ and $d'$ of $\mathsf{q}^+$ and $\mathsf{q}^-$ are defined by

(1.23.2) $$d(x, u) = \sup \{w \in R \mid \operatorname{card} ({}_t\mathsf{q}^+{}_u x) \leqq 1 \text{ for all } t \in T \cap \langle u, w\rangle\} \quad \text{for all} \quad (x, u) \in D_{\mathsf{q}^+},$$

and

(1.23.3) $$d'(x, u) = \sup \{w \in R \mid \operatorname{card} ({}_t\mathsf{q}^-{}_u x) \leqq 1 \text{ for all } t \in (-T) \cap \langle u, w\rangle\} \quad \text{for all} \quad (x, u) \in D_{\mathsf{q}^-},$$

respectively. Then for each $(x, u) \in D_{\mathsf{q}^+} \subset D_\mathsf{q}$ both $d(x, u)$ and $d^+(x, u)$ are defined and

(1.23.4) $$d^+(x, u) = d(x, u) \quad \text{for all} \quad (x, u) \in D_{\mathsf{q}^+} .$$

Analogously, for each $(x, -u) \in D_{\mathsf{q}^-}$ it holds $(x, u) \in D_\mathsf{q}$ so that $d'(x, -u)$ as well as $d^-(x, u)$ are defined and the equality

(1.23.5) $$d^-(x, u) = -d'(x, -u) \quad \text{for all} \quad (x, -u) \in D_{\mathsf{q}^-}$$

takes place.

**1.24. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *local unicity at a point* $(x, u) \in D_{\mathsf{q}}$ iff $d^+(x, u) > u$ $(d'(x, u) < u, d'(x, u) < u < d^+(x, u))$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *local unicity* iff it has positive (negative, bilateral) local unicity at each point $(x, u) \in D_{\mathsf{q}}$.

The symmetric pseudoprocess $\mathsf{q}$ is siad to have *positive* (*negative, bilateral*) *global untcity at a point* $(x, u) \in D_{\mathsf{q}}$ iff $d^+(x, u) = +\infty$ $(d^-(x, u) = -\infty, d^+(x, u) = -d^-(x, u) = +\infty)$.

The symmetric pseudoprocess $\mathsf{q}$ is said to have *positive* (*negative, bilateral*) *global unicity* iff it has positive (negative, bilateral) global unicity at each point $(x, u) \in D_{\mathsf{q}}$.

If $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ is a right pseudoprocess positively induced by a symmetric pseudoprocess $\mathsf{q}$, then $\mathsf{q}^+$ is said to have some of the properties described above iff $\mathsf{q}$ has the property.

**1.25. Lemma.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$. *Then the following assertions hold*:

(i) *If* $(x, u) \in D_{\mathsf{q}}$ *is a start or an end point of* $\mathsf{q}$, *then* $d^-(x, u) = -\infty$ *or* $d^+(x, u) = +\infty$, *respectively.*

(ii) *If* $(x, u) \in D_{\mathsf{q}}$ *and* $u < d^+(x, u) < +\infty$ *or* $-\infty < d^-(x, u) < u$, *then* $d^+(x, u) < e^+(x, u)$ *or* $e^-(x, u) < d^-(x, u)$, *respectively.*

**1.26. Lemma.** *Let* $\mathsf{q}, \tilde{\mathsf{q}} \in \mathsf{Ss}(P, T)$, $\mathsf{q} \subset \tilde{\mathsf{q}}$. *Let* $e^+, e^-, d^+, d^-$ *and* $\tilde{e}^+, \tilde{e}^-, \tilde{d}^+, \tilde{d}^-$ *be the corresponding extents of existence and unicity of* $\mathsf{q}$ *and* $\tilde{\mathsf{q}}$, *respectively. Then the following assertions hold*:

(i) *If* $(x, u) \in D_{\mathsf{q}}$, *then*

$$e^+(x, u) \leqq \tilde{e}^+(x, u), \quad \tilde{e}^-(x, u) \leqq e^-(x, u), \tag{1.26.1}$$

$$\tilde{d}^+(x, u) \leqq d^+(x, u), \quad d^-(x, u) \leqq \tilde{d}^-(x, u). \tag{1.26.2}$$

(ii) *If* $(x, u) \in D_{\tilde{\mathsf{q}}}$ *is a start point or an end point of* $\tilde{\mathsf{q}}$, *then it is a start point or an end point of* $\mathsf{q}$, *respectively.*

(iii) *If* $\mathsf{q}$ *has positive, negative, bilateral local or global existence at a point* $(x, u) \in D_{\mathsf{q}}$, *then* $\tilde{\mathsf{q}}$ *has the same property.*

(iv) *If* $\tilde{\mathsf{q}}$ *has positive, negative, bilateral local or global unicity at a point* $(x, u) \in D_{\tilde{\mathsf{q}}}$, *then* $\mathsf{q}$ *has the same property.*

**1.27. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$. The symmetric pseudoprocess $\mathsf{q}$ is said to be *solution complete* iff for each pair $((y, v), (x, u)) \in \mathsf{q}$ there exists $s \in S_{\mathsf{q}}$ such that $s(u) = x$, $s(v) = y$.

**1.28. Theorem.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$. *Then the symmetric pseudoprocess* $\mathsf{q}$ *is solution complete iff the right pseudoprocess* $\mathsf{q}^+$ *positively induced by* $\mathsf{q}$ *is solution complete.*

Proof follows directly from 1.7 (i), 1.14 (iii) and 1.27.

## 2. SYMMETRIC PROCESSES

**2.1. Definition.** Let $\mathsf{q} \in \mathsf{Ss}(P, T)$ and let $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ be positively induced by $\mathsf{q}$. The symmetric pseudoprocess $\mathsf{q}$ is said to be *compositive* or *transitive* iff the right pseudoprocess $\mathsf{q}^+$ is compositive or transitive, respectively.

The symmetric pseudoprocess $\mathsf{q}$ is called a *symmetric process in* $P$ *over* $T$ iff it is compositive and transitive.

The sets of all compositive symmetric pseudoprocesses, of all transitive symmetric pseudoprocesses and of all symmetric processes in $P$ over $T$ will be denoted by $\mathsf{Ssc}(P, T)$, $\mathsf{Sst}(P, T)$ and $\mathsf{S}(P, T)$, respectively.

**2.2. Lemma.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$, *let* $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ *be positively induced by* $\mathsf{q}$ *and let* $\mathsf{q}^- \in \mathsf{Ps}(P, -T)$ *be negatively induced by* $\mathsf{q}$. *Then the following assertions are equivalent*:

(i) $\mathsf{q} \in \mathsf{Ssc}(P, T)$.

(ii) $\mathsf{q}^+ \in \mathsf{Psc}(P, T)$.

(iii) $\mathsf{q}^- \in \mathsf{Psc}(P, -T)$.

(iv) ${}_v\mathsf{q}_u \subset {}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u$ *for all* $u \leqq t \leqq v$ *in* $T$.

(v) ${}_v\mathsf{q}_u \subset {}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u$ *for all* $v \leqq t \leqq u$ *in* $T$.

(vi) ${}_v\mathsf{q}_u \subset {}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u$ *for all* $u, v, t \in T$, $t$ *between* $u, v$.

Proof follows from 2.1, (1.10.2), (RC) in 1.1, (1.5.1) and (1.10.4).

**2.3. Lemma.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$, *let* $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ *be positively induced by* $\mathsf{q}$ *and let* $\mathsf{q}^- \in \mathsf{Ps}(P, -T)$ *be negatively induced by* $\mathsf{q}$. *Then the following assertions are equivalent*:

(i) $\mathsf{q} \in \mathsf{Sst}(P, T)$.

(ii) $\mathsf{q}^+ \in \mathsf{Pst}(P, T)$.

(iii) $\mathsf{q}^- \in \mathsf{Pst}(P, -T)$.

(iv) ${}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u \subset {}_v\mathsf{q}_u$ *for all* $u \leqq t \leqq v$ *in* $T$.

(v) ${}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u \subset {}_v\mathsf{q}_u$ *for all* $v \leqq t \leqq u$ *in* $T$.

(vi) ${}_v\mathsf{q}_t \circ {}_t\mathsf{q}_u \subset {}_v\mathsf{q}_u$ *for all* $u, v, t \in T$, $t$ *between* $u, v$.

**2.4. Lemma.** *Let* $\mathfrak{q} \in \mathsf{Ss}(P, T)$. *Then the following assertions are equivalent:*

(i) $\mathfrak{q} \in \mathsf{S}(P, T)$.

(ii) ${}_v\mathfrak{q}_u = {}_v\mathfrak{q}_t \circ {}_t\mathfrak{q}_u$ *for all* $u, v, t \in T$, $t$ *between* $u, v$.

**2.5. Lemma.** *If* $\mathfrak{q} \in \mathsf{Ssc}(P, T)$, *then*

$$D_{\mathfrak{q}^+} = D_{\mathfrak{q}} = I_{\mathfrak{q}} ,$$
$$D_{\mathfrak{q}^-} = \{(x, u) \in P \times (-T) \mid (x, -u) \in D_{\mathfrak{q}}\} .$$

**2.6. Lemma.** *Let* $\mathfrak{q} \in \mathsf{Ss}(P, T)$ *and let* $I$ *be an arbitrary set. Then the following assertions hold:*

(i) *If* $s \in S_{\mathfrak{q}}$, $J$ *an interval in* $T$, *then* $s|_J \in S_{\mathfrak{q}}$.

(ii) *If* $s_i \in S_{\mathfrak{q}}$ *for* $i \in I$ *are such that* $D_{\cap s_i}$ *is an interval in* $T$, *then* $\cap s_i \in S_{\mathfrak{q}}$.

(iii) *If* $\mathfrak{q}$ *is transitive and if* $s_i \in S_{\mathfrak{q}}$ *with* $i \in I$ *are such that* $D_{s_i} \cap D_{s_j} \neq \emptyset$ *and* $s_i \cup s_j$ *is a map for all* $i, j \in I$, *then* $s_i \in S_{\mathfrak{q}}$.

**2.7. Lemma.** *Let* $\mathfrak{q} \in \mathsf{Sst}(P, T)$, $(x, u) \in D_{\mathfrak{q}}$. *Then the following assertions hold:*

(i) *If* $v, w \in T$, $v$ *between* $u, w$, $z_w\mathfrak{q}_v y$, $y_v\mathfrak{q}_u x$, *then also* $z_w\mathfrak{q}_u x$.

(ii) *Let* $(y, v) \in D_{\mathfrak{q}}$ *be such that* $y_v\mathfrak{q}_u x$. *If* $v = e^-(x, u)$ *or* $v = e^+(x, u)$, *then* $(y, v)$ *is a start point or an end point of* $\mathfrak{q}$, *respectively.*

**2.8. Lemma.** *Let* $\mathfrak{q} \in \mathsf{Ssc}(P, T)$, $(x, u) \in D_{\mathfrak{q}}$, $u, v, w \in T \cap \langle d^-(x, u), d^+(x, u)\rangle$, $v$ *between* $u, w$. *If* $y_v\mathfrak{q}_u x$, $z_w\mathfrak{q}_u x$, *then also* $z_w\mathfrak{q}_v y$.

**2.9. Theorem.** *Let* $\mathfrak{q} \in \mathsf{Ssc}(P, T)$ *have global unicity and let* $\mathsf{s} : T \to P$. *Then* $\mathsf{s} \in S_{\mathfrak{q}}$ *iff the following two conditions are satisfied:*

(i) $D_{\mathsf{s}}$ *is an interval in* $T$;

(ii) *there exists* $u \in D_{\mathsf{s}}$ *such that* $\mathsf{s}(v)_v\mathfrak{q}_u\mathsf{s}(u)$ *holds for all* $v \in D_{\mathsf{s}}$.

Proof follows easily from Definition 1.13 and Lemma 2.8.

**2.10. Theorem.** *Let* $\mathfrak{q} \in \mathsf{Ss}(P, T)$. *If* $\mathfrak{q}$ *is solution complete, then it is compositive.*

Proof. According to Theorem 1.28 the right pseudoprocess $\mathfrak{q}^+$ is solution complete so that it is compositive (see Theorem 3.8 in [5]). Now apply Definition 2.1.

**2.11. Definition.** Let $\mathfrak{q} \in \mathsf{Ss}(P, T)$, $(x, u) \in D_{\mathfrak{q}}$, $\mathsf{s} \subset P \times T$. The relation $\mathsf{s}$ is called a *characteristic solution of* $\mathfrak{q}$ *through the point* $(x, u)$ iff it satisfies the following two conditions:

(i) $D_{\mathsf{s}} = \{v \in T \mid \operatorname{card}({}_t\mathfrak{q}_u x) = 1$ *for all* $t \in T$, $t$ *between* $u, v\}$;

(ii) $\mathsf{s}(v)_v\mathfrak{q}_u x$ *holds for all* $v \in D_{\mathsf{s}}$.

**2.12. Lemma.** *Let* $\mathsf{q} \in \mathsf{Ssc}(P, T)$ *have bilateral local unicity at a point* $(x, u) \in D_{\mathsf{q}}$ *and let* $\mathsf{s}$ *be the characteristic solution of* $\mathsf{q}$ *through* $(x, u)$. *Then* $\mathsf{s} \in S_{\mathsf{q}}$ *with*

$$\sup D_s = \min\{e^+(x, u), d^+(x, u)\} = \begin{cases} e^+(x, u) & \text{if } d^+(x, u) = +\infty\,, \\ d^+(x, u) & \text{if } d^+(x, u) < +\infty\,, \end{cases}$$

$$\inf D_s = \max\{e^-(x, u), d^-(x, u)\} = \begin{cases} e^-(x, u) & \text{if } d^-(x, u) = -\infty\,, \\ d^-(x, u) & \text{if } d^-(x, u) > -\infty\,. \end{cases}$$

**2.13. Remark.** In Definition 3.10 in [5], each right pseudoprocess $\mathsf{p}$ in $P$ over $T$ is associated with the maximal compositive right pseudoprocess $^\wedge\mathsf{p}$ in $P$ over $T$ contained in $\mathsf{p}$, the so called lower modification of $\mathsf{p}$. The construction of the lower modification of a right pseudoprocess described in item 3.12 in [5] is applicable to symmetric pseudoprocesses as well. However, this is not necessary, because, as will be shown, the maximal compositive symmetric pseudoprocess contained in a given symmetric pseudoprocess $\mathsf{q}$, which will be called again the lower modification of $\mathsf{q}$, can be constructed directly from the lower modifications of the pertinent right pseudoprocesses $\mathsf{q}^+$ and $\mathsf{q}^-$.

**2.14. Theorem.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$, *let* $\mathsf{q}^+ \in \mathsf{Ps}(P, T)$ *be positively induced by* $\mathsf{q}$, *let* $\mathsf{q}^- \in \mathsf{Ps}(P, T)$ *be negatively induced by* $\mathsf{q}$, *let* $^\wedge\mathsf{q}^+$ *be the lower modification of* $\mathsf{q}^+$ *and let* $^\wedge\mathsf{q}^-$ *be orientation-change produced from* $^\wedge\mathsf{q}^+$. *Then* $^\wedge\mathsf{q}^-$ *is the lower modification of* $\mathsf{q}^-$.

Proof. According to Definition 1.8, $^\wedge\mathsf{q}^- \in \mathsf{Ps}(P, -T)$. First we shall prove that $^\wedge\mathsf{q}^-$ is compositive.

The right pseudoprocess $^\wedge\mathsf{q}^+$ being the lower modification of $\mathsf{q}^+$ is compositive, i.e.

(2.14.1) $\quad {}_v{}^\wedge\mathsf{q}^+{}_u \supset {}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u$ for all $u \leqq t \leqq v$ in $T$.

Hence

(2.14.2) $\quad ({}_v{}^\wedge\mathsf{q}^+{}_u)^{-1} \supset ({}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u)^{-1}$ for all $u \leqq t \leqq v$ in $T$.

According to (1.10.1) it is

(2.14.3) $$({}_v{}^\wedge\mathsf{q}^+{}_u)^{-1} = {}_{-u}{}^\wedge\mathsf{q}^-{}_{-v}$$

so that

(2.14.4) $\quad ({}_v{}^\wedge\mathsf{q}^+{}_t \circ {}_t{}^\wedge\mathsf{q}^+{}_u)^{-1} = ({}_t{}^\wedge\mathsf{q}^+{}_u)^{-1} \circ ({}_v{}^\wedge\mathsf{q}^+{}_t)^{-1} = {}_{-u}{}^\wedge\mathsf{q}^-{}_{-t} \circ {}_{-t}{}^\wedge\mathsf{q}^-{}_{-v}$

holds for all $u \leqq t \leqq v$ in $T$. Substituting from (2.14.3) and (2.14.4) into (2.14.2) we obtain

$${}_{-u}{}^\wedge\mathsf{q}^-{}_{-v} \subset {}_{-u}{}^\wedge\mathsf{q}^-{}_{-t} \circ {}_{-t}{}^\wedge\mathsf{q}^-{}_{-v} \quad \text{for all} \quad -v \leqq -t \leqq -u \quad \text{in} \quad -T.$$

Thus $^\wedge\mathsf{q}^-$ is compositive.

Now we shall prove that ${}^{\wedge}q^-$ is the maximal compositive right pseudoprocess in $P$ over $-T$ contained in $q^-$.

Let $p \in Psc(P, -T)$ be such that ${}^{\wedge}q^- \subset p \subset q^-$. Then

$$(2.14.5) \qquad {}_v{}^{\wedge}q^-{}_u \subset {}_vp_u \subset {}_vq^-{}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad -T .$$

Denote by $p'$ the right pseudoprocess in $P$ over $T$ which is orientation-change produced from $p$. From (2.14.5) and (1.10.1) one obtains

$$({}_{-u}{}^{\wedge}q^+{}_{-v})^{-1} \subset {}_{-u}p'{}_{-v} \subset {}_{-u}q^+{}_{-v} \quad \text{for all} \quad -v \leqq -u \quad \text{in} \quad T ,$$

which can be written equivalently as

$${}^{\wedge}q^+ \subset p' \subset q^+ .$$

Since ${}^{\wedge}q^+$ and $p'$ are compositive, it is necessarily ${}^{\wedge}q^+ = p'$, hence ${}^{\wedge}q^- = p$.

We have proved that ${}^{\wedge}q^-$ is the lower modification of $q^-$.

**2.15. Theorem.** *Let* $q \in Ss(P, T)$, *let* $q^+ \in Ps(P, T)$ *be positively induced by* $q$ *and let* ${}^{\wedge}q^+$ *be the lower modification of* $q^+$. *Then the symmetric pseudoprocess*

$$(2.15.1) \qquad {}^{\wedge}q = {}^{\wedge}q^+ \cup ({}^{\wedge}q^+)^{-1}$$

*is the maximal compositive symmetric pseudoprocess in* $P$ *over* $T$ *contained in* $q$.

Proof. Since ${}^{\wedge}q^+$ is compositive, ${}^{\wedge}q$ is compositive as well.

Let ${}^{\sim}q \in Ssc(P, T)$ be such that ${}^{\wedge}q \subset {}^{\sim}q \subset q$. Then

$$({}^{\wedge}q^+ \cup ({}^{\wedge}q^+)^{-1}) \subset ({}^{\sim}q^+ \cup ({}^{\sim}q^+)^{-1}) \subset (q^+ \cup (q^+)^{-1}) .$$

Hence, according to 1.12, one obtains ${}^{\wedge}q^+ \subset {}^{\sim}q^+ \subset q^+$. Since ${}^{\wedge}q^+$ is the lower modification of $q^+$ and ${}^{\sim}q^+$ is compositive, it is necessarily ${}^{\wedge}q^+ = {}^{\sim}q^+$, hence ${}^{\wedge}q = {}^{\sim}q$ follows by virtue of Lemma 1.12.

**2.16. Definition.** Let $q \in Ss(P, T)$. The symmetric compositive pseudoprocess ${}^{\wedge}q$ in $P$ over $T$ defined by

$$(2.16.1) \qquad {}^{\wedge}q = {}^{\wedge}q^+ \cup ({}^{\wedge}q^+)^{-1}$$

is called the *lower modification* of the symmetric pseudoprocess $q$.

**2.17. Remark.** One may verify easily that

$$D_{{}^{\wedge}q} = D_{{}^{\wedge}q^+} \subset D_{q^+} \subset D_q .$$

If, in addition, $D_q = I_q$, then

$$D_{{}^{\wedge}q} = D_{{}^{\wedge}q^+} = D_{q^+} = D_q .$$

**2.18. Theorem.** *Let* $\mathfrak{q}_i \in \mathrm{Ss}(P, T)$ *for* $i = 1, 2$ *and let* $\mathfrak{q}_i^+$ *be the right pseudoprocess positively induced by* $\mathfrak{q}_i$. *Then* $\mathfrak{q}_1 \cap \mathfrak{q}_2 \in \mathrm{Ss}(P, T)$ *and its lower modification is the compositive symmetric pseudoprocess* $\mathfrak{q}_1 \wedge \mathfrak{q}_2$ *in* $P$ *over* $T$ *defined by*

$$\mathfrak{q}_1 \wedge \mathfrak{q}_2 = (\mathfrak{q}_1^+ \wedge \mathfrak{q}_2^+) \cup (\mathfrak{q}_1^+ \wedge \mathfrak{q}_2^+)^{-1}, \tag{2.18.1}$$

*where* $\mathfrak{q}_1^+ \wedge \mathfrak{q}_2^+$ *denotes the lower modification of the right pseudoprocess* $\mathfrak{q}_1^+ \cap \mathfrak{q}_2^+$.

Proof. According to the assertion (i) of Lemma 1.7 it is

$$\mathfrak{q}_1 = \mathfrak{q}_1^+ \cup (\mathfrak{q}_1^+)^{-1}, \quad \mathfrak{q}_2 = \mathfrak{q}_2^+ \cup (\mathfrak{q}_2^+)^{-1}$$

so that

$$\mathfrak{q}_1 \cap \mathfrak{q}_2 = (\mathfrak{q}_1^+ \cap \mathfrak{q}_2^+) \cup (\mathfrak{q}_1^+ \cap \mathfrak{q}_2^+)^{-1}.$$

Hence and from 1.7 (i) one obtains

$$(\mathfrak{q}_1 \cap \mathfrak{q}_2)^+ = \mathfrak{q}_1^+ \cap \mathfrak{q}_2^+. \tag{2.18.2}$$

Since the lower modification of $\mathfrak{q}_1^+ \cap \mathfrak{q}_2^+$ is $\mathfrak{q}_1^+ \wedge \mathfrak{q}_2^+$, the equality (2.18.1) follows now directly from (2.18.2) and (2.16.1).

**2.19. Theorem.** *Let* $\mathfrak{q} \in \mathrm{Ss}(P, T)$, *let* $^\wedge\mathfrak{q}$ *be its lower modification. Then* $S_{^\wedge\mathfrak{q}} = S_{\mathfrak{q}}$.

Proof. Applying Theorem 6.14 to the equalities

$$\mathfrak{q} = \mathfrak{q}^+ \cup (\mathfrak{q}^+)^{-1}, \quad ^\wedge\mathfrak{q} = {^\wedge\mathfrak{q}^+} \cup (^\wedge\mathfrak{q}^+)^{-1}$$

we obtain $S_{\mathfrak{q}} = S_{\mathfrak{q}^+}$, $S_{^\wedge\mathfrak{q}} = S_{^\wedge\mathfrak{q}^+}$. Theorem 3.14 in [5] yields $S_{^\wedge\mathfrak{q}^+} = S_{\mathfrak{q}^+}$. Thus $S_{^\wedge\mathfrak{q}} = S_{\mathfrak{q}}$.

**2.20. Corollary.** *Let* $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Ss}(P, T)$. *Then* $S_{\mathfrak{q}_1 \wedge \mathfrak{q}_2} = S_{\mathfrak{q}_1 \cap \mathfrak{q}_2}$.

## 3. LOCAL BEHAVIOUR OF SYMMETRIC PSEUDOPROCESSES

**3.1.** In Section 5 of the paper [5] we have investigated the local behaviour of right pseudoprocesses. Let us recall the basic notions and notation which will be used in the sequel.

Given a (right or symmetric) pseudoprocess $\mathfrak{r}$ in $P$ over $T$, the symbol $L_{\mathfrak{r}}$ will denote the set

$$L_{\mathfrak{r}} = \{(s, u) \in S_{\mathfrak{r}} \times T \mid u \in D_s\}. \tag{3.1.1}$$

Let $\mathfrak{p} \in \mathrm{Ps}(P, T)$, $(x, u) \in D_{\mathfrak{p}}$. Then $\mathfrak{p}$ is said to have r i g h t (or l e f t) l o c a l e x i s t e n c e o f s o l u t i o n s a t t h e p o i n t $(x, u)$ iff the following conditions are fulfilled:

(i) $(x, u) \notin \mathscr{E}_{\mathfrak{p}}$ (or $(x, u) \notin \mathscr{S}_{\mathfrak{p}}$);

(ii) There exist $\varepsilon > 0$ and $s \in S_{\mathfrak{p}}$ such that

$$\langle u, u + \varepsilon \rangle \cap T \subset D_s \quad (\text{or } \langle u - \varepsilon, u \rangle \cap T \subset D_s).$$

A pseudoprocess p is said to have bilateral local existence of solutions at the point $(x, u)$ iff it has right local existence of solutions at the point $(x, u)$ if $(x, u) \in D_p - \mathscr{E}_p$ and left local existence of solutions at the point $(x, u)$ if $(x, u) \in \in D_p - \mathscr{S}_p$. A pseudoprocess p is said to have right or left or bilateral local existence of solutions iff it has the property at each point $(x, u) \in D_p$.

Let p, $^{\sim}$p $\in$ Ps$(P, T)$. Then $^{\sim}$p is said to determine the local behaviour of p (which is shortly written as $^{\sim}p \prec p$) iff $^{\sim}p \subset p$ and there exists a map

$$k : L_p \to R \tag{3.1.2}$$

such that $k(s, u) > u$ for $u < \sup D_s$, $k(s, u) = u$ for $u = \max D_s$ and

$$s|_{\langle u, k(s,u)\rangle} \in S_{\sim p} . \tag{3.1.3}$$

A pseudoprocess $^{\sim}$p is said to determine the bilateral local behaviour of p (which is shortly written as $^{\sim}p \preccurlyeq p$) iff $^{\sim}p \subset p$ and there exist maps

$$k_1, k_2 : L_p \to R \tag{3.1.4}$$

such that

$$k_1(s, u) < u \quad \text{for} \quad \inf D_s < u\,, \qquad k_1(s, u) = u \quad \text{for} \quad \min D_s = u\,,$$

$$k_2(s, u) > u \quad \text{for} \quad \sup D_s > u\,, \qquad k_2(s, u) = u \quad \text{for} \quad \max D_s = u$$

and

$$s|_{\langle k_1(s,u), k_2(s,u)\rangle} \in S_{\sim p} . \tag{3.1.5}$$

Now, let q $\in$ Ss$(P, T)$ and $q^+ \in$ Ps$(P, T)$ be positively induced by q. According to Theorem 1.14 (iii) it holds $S_q = S_{q^+}$. This enables us to define the corresponding notions related to the local existence of solutions and to the local behaviour of pseudoprocesses for symmetric pseudoprocesses in a natural way as follows.

**3.2. Definition.** Let q $\in$ Ss$(P, T)$, let $q^+ \in$ Ps$(P, T)$ be positively induced by q and let $(x, u) \in D_q$. The symmetric pseudoprocess q is said to have *right* or *left* or *bilateral local existence of solutions at the point* $(x, u)$ iff the right pseudoprocess $q^+$ has right or left or bilateral local existence of solutions at the point $(x, u)$, respectively.

The symmetric pseudoprocess q is said to have *right* or *left* or *bilateral local existence of solutions* iff it has the property at each point $(x, u) \in D_q$.

**3.3. Definition.** Let q $\in$ Ss$(P, T)$, p, $^{\sim}$p, $q^+ \in$ Ps$(P, T)$, $^{\approx}$p, $q^- \in$ Ps$(P, -T)$, where $q^+$ is positively and $q^-$ negatively induced by q. The right pseudoprocess $^{\approx}$p, $^{\sim}$p or p is said to *determine the negative local behaviour, the positive local behaviour* or *the local behaviour* of the symmetric pseudoprocess q (which is shortly written as $^{\approx}p \leqq q$, $^{\sim}p \prec q$ or $p \preccurlyeq q$) iff $^{\approx}p \prec q^-$, $^{\sim}p \prec q^+$, or $p \preccurlyeq q^+$.

Let $q$, $\tilde{q} \in Ss(P, T)$. The symmetric pseudoprocess $\tilde{q}$ is said to *determine the local behaviour* of the symmetric pseudoprocess $q$ (shortly written as $\tilde{q} \preccurlyeq q$) iff $\tilde{q}^+ \preccurlyeq q$, where $\tilde{q}^+$ is the right pseudoprocess positively induced by $\tilde{q}$.

**3.4. Remark.** If $q$, $\tilde{q} \in Ss(P, T)$, $\tilde{q} \preccurlyeq q$, then $D_{\tilde{q}} \subset D_q$. If, in addition, $D_q = I_q$, then $D_{\tilde{q}} = D_q$. This equality holds in particular if $q \in Ssc(P, T)$.

Since Definitions 3.2 and 3.3 are immediate generalizations of Definitions 5.2, 5.3, 5.5 and 5.6 from [5], it is natural that many results valid for right pseudoprocesses remain valid as well when formulated for symmetric pseudoprocesses. Some results of this kind are given in what follows.

Similarly as in [5] many assertions concerning the relations $\prec$ and $\preccurlyeq$ may be formulated simultaneously. It will be done using the symbol $<$. In these assertions the symbol $<$ has to be replaced either by $\prec$ or by $\preccurlyeq$.

**3.5. Lemma.** *Let* $q \in Ss(P, T)$, $p \in Ps(P, T)$, $\tilde{p} \in Ps(P, -T)$. *Then the following assertions hold*:

(i) *If* $p < q$, *then* $S_p \subset S_q$, $D_p \subset D_q$, $\mathscr{E}_p = \mathscr{E}_q \cap D_p$.

(ii) *If* $\tilde{p} \leqq q$, *then* $S_{\tilde{p}} \subset S_{q^-}$.

(iii) *If* $p \leqq q$, *then* $\mathscr{E}_p = \mathscr{E}_q \cap D_p$, $\mathscr{S}_p = \mathscr{S}_q \cap D_p$.

(iv) *If* $p \prec q$ *and* $p$ *has right or left local existence of solutions at each point, then* $D_p = D_q$, $\mathscr{E}_p = \mathscr{E}_q$; *if, in addition,* $p \preccurlyeq q$, *then also* $\mathscr{S}_p = \mathscr{S}_q$.

**3.6. Lemma.** *Let* $q$, $\tilde{q}$, $\approx q \in Ss(P, T)$. *If* $q \approx\!\preccurlyeq \tilde{q}$, $\tilde{q} \preccurlyeq q$, *then* $\approx q \preccurlyeq q$.

**3.7. Lemma.** *Let* $p$, $\tilde{p} \in Ps(P, T)$, $q \in Ss(P, T)$, $\tilde{p} \subset p \subset q$. *If* $\tilde{p} < q$, *then* $p < q$.

**3.8. Lemma.** *Let* $p \in Ps(P, T)$, $q$, $\tilde{q} \in Ss(P, T)$, $p \subset \tilde{q} \subset q$. *If* $p < q$, *then* $p < \tilde{q}$.

**3.9. Lemma.** *Let* $p \in Ps(P, T)$, $q \in Ss(P, T)$. *Then* $p < q$ *iff* $\hat{p} < q$, *where* $\hat{p}$ *is the lower modification of* $p$.

**3.10. Lemma.** *Let* $p$, $\tilde{p} \in Ps(P, T)$, $q \in Ss(P, T)$. *Then the following assertions are equivalent.*

(i) $p < q$, $\tilde{p} < q$.

(ii) $p \cap \tilde{p} < q$.

(iii) $p \wedge \tilde{p} < q$.

**3.11. Lemma.** *Let* $q$, $\tilde{q}$, $\approx q \in Ss(P, T)$. *If* $\tilde{q} \preccurlyeq q$, *then* $\tilde{q} \cap q^{\approx} \preccurlyeq q \cap \approx q$. *Especially, if* $\tilde{q} \preccurlyeq q$, $\approx q \preccurlyeq q$, *then* $\tilde{q} \cap \approx q \preccurlyeq \tilde{q}$, $\tilde{q} \cap \approx q \preccurlyeq \approx q$.

**3.12. Lemma.** *Let* $\mathsf{q} \in \mathsf{Ssc}(P, T)$, $^{\sim}\mathsf{q}$, $^{\approx}\mathsf{q} \in \mathsf{Ss}(P, T)$, $\mathsf{q} \preccurlyeq {}^{\sim}\mathsf{q}$, $\mathsf{q} \preccurlyeq {}^{\approx}\mathsf{q}$. *Then* $\mathsf{q}^+ \subset \subset {}^{\sim}\mathsf{q}^+ \wedge {}^{\approx}\mathsf{q}^+$, $\mathsf{q}^- \subset {}^{\sim}\mathsf{q}^- \wedge {}^{\approx}\mathsf{q}^-$, $\mathsf{q} \subset {}^{\sim}\mathsf{q} \wedge {}^{\approx}\mathsf{q}$.

**3.13. Theorem.** *Let* $\mathsf{q} \in \mathsf{Ss}(P, T)$, $\mathsf{p} \in \mathsf{Ps}(P, T)$ *and let* $\mathsf{p}' \in \mathsf{Ps}(P, -T)$ *be orientation change produced from* $\mathsf{p}$. *Then the following assertions hold:*

(i) $\mathsf{p}' \leqq \mathsf{q}$ *iff* $\mathsf{p} \subset \mathsf{q}$ *and there exists a map* $h^- : L_\mathsf{q} \to R$ *such that*

$$h^-(s, u) < u \quad for \quad \inf D_s < u\,, \quad h^-(s, u) = u \quad for \quad \min D_s = u$$

*and*

$$(3.13.1) \qquad s(t)\,{}_t\mathsf{p}_v\, s(v) \quad for\ all \quad h^-(s, u) \leqq v \leqq t \leqq u \quad in \quad D_s\,.$$

(ii) $\mathsf{p} \prec \mathsf{q}$ *iff* $\mathsf{p} \subset \mathsf{q}$ *and there exists a map* $h^+ : L_\mathsf{q} \to R$ *such that*

$$h^+(s, u) > u \quad for \quad u < \sup D_s\,, \quad h^+(s, u) = u \quad for \quad u = \max D_s$$

*and*

$$(3.13.2) \qquad s(t)\,{}_t\mathsf{p}_v\, s(v) \quad for\ all \quad u \leqq v \leqq t \leqq h^+(s, u) \quad in \quad D_s\,.$$

(iii) $\mathsf{p} \preccurlyeq \mathsf{q}$ *iff* $\mathsf{p} \subset \mathsf{q}$ *and there exist maps* $h^+, h^- : L_\mathsf{q} \to R$ *such that*

$$h^+(s, u) > u \quad for \quad u < \sup D_s\,, \qquad h^+(s, u) = u \quad for \quad u = \max D_s\,,$$

$$h^-(s, u) < u \quad for \quad \inf D_s < u\,, \qquad h^-(s, u) = u \quad for \quad u = \min D_s$$

*and*

$$(3.13.3) \qquad s(t)\,{}_t\mathsf{p}_v\, s(v) \quad for\ all \quad h^-(s, u) \leqq v \leqq t \leqq h^+(s, u) \quad in \quad D_s\,.$$

Proof. Before proving the assertion (i) let us recall that (1.8.1) and (1.10.1) yield $\mathsf{p}' \subset \mathsf{q}^-$ iff $\mathsf{p} \subset \mathsf{q}^+$ and according to 1.14 (ii), $s \in S_\mathsf{q} = S_{\mathsf{q}^+}$ iff there exists $s' \in S_{\mathsf{q}^-}$ such that $s'(t) = s(-t)$ for all $-t \in D_s$.

Suppose $\mathsf{p}' \leqq \mathsf{q}$, i.e. $\mathsf{p}' \prec \mathsf{q}^-$ and prove that (3.13.1) is fulfilled. Take $(s, u) \in L_\mathsf{q}$ arbitrary. According to the assumption there exists a real $h'(s', -u)$ such that

$$(3.13.4) \qquad s'(-v)\,{}_{-v}\mathsf{p}'_{-t}\, s'(-t) \quad \text{for all} \quad -u \leqq -t \leqq -v \leqq h'(s', -u) \quad \text{in} \quad D_s\,.$$

Hence, setting $h^-(s, u) = -h'(s', -u)$ and using (1.8.1), one easily obtains (3.13.1).

Suppose now that the condition (3.13.1) is fulfilled and prove that $\mathsf{p}' \prec \mathsf{q}^-$. The condition (3.13.1) can be written in the form

$$(3.13.5) \qquad s(-t)\,{}_{-t}\mathsf{p}_{-v}\, s(-v) \quad \text{for all} \quad h^-(s, u) \leqq -v \leqq -t \leqq u \quad \text{with} \quad v, t \in D_{s'}\,,$$

i.e.

$$s'(v)\,{}_v\mathsf{p}'_t\, s'(t) \quad \text{for all} \quad -u \leqq t \leqq v \leqq -h^-(s, u) \quad \text{with} \quad v, t \in D_{s'}\,.$$

Setting $h'(s', u) = -h^-(s, -u)$ we conclude that for each $(s', u) \in L_{\mathsf{q}^-}$ there exists a real $h'(s', u)$ such that

$$s'(t)\,{}_t\mathsf{p}_v\, s'(v) \quad \text{for all} \quad u \leqq v \leqq t \leqq h'(s', u) \quad \text{in} \quad D_{s'}\,.$$

Thus $\mathsf{p}' \prec \mathsf{q}^-$.

The assertions (ii) and (iii) follow immediately from Definition 3.3.

**3.14. Theorem.** *Let* $\mathsf{q}, \tilde{\mathsf{q}} \in \mathrm{Ss}(P, T)$. *Then the following three assertions are equivalent*:

(i) $\tilde{\mathsf{q}} \preccurlyeq \mathsf{q}$;

(ii) $\tilde{\mathsf{q}}^+ \preccurlyeq \mathsf{q}^+$;

(iii) $\tilde{\mathsf{q}}^- \preccurlyeq \mathsf{q}^-$.

Proof follows easily from Theorem 1.14 and Definition 3.3.

**3.15. Theorem.** *Let* $\mathsf{q} \in \mathrm{Ss}(P, T)$, $\tilde{\mathsf{q}} \in \mathrm{Sst}(P, T)$. *If* $\tilde{\mathsf{q}}^+ \prec \mathsf{q}$, $\tilde{\mathsf{q}}^- \precsim \mathsf{q}$, *then* $\tilde{\mathsf{q}} \preccurlyeq \mathsf{q}$.

Proof. According to 1.12 the inclusion $\tilde{\mathsf{q}} \subset \mathsf{q}$ is equivalent to any one of the inclusions $\tilde{\mathsf{q}}^+ \subset \mathsf{q}^+$ and $\tilde{\mathsf{q}}^- \subset \mathsf{q}^-$.

To each $(s, u) \in L_{\mathsf{q}}$ we can assign reals $h^+(s, u)$ and $h^-(s, u)$ as in Theorem 3.13 such that

(3.15.1) $\qquad s(t)\,{}_t\tilde{\mathsf{q}}_v\, s(v) \quad \text{for all} \quad u \leqq v \leqq t \leqq h^+(s, u) \quad \text{in} \quad T$

and

(3.15.2) $\qquad s(t)\,{}_t\tilde{\mathsf{q}}_v\, s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq t \leqq u \quad \text{in} \quad T\,.$

Especially,

$$s(t)\,{}_t\tilde{\mathsf{q}}_u\, s(u)\,, \quad s(u)\,{}_u\tilde{\mathsf{q}}_v\, s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq u \leqq t \leqq h^+(s, u) \quad \text{in} \quad T\,.$$

Since $\tilde{\mathsf{q}}$ is transitive, it holds also

(3.15.3) $\quad s(t)\,{}_t\tilde{\mathsf{q}}_v\, s(v) \quad \text{for all} \quad h^-(s, u) \leqq v \leqq u \leqq t \leqq h^+(s, u) \quad \text{in} \quad T\,.$

Finally, $({}_t\tilde{\mathsf{q}}_v)^{-1} = {}_v\tilde{\mathsf{q}}_t$ so that $s(v)\,{}_v\tilde{\mathsf{q}}_t\, s(t)$ holds iff $s(t)\,{}_t\tilde{\mathsf{q}}_v\, s(v)$. This together with (3.15.1), (3.15.2) and (3.15.3) yields

$$s(t)\,{}_t\tilde{\mathsf{q}}_v\, s(v) \quad \text{for all} \quad v, t \in D_s \cap \langle h^-(s, u), h^+(s, u)\rangle\,,$$

which was to be proved.

**3.16. Definition.** Let $q, {}^{\sim}q \in Ss(P, T)$. The symmetric pseudoprocesses $q$ and ${}^{\sim}q$ are said to be *negatively locally equivalent* or *positively locally equivalent* or *locally equivalent* (which is shortly written as $q \leqq\geqq {}^{\sim}q$ or $q \prec\succ {}^{\sim}q$ or $q \preccurlyeq\succcurlyeq {}^{\sim}q$) iff there exists $p \in Ps(P, -T)$ or ${}^{\sim}p \in Ps(P, T)$ or ${}^{\approx}p \in Ps(P, T)$ such that $p \leqq q$ and $p \leqq {}^{\sim}q$ or ${}^{\sim}p \prec q$ and ${}^{\sim}p \prec {}^{\sim}q$ or ${}^{\approx}p \preccurlyeq q$ and ${}^{\approx}p \preccurlyeq {}^{\sim}q$, respectively.

**3.17. Lemma.** *Let* $q, {}^{\sim}q \in Ss(P, T)$. *Then the following assertions hold*:

(i) $q \leqq\geqq {}^{\sim}q$ iff $q^- \prec\succ {}^{\sim}q^-$.

(ii) $q \prec\succ {}^{\sim}q$ iff $q^+ \prec\succ {}^{\sim}q^+$.

(iii) $q \preccurlyeq\succcurlyeq {}^{\sim}q$ iff $q^+ \preccurlyeq\succcurlyeq {}^{\sim}q^+$.

**3.18. Lemma.** *Let* $q, {}^{\sim}q \in Ss(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \leqq\geqq {}^{\sim}q$;

(ii) $q^- \cap {}^{\sim}q^- \prec q^-,\ q^- \cap {}^{\sim}q^- \prec {}^{\sim}q^-$;

(iii) $q^- \wedge {}^{\sim}q^- \prec q^-,\ q^- \wedge {}^{\sim}q^- \prec {}^{\sim}q^-$.

**3.19. Lemma.** *Let* $q, {}^{\sim}q \in Ss(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \prec\succ {}^{\sim}q$;

(ii) $q^+ \cap {}^{\sim}q^+ \prec q^+,\ q^+ \cap {}^{\sim}q^+ \prec {}^{\sim}q^+$;

(iii) $q^+ \wedge {}^{\sim}q^+ \prec q^+,\ q^+ \wedge {}^{\sim}q^+ \prec {}^{\sim}q^+$.

**3.20. Lemma.** *Let* $q, {}^{\sim}q \in Ss(P, T)$. *Then the following three assertions are equivalent*:

(i) $q \preccurlyeq\succcurlyeq {}^{\sim}q$;

(ii) $q^+ \cap {}^{\sim}q^+ \preccurlyeq q^+,\ q^+ \cap {}^{\sim}q^+ \preccurlyeq {}^{\sim}q^+$;

(iii) $q^+ \wedge {}^{\sim}q^+ \preccurlyeq q^+,\ q^+ \wedge {}^{\sim}q^+ \preccurlyeq {}^{\sim}q^+$.

**3.21. Theorem.** *The positive local equivalence, the negative local equivalence and the local equivalence of symmetric pseudoprocesses in* $P$ *over* $T$ *are equivalence relations in the set* $Ss(P, T)$.

Proof. See 3.17 and Theorem 5.15 in [5].

**3.22. Theorem.** *Let* $q, {}^{\sim}q \in Ss(P, T)$ *have right local existence of solutions. Then* $q \prec\succ {}^{\sim}q$ *iff the following conditions are fulfilled*:

(i) $D_q = D_{\sim q},\ \mathscr{E}_q = \mathscr{E}_{\sim q}$;

(ii) *there exists a map*

$$r^{+} : L_{q} \to R$$

*such that*

$$r^{+}(s, u) > u \quad \textit{for} \quad u < \sup D_{s}\,, \qquad r^{+}(s, u) = u \quad \textit{for} \quad u = \max D_{s}$$

*and*

$$s|_{\langle u, r^{+}(s,u)\rangle} \in S_{\sim q}\,.$$

Proof follows from (1.6.3), 1.21, 1.14 (iii), 3.17 (ii) and Theorem 5.16 in [5].

**3.23. Theorem.** *Let* q, $^{\sim}$q $\in$ Ss($P$, $T$) *have left local existence of solutions. Then* q $\leqq\geqq$ $^{\sim}$q *iff the following conditions are fulfilled*:

(i) $D_{q} = D_{\sim q}$, $\mathcal{S}_{q} = \mathcal{S}_{\sim q}$;

(ii) *there exists a map*

$$r^{-} : L_{q} \to R$$

*such that*

$$r^{-}(s, u) < u \quad \textit{for} \quad \inf D_{s} < u\,, \qquad r^{-}(s, u) = u \quad \textit{for} \quad \min D_{s} = u$$

*and*

$$s|_{\langle r^{-}(s,u), u\rangle} \in S_{\sim q}\,.$$

Proof is similar to that of Theorem 5.16 in [5].

**3.24. Theorem.** *Let* q, $^{\sim}$q $\in$ Ss($P$, $T$) *have bilateral local existence of solutions. Then* q $\preccurlyeq\succcurlyeq$ $^{\sim}$q *iff the following conditions are fulfilled*:

(i) $D_{q} = D_{\sim q}$, $\mathcal{E}_{q} = \mathcal{E}_{\sim q}$, $\mathcal{S}_{q} = \mathcal{S}_{\sim q}$;

(ii) *there exist maps*

$$r^{+}, r^{-} : L_{q} \to R$$

*such that*

$$r^{+}(s, u) > u \quad \textit{for} \quad \sup D_{s} > u\,, \qquad r^{+}(s, u) = u \quad \text{for} \quad \max D_{s} = u\,,$$

$$r^{-}(s, u) < u \quad \textit{for} \quad \inf D_{s} < u\,, \qquad r^{-}(s, u) = u \quad \textit{for} \quad \min D_{s} = u$$

and

$$s|_{\langle r^{-}(s,u), r^{+}(s,u)\rangle} \in S_{\sim q}\,.$$

Proof follows from (1.6.3), 1.21, 1.14, 3.17 and Theorem 5.21 in [5].

**3.25. Theorem.** *Let* $T$ *be a closed subset of* R *and let* q, $^{\sim}$q $\in$ S($P$, $T$) *be solution complete processes. Then* q $\preccurlyeq\succcurlyeq$ $^{\sim}$q *iff* q = $^{\sim}$q.

Proof follows from 3.17 (iii) and Theorem 5.21 in [5].