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THE LAPLACE TRANSFORM OF ANALYTIC VECTOR-VALUED FUNCTIONS (REAL CONDITIONS)

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This paper is concerned with the problem of characteristic properties of the Laplace transform of vector-valued exponentially bounded functions on positive half-axis which are analytic in the sense that they are, roughly speaking, developable in power series at the points of positive half-axis with a linearly increasing radius of convergence. The properties can be described in a simple way: the functions in question are infinitely differentiable on the positive half-axis R^+ and their derivatives satisfy the inequalities (B) (I) in Theorem 5 with certain constants $M \geqq 0$, $\omega \geqq 0$ and $\varrho \geqq 0$.

We give necessary and sufficient representability conditions of purely real (Widder's) type in terms of the behavior of derivatives of the Laplace image on the real half-axis as shown in Theorem 5.

The proof of Theorem 5 is mainly based on the representability theorems for Lipschitzian functions [1], [2]. In a reflexive space, the representability theorems from [3], [4] may be used, in the numerical case, the original representability theorem of Widder [5] is sufficient (see Remark 7).

The analyticity of semigroups is examined in Theorem 9.

It seems that the results presented are new even in the simplest, i.e. the numerical case.

1. In the sequel, R will denote the real number field and R^+ the set of all positive numbers. If M_1, M_2 are arbitrary sets, then $M_1 \to M_2$ will denote the set of all mappings of the whole set M_1 into the set M_2.

2. By E we denote a Banach space over R with the norm $\|\cdot\|$.

3. The notions of differentiability, measurability and integrability of functions with values in E are used in the strong (norm) sense.

4. Lemma. $(p + 1)!/\lambda^{p+2} \leqq p!/(\lambda - 1)^{p+1}$ *for any* $\lambda > 1$ *and* $p \in \{0, 1, \ldots\}$.

5. Fundamental theorem (real form). *Let M, ω, ϱ be nonnegative constants and let $F \in (\omega, \infty) \to E$. Then the following two conditions (A) and (B) are equivalent:*

(A) (I) *the function F is infinitely differentiable in $(0, \infty)$,*

(II) $\left\| \frac{d^{p+q}}{d\lambda^{p+q}} (\lambda^q F(\lambda)) \right\| \leqq \frac{Mp!\, q!\varrho^q}{(\lambda - \omega)^{p+1}}$ *for any $\lambda > \omega$ and $p, q \in \{0, 1, \ldots\}$,*

(B) *there exists an infinitely differentiable function $f \in R^+ \to E$ such that*

(I) $\|f^{(q)}(t)\| \leqq \frac{Me^{\omega t} q!\, \varrho^q}{t^q}$ *for any $t \in R^+$ and $q \in \{0, 1, \ldots,\}$,*

(II) $F(\lambda) = \int_0^\infty e^{-\lambda\tau} f(\tau)\, d\tau$ *for any $\lambda > \omega$.*

Proof. (A) ⇒ (B): Let us first denote

$$G_q(\lambda) = -\frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^q F(\lambda))$$

for any $\lambda > \omega$ and $q \in \{0, 1, \ldots\}$.

It follows easily from (A) by means of Lemma 4 that

$$(1) \quad \left\| \frac{d^p}{d\lambda^p} G_q(\lambda) \right\| = \left\| \frac{d^{p+q+1}}{d\lambda^{p+q+1}} (\lambda^q F(\lambda)) \right\| \leqq \frac{M(p+1)!\, q!\, \varrho^q}{(\lambda - \omega)^{p+2}} \leqq \frac{Mp!\, q!\, \varrho^q}{(\lambda - \omega - 1)^{p+1}}$$

for any $\lambda > \omega + 1$ and $p, q \in \{0, 1, \ldots\}$,

$$(2) \quad \left\| \frac{d^p}{d\lambda^p} (\lambda\, G_q(\lambda)) \right\| = \left\| \frac{d^p}{d\lambda^p} \left[\lambda \frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^q F(\lambda)) \right] \right\| =$$

$$= \left\| \frac{d^p}{d\lambda^p} \left[\frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^{q+1} F(\lambda)) - (q+1) \frac{d^q}{d\lambda^q} (\lambda^q F(\lambda)) \right] \right\| \leqq$$

$$\leqq \left\| \frac{d^{p+q+1}}{d\lambda^{p+q+1}} (\lambda^{q+1} F(\lambda)) \right\| + \left\| (q+1) \frac{d^{p+q}}{d\lambda^{p+q}} (\lambda^q F(\lambda)) \right\| \leqq$$

$$\leqq \frac{Mp!\, (q+1)!\, \varrho^{q+1}}{(\lambda - \omega)^{p+1}} + (q+1) \frac{Mp!\, q!\, \varrho^q}{(\lambda - \omega)^{p+1}} \leqq$$

$$\leqq \frac{Mp!\, (q+1)!\, \varrho^q (1 + \varrho)}{(\lambda - \omega)^{p+1}} \leqq \frac{Mp!\, (q+1)!\, \varrho^q (1 + \varrho)}{(\lambda - \omega - 1)^{p+1}}$$

for any $\lambda > \omega + 1$ and $p, q \in \{0, 1, \ldots\}$.

With regard to (1) and (2), we obtain from Theorem 4 in [2] (*) that there exists

(*) Let us remark that the property (4) serves only to deduce the properties (6) and (7). Hence only the properties (3), (5), (6), (7) are needed and used in the sequel. But these properties can be also obtained from Proposition 4.17 in [1] since it is easy to see from its proof that the function f has also the property that $f(0_+)$ exists, though this fact is not explicitly stated.

a sequence $\psi_q \in R^+ \to E$, $q \in \{0, 1, \dots\}$ such that

(3) $$\|\psi_q(t)\|_L \leqq Mq!\,\varrho^q e^{(\omega+1)t} \quad \text{for any} \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \dots\},$$

(4) $$\|\psi_q(t_1) - \psi_q(t_2)\| \leqq M(q+1)!\,\varrho^q(1+\varrho)\int_{t_1}^{t_2} e^{(\omega+1)\tau}\,\mathrm{d}\tau$$
$$\text{for any} \quad t_1, t_2 \in R^+, \quad t_1 < t_2 \quad \text{and} \quad q \in \{0, 1, \dots\},$$

(5) $$\int_0^\infty e^{-\lambda\tau}\psi_q(\tau)\,\mathrm{d}\tau = -\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^q F(\lambda)) \quad \text{for any} \quad \lambda > \omega + 1 \quad \text{and} \quad q \in \{0, 1, \dots\}.$$

It follows from (4) that

(6) the function ψ_q is continuous on R^+ for any $q \in \{0, 1, \dots\}$,

(7) $\psi_q(0_+)$ exists for any $q \in \{0, 1, \dots\}$.

Now we shall prove that

(8) $$\lambda\int_0^\infty e^{-\lambda\tau}\psi_q(\tau)\,\mathrm{d}\tau - \psi_q(0_+) \xrightarrow[\lambda\to\infty]{} 0 \quad \text{for any} \quad q \in \{0, 1, \dots\}.$$

Indeed, let $q \in \{0, 1, \dots\}$ be fixed. By (3), we have

$$\left\|\lambda\int_0^\infty e^{-\lambda\tau}\psi_q(\tau)\,\mathrm{d}\tau - \psi_q(0_+)\right\| = \left\|\lambda\int_0^\infty e^{-\lambda\tau}(\psi_q(\tau) - \psi_q(0_+))\,\mathrm{d}\tau\right\| =$$
$$= \left\|\lambda\int_0^T e^{-\lambda\tau}(\psi_q(\tau) - \psi_q(0_+))\,\mathrm{d}\tau + \lambda\int_T^\infty e^{-\lambda\tau}(\psi_q(\tau) - \psi_q(0_+))\,\mathrm{d}\tau\right\| \leqq$$
$$\leqq \lambda\int_0^T e^{-\lambda\tau}\,\mathrm{d}\tau \sup_{0<\tau<T}\|\psi_q(\tau) - \psi_q(0_+)\| + \lambda\int_T^\infty e^{-\lambda\tau}\|\psi_q(\tau) - \psi_q(0_+)\|\,\mathrm{d}\tau \leqq$$
$$\leqq \lambda\int_0^\infty e^{-\lambda\tau}\,\mathrm{d}\tau \sup_{0<\tau<T}\|\psi_q(\tau) - \psi_q(0_+)\| + \lambda\int_T^\infty e^{-\lambda\tau}(2Mq!\,\varrho^q e^{(\omega+1)\tau})\,\mathrm{d}\tau =$$
$$= \sup_{0<\tau<T}(\|\psi_q(\tau) - \psi_q(0_+)\|) + 2Mq!\,\varrho^q\lambda\int_T^\infty e^{-(\lambda-\omega-1)\tau}\,\mathrm{d}\tau =$$
$$= \sup_{0<\tau<T}(\|\psi_q(\tau) - \psi_q(0_+)\|) + 2Mq!\,\varrho^q\lambda\frac{e^{-(\lambda-\omega-1)T}}{\lambda-\omega-1}$$
$$\text{for any} \quad T > 0 \quad \text{and} \quad \lambda > \omega + 1.$$

Let now $\varepsilon > 0$. By (7), we choose $T > 0$ so small that $\sup_{0<\tau<T}\|\psi_q(\tau) - \psi_q(0_+)\| \leqq \frac{1}{2}\varepsilon$

and for this T, we choose $\lambda > \omega + 1$ so large that

$$2Mq!\,\varrho^q \lambda \frac{e^{-(\lambda-\omega-1)T}}{\lambda - \omega - 1} \leqq \frac{\varepsilon}{2}.$$

Then we obtain immediately from the preceding formula that $\|\lambda \int_0^\infty e^{-\lambda\tau}\,\psi_q(\tau)\,\mathrm{d}\tau - \psi_q(0_+)\| \leqq \varepsilon$ for sufficiently large λ which proves (8).

On the other hand, by (A) we have

$$\left\|\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}\left(\lambda^q F(\lambda)\right)\right\| \leqq \frac{Mq!\,\varrho^q}{(\lambda-\omega)^2}$$

and consequently

(9) $$\lambda \frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}\left(\lambda^q F(\lambda)\right) \xrightarrow[\lambda\to\infty]{} 0 \quad \text{for any} \quad q \in \{0, 1, \ldots\}\,.$$

It follows from (5), (8) and (9) that

(10) $$\psi_q(0_+) = 0 \quad \text{for any} \quad q \in \{0, 1, \ldots\}\,.$$

Further, by (4) and (10) we find that

(11) $$\|\psi_q(t)\| \leqq M(q+1)!\,\varrho^q(1+\varrho)\,t e^{(\omega+1)t} \quad \text{for any} \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \ldots\}\,.$$

Let us now take $\varphi_q = (1/t)\,\psi_q(t)$ for any $t \in R^+$ and $q \in \{0, 1, \ldots\}$.

It follows from (6) and (11) that

(12) the function φ_q is continuous for any $q \in \{0, 1, \ldots\}$,

(13) $$\|\varphi_q(t)\| \leqq M(q+1)!\,\varrho^q(1+\varrho)\,e^{(\omega+1)t} \quad \text{for any} \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \ldots\}\,.$$

Now (12) and (13) give easily

(14) $$\int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau \xrightarrow[\lambda\to\infty]{} 0 \quad \text{for any} \quad q \in \{0, 1, \ldots\}\,.$$

On the other hand, we get easily from (3) and (13) by means of Proposition 4.4 in [1] that

(15) $$\int_0^\infty e^{-\lambda\tau}\,\psi_q(\tau)\,\mathrm{d}\tau = \int_0^\infty e^{-\lambda\tau}\tau\,\varphi_q(\tau)\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau$$

$$\text{for any} \quad \lambda > \omega + 1 \quad \text{and} \quad q \in \{0, 1, \ldots\}\,.$$

By (5) and (15) we have

(16) $$\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}\left(\lambda^q F(\lambda)\right) \text{ for any } \lambda > \omega + 1 \text{ and } q \in \{0, 1, \ldots\}\,.$$

Using (9), (14) and (16) we conclude finally that

$$(17)\quad \int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau = \frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}(\lambda^q F(\lambda)) \quad \text{for any} \quad \lambda > \omega + 1 \quad \text{and} \quad q \in \{0, 1, \ldots\}.$$

Using Propositions 4.4, 4.6 and 4.12 in [1] we get from (A) and from (17) that

$$(18)\qquad \|\varphi_q(t)\| \leqq M e^{\omega t} q!\, \varrho^q \quad \text{for any} \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \ldots\}.$$

Now let us denote

$$(19)\qquad f = \varphi_0 .$$

We shall prove that

(20) the function f is q-times differentiable for any $q \in \{0, 1, \ldots\}$,

$$(21)\qquad t^q f^{(q)}(t) = (-1)^q\,\varphi_q(t) \quad \text{for any} \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \ldots\}.$$

To this goal we proceed by induction on q.

Clearly, for $q = 0$, the statements (20) and (21) are true by (19).

Now we shall suppose the validity of (20), (21) for a fixed $q \in \{0, 1, \ldots\}$ and prove it for $q + 1$.

According to the induction hypothesis just made, and according to (12), (13) and (17) we can establish the following identities for any $\lambda > \omega + 1$:

$$(22)\quad \int_0^\infty e^{-\lambda\tau}\tau^{q+1} f^{(q)}(\tau)\,\mathrm{d}\tau = \int_0^\infty e^{-\lambda\tau}\tau(\tau^q f^{(q)}(\tau))\,\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^\infty e^{-\lambda\tau}\tau^q f^{(q)}(\tau)\,\mathrm{d}\tau =$$
$$= (-1)^{q+1}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau = (-1)^{q+1}\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^q F(\lambda)),$$

$$(23)\quad \int_0^\infty e^{-\lambda\tau}\left(\int_0^\tau \sigma^q f^{(q)}(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau = (-1)^q\int_0^\infty e^{-\lambda\tau}\left(\int_0^\tau \varphi_q(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\tau =$$
$$= \frac{(-1)^q}{\lambda}\int_0^\infty e^{-\lambda\tau}\,\varphi_q(\tau)\,\mathrm{d}\tau = \frac{(-1)^q}{\lambda}\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}(\lambda^q F(\lambda)),$$

$$(24)\quad \int_0^\infty e^{-\lambda\tau}\int_0^\tau \varphi_{q+1}(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \frac{1}{\lambda}\int_0^\infty e^{-\lambda\tau}\,\varphi_{q+1}(\tau)\,\mathrm{d}\tau = \frac{1}{\lambda}\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^{q+1} F(\lambda)).$$

Further, a simple calculation shows that

$$(25)\qquad \frac{1}{\lambda}\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^{q+1} F(\lambda)) = \frac{1}{\lambda}\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda\lambda^q F(\lambda)) =$$
$$= \frac{1}{\lambda}\left[\lambda\frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^q F(\lambda)) + (q+1)\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}(\lambda^q F(\lambda))\right] =$$
$$= \frac{\mathrm{d}^{q+1}}{\mathrm{d}\lambda^{q+1}}(\lambda^q F(\lambda)) + \frac{q+1}{\lambda}\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}(\lambda^q F(\lambda)) \quad \text{for any} \quad \lambda > \omega + 1 .$$

Using now the uniqueness theorem for the Laplace transform, see 4.11 in [1], we get from (22)–(25) that

(26) $t^{q+1} f^{(q)}(t) = (q + 1) \int_0^t \tau^q f^{(q)}(\tau)\, d\tau + (-1)^{q+1} \int_0^t \varphi_{q+1}(\tau)\, d\tau$ for any $t \in R^+$.

Now (26) implies that

(27) the function $t^{q+1} f^q(t)$, $t \in R^+$, is differentiable,

(28) $(t^{q+1} f^{(q)}(t))' = (q + 1)\, t^q f^{(q)}(t) + (-1)^{q+1} \varphi_{q+1}(t)$ for all $t \in R^+$.

But we see from (27) that the function $f^{(q)}$ itself is differentiable and consequently

(29) f is $(q + 1)$-times differentiable.

On the other hand, (28) and (29) imply

(30) $t^{q+1} f^{(q+1)}(t) = (-1)^{q+1} \varphi_{q+1}(t)$ for every $t \in R^+$.

By (29) and (30), the induction step for the properties (20) and (21) is verified and consequently (20) and (21) hold for $q \in \{0, 1, 2, \ldots\}$.

The desired property (B) follows from (18), (20) and (21).

(B) ⇒ (A): Let f be a fixed function satisfying the condition (B).

It follows at once from this property that

(1) the function F is infinitely differentiable on (ω, ∞).

Now we will prove by induction on q that

(2) $\int_0^\infty e^{-\lambda\tau} \tau^q f^{(q)}(\tau)\, d\tau = (-1)^q \frac{d^q}{d\lambda^q} (\lambda^q F(\lambda))$ for any $\lambda > \omega$ and $q \in \{0, 1, \ldots\}$.

The identity (2) is evidently true by the assumption (B) for $q = 0$.

Now let us suppose its validity for a fixed $q \in \{0, 1, \ldots\}$ and proceed to prove it for $q + 1$. We have clearly $(t^{q+1} f^{(q)}(t))' = t^{q+1} f^{(q+1)}(t) + (q + 1)\, t^q f^{(q)}(t)$ for any $t \in R^+$.

Using this identity, we can write by (1), (2) and (B) that

$$\int_0^\infty e^{-\lambda\tau} \tau^{q+1} f^{(q+1)}(\tau)\, d\tau = -(q+1) \int_0^\infty e^{-\lambda\tau} \tau^q f^{(q)}(\tau)\, d\tau + \int_0^\infty e^{-\lambda\tau} \frac{d}{d\tau} (\tau^{q+1} f^{(q)}(\tau))\, d\tau =$$

$$= -(q + 1) \int_0^\infty e^{-\lambda\tau} \tau^q f^{(q)}(\tau)\, d\tau + \lambda \int e^{-\lambda\tau} \tau^{q+1} f^{(q)}(\tau)\, d\tau =$$

$$= (-1)^{q+1} (q + 1) \frac{d^q}{d\lambda^q} (\lambda^q F(\lambda)) - \lambda \frac{d}{d\lambda} \int_0^\infty e^{-\lambda\tau} \tau^q f^{(q)}(\tau)\, d\tau =$$

$$= (-1)^{q+1} (q + 1) \frac{d^q}{d\lambda^q} (\lambda^q F(\lambda)) - (-1)^q \lambda \frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^q F(\lambda)) =$$

$$= (-1)^{q+1} \left[(q + 1) \frac{d^q}{d\lambda^q} (\lambda^q F(\lambda)) + \lambda \frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^q F(\lambda)) \right] = (-1)^{q+1} \frac{d^{q+1}}{d\lambda^{q+1}} (\lambda^{q+1} F(\lambda))$$

which verifies the induction step.

Now, using (2), we obtain from (B) that

$$(3) \quad \left\| \frac{d^{p+q}}{d\lambda^{p+q}} (\lambda^q F(\lambda)) \right\| = \left\| \frac{d^p}{d\lambda^p} \int_0^\infty e^{-\lambda\tau} (-\tau)^q f^{(q)} \, d\tau \right\| = \left\| \int_0^\infty e^{-\lambda\tau} (-\tau)^{p+q} f^{(q)}(\tau) \, d\tau \right\| \leqq$$

$$\leqq \int_0^\infty e^{-\lambda\tau} \tau^p (\tau^q \| f^{(q)}(\tau) \|) \, d\tau \leqq \int_0^\infty e^{-\lambda\tau} \tau^p M e^{\omega\tau} q! \, \varrho^q = \frac{M p! \, q! \, \varrho^q}{(\lambda - \omega)^{p+1}}$$

$$\text{for any} \quad t \in R^+ \quad \text{and} \quad p, q \in \{0, 1, \ldots\} .$$

But (1) and (3) give (A).

6. Remark. The preceding theorem has the advantage that the chatacteristic growth constants (M, ω, ϱ) are preserved in the course of the transformation in both directions. Nevertheless, technically it is difficult to verify the determining property (A).

7. Remark. Some words about the proof of the implication (A) ⇒ (B) in the preceding theorem. The first part of this proof deals with the existence of a sequence of functions $\varphi_q \in R^+ \to E$ with the properties (12), (17) and (18). In the case of a reflexive space E, these properties can be obtained directly from Corollary 9 in [3] or from Theorem 4 in [4], in the numerical case, $E = R$, from Theorems 16a and 16b in [5], if we replace in (12) "continuous" by "measurable". In the remaining part of the proof we must replace "differentiable" by "feebly differentiable" in the following sense: a function $f \in R^+ \to E$ is feebly differentiable if there exists a function $g \in \in R^+ \to E$ such that for any $0 < \alpha < \beta$, g is integrable over (α, β) and $f(\beta) - f(\alpha) = = \int_\alpha^\beta g(\sigma) \, d\sigma$. Moreover, some identities are valid not for all $t \in R^+$ but only for almost all $t \in R^+$. All these technicalities do not influence the final result.

8. Remark. The infinitely differentiable functions satisfying the inequality

$$(1) \qquad \| f^{(q)}(t) \| \leqq M e^{\omega t} \frac{q! \, \varrho^q}{t^q}$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$ with some nonnegative constants M, ω, ϱ represent a special class of functions analytic on the positive half-axis. It is clear from Taylor's theorem that (1) implies

$$(2) \qquad f(\tau) = \sum_{q=0}^\infty \frac{f^{(q)}(t)}{q!} (\tau - t)^q$$

for every $t, \tau \in R^+$ such that $|\tau - t| < t/\varrho$ (supposing $\varrho > 0$ because the case $\varrho = 0$ is uninteresting). We see from (2) that the radius of convergence of the power series in (2) depends linearly on the value $t \in R^+$ (more precisely, it is not less than a positive linear function of t). This is a remarkable property of the class of functions considered but not characteristic because it does not imply the exponential growth of derivatives.

9. A function $\mathscr{T}$ defined on R^+ with values in the Banach space of bounded linear operators on E into E will be called a semigroup of linear operators in E if

(T_1) $\mathscr{T}(t_1 + t_2) = \mathscr{T}(t_1)\,\mathscr{T}(t_2)$ for any $t_1, t_2 \in R^+$,

(T_2) $\mathscr{T}(t)\,x \to x$ $(t \to 0_+)$ for any x from a dense subset of E.

10. Theorem. *Let $\mathscr{T}$ be a semigroup of linear operators in E. Then the following two statements* (A), (B) *are equivalent:*

(A) *there exist $M \geqq 0$, $\omega \geqq 0$ and $\varrho \geqq 0$ so that for every $x \in E$ for which the function $\mathscr{T}(\cdot)\,x$ is infinitely differentiable on R^+, we have*

$$\left\|\frac{\mathrm{d}^q}{\mathrm{d}t^q}\,\mathscr{T}(t)\,x\right\| \leqq M e^{\omega t}\,\frac{q!\,\varrho^q}{t^q}\,\|x\|$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$;

(B) *there exist $\delta > 0$ and $K \geqq 0$ so that $\|\mathscr{T}(t)\| \leqq K$ for every $0 < t \leqq \delta$ and $\|\mathscr{T}(t_1) - \mathscr{T}(t_2)\| \leqq K|\log t_1 - \log t_2|$ for every $0 < t_1, t_2 \leqq \delta$.*

Proof. It follows easily from (T_1), (T_2) (see the proof of Theorem 10.3.4 in [8]) that there is a subset $D \subseteq E$ such that

(1) $\qquad D$ is dense in E,

(2) $\qquad$ the function $\mathscr{T}(\cdot)\,x$ is infinitely differentiable for every $x \in D$.

For $x \in E$ and $q \in \{0, 1, \ldots\}$, we shall write simply $\mathscr{T}^{(q)}(t)\,x$ instead of $(\mathrm{d}^q/\mathrm{d}t^q)\,\mathscr{T}(t)\,x$, if this derivative exists. Instead of $\mathscr{T}^{(1)}(t)\,x$ we write $\mathscr{T}'(t)\,x$.

(A) $\Rightarrow$ (B): It follows from (A) and (2) that

$$\|\mathscr{T}(t)\,x\| \leqq M e^{\omega t}\|x\| \quad \text{for every} \quad x \in D, \tag{3}$$

$$\|\mathscr{T}(t_1)\,x - \mathscr{T}(t_2)\,x\| = \left\|\int_{t_1}^{t_2} \mathscr{T}'(\tau)\,x\,\mathrm{d}\tau\right\| \leqq \int_{t_1}^{t_2} \|\mathscr{T}'(\tau)\,x\|\,\mathrm{d}\tau \leqq \tag{4}$$

$$\leqq \int_{t_1}^{t_2} M e^{\omega\tau}\,\frac{\varrho}{\tau}\,\mathrm{d}\tau\,\|x\| \leqq M\varrho e^{\omega t_2} \int_{t_1}^{t_2} \frac{1}{\tau}\,\mathrm{d}\tau\,\|x\| \leqq$$

$$\leqq M\varrho e^{\omega t_2}(\log t_1 - \log t_2)\,\|x\| \quad \text{for every} \quad x \in D \quad \text{and} \quad 0 < t_1 \leqq t_2\,.$$

But (1), (3) and (4) imply immediately (B) if we take, for example, $\delta = 1$, $K = M + M\varrho e^{\omega}$.

(B) $\Rightarrow$ (A): Let us first fix a $\delta > 0$ and a $K \geqq 0$ so that (B) holds.

It is easy to see that for any $t \in R^+$, there is a positive integer n_t such that

$$(n_t - 1)\,\delta < t \leqq n_t\delta \quad \text{for any} \quad t \in R^+ . \tag{5}$$

Since (5) yields $0 < t - (n_t - 1)\,\delta \leqq \delta$ and $n_t < t/\delta + 1$, we get from (T_1) and (B)

$$\begin{aligned}\|\mathscr{T}(t)\| &= \|\mathscr{T}((n_t - 1)\,\delta + t - (n_t - 1)\,\delta)\| = \\ &= \|\mathscr{T}(\delta)^{n_t - 1}\,\mathscr{T}(t - (n_t - 1)\,\delta)\| \leqq K^{n_t - 1}K = \\ &= K^{n_t} \leqq K^{t/\delta + 1} = K(K^{1/\delta})^t = Ke^{(\log K^{1/\delta})t} \quad \text{for every} \quad t \in R^+ .\end{aligned} \tag{6}$$

Let us now take $\varkappa = \log K^{1/\delta}$. Then (6) implies

$$\|\mathscr{T}(t)\| \leqq Ke^{\varkappa t} \quad \text{for every} \quad t \in R^+ . \tag{7}$$

On the other hand, it is easy to see from (T_1) and (B) that

$$\mathscr{T}'(s + t)\,x = \mathscr{T}(s)\,\mathscr{T}'(t)\,x = \mathscr{T}(t)\,\mathscr{T}'(s)\,x \quad \text{for every} \quad t, s \in R^+ \quad \text{and} \quad x \in D . \tag{8}$$

Further, by (5) we can write

$$n_t < t/\delta + 1 \quad \text{and} \quad t/n_t \leqq \delta \quad \text{for every} \quad t \in R^+ . \tag{9}$$

It follows from (B) and (2) that

$$\begin{aligned}\|\mathscr{T}'(t)\,x\| &= \left\|\lim_{h\to 0+} \frac{1}{-h}\,(\mathscr{T}(t - h)\,x - \mathscr{T}(t)\,x\right\| \leqq \\ &\leqq \lim_{h\to 0+} \frac{1}{h}\,\|\mathscr{T}(t - h)\,x - \mathscr{T}(t)\,x\| \leqq K \lim_{h\to 0+} \frac{1}{-h}\,(\log(t - h) - \log t)\,\|x\| = \frac{K}{t}\,\|x\| \\ &\qquad \text{for every} \quad x \in D \quad \text{and} \quad 0 < t \leqq \delta .\end{aligned} \tag{10}$$

Using (T_1), (2) and (7)–(10) we get

$$\begin{aligned}\|\mathscr{T}'(t)\,x\| &= \left\|\mathscr{T}\left(t - \frac{t}{n_t}\right)\mathscr{T}'\left(\frac{t}{n_t}\right)x\right\| \leqq Ke^{\varkappa(t - t/n_t)}\,K\,\frac{n_t}{t}\,\|x\| \leqq \\ &\leqq K^2e^{\varkappa t}\,\frac{1}{t}\left(\frac{t}{\delta} + 1\right)\|x\| \leqq K^2e^{\varkappa t}e^{t/\delta}\,\frac{1}{t}\,\|x\| = K^2e^{(\varkappa + \delta^{-1})t}\,\frac{1}{t}\,\|x\| \\ &\qquad \text{for every} \quad x \in D \quad \text{and} \quad t \in R^+ .\end{aligned} \tag{11}$$

It follows from (1) and (11) by virtue of the theorem on differentiation of limit that

(12) the function $\mathscr{T}(\cdot)\, x$ is continuously differentiable for every $x \in E$,

(13) $$\|\mathscr{T}'(t)\, x\| \leqq K^2 e^{(\varkappa+\delta^{-1})t} \frac{1}{t} \|x\| \quad \text{for every} \quad x \in E \quad \text{and} \quad t \in R^+ .$$

We see from (T_1) and (12) that

(14) $$\mathscr{T}'(t + s)\, x = \mathscr{T}(s)\, \mathscr{T}'(t)\, x = \mathscr{T}(t)\, \mathscr{T}'(s)\, x$$
$$\text{for every} \quad x \in E \quad \text{and} \quad t, s \in R^+ .$$

We shall now prove that

(15) the function $\mathscr{T}(\cdot)\, x$ is n-times differentiable for every $x \in E$ and $n \in \{1, 2, \ldots\}$,

(16) $$\mathscr{T}^{(n)}(t)\, x = (\mathscr{T}'(t/n))^n\, x \quad \text{for every} \quad x \in E, \quad t \in R^+ \quad \text{and} \quad n \in \{1, 2, \ldots\} .$$

We proceed by induction. The case $n = 1$ is true by (12). Let now (15) and (16) hold for a fixed $n \in \{1, 2, \ldots\}$. Then we see from (14) that, for every $x \in E$, the function $\mathscr{T}(\cdot)\, x$ is $(n + 1)$-times differentiable and

$$\mathscr{T}^{(n+1)}(\tau + \sigma)\, x = \frac{\mathrm{d}^n}{\mathrm{d}\tau^n} [\mathscr{T}(\tau)\, \mathscr{T}'(\sigma)\, x] = \mathscr{T}^{(n)}(\tau)\, \mathscr{T}'(\sigma)\, x \quad \text{for every} \quad \tau, \sigma \in R^+ .$$

Hence (15) holds with $n + 1$ instead of n and moreover, taking $\tau = tn/(n + 1)$ and $\sigma = t/(n + 1)$ in the preceding formula, we get immediately (16) with $n + 1$ instead of n again.

Making use of (13) and (16) we get

(17) $$\|\mathscr{T}^{(n)}(t)\, x\| = \left\|\left(\mathscr{T}'\left(\frac{t}{n}\right)\right)^n x\right\| \leqq \left[K^2 e^{(\varkappa+\delta^{-1})t/n} \frac{n}{t}\right]^n \|x\| = K^{2n} e^{(\varkappa+\delta^{-1})t} \frac{n^n}{t^n} \|x\| =$$
$$= K^{2n} e^{(\varkappa+\delta^{-1})t} \frac{1}{t^n} \frac{n^n}{n!}\, n!\, \|x\| \leqq K^{2n} e^{(\varkappa+\delta^{-1})t} \frac{1}{t^n}\, e^n n!\, \|x\| = e^{(\varkappa+\delta^{-1})t} \frac{n!\, (K^2 e)^n}{t^n} \|x\|$$
$$\text{for every} \quad x \in E, \quad t \in R^+ \quad \text{and} \quad n \in \{1, 2, \ldots\} .$$

Let us now take $M = \max(1, K)$, $\omega = \varkappa + \delta^{-1}$ and $\varrho = K^2 e$. Then we get from (7) and (17) that

(18) $$\|\mathscr{T}^{(q)}(t)\, x\| \leqq M e^{\omega t} \frac{q!\, \varrho^q}{t^q} \|x\| \quad \text{for every} \quad x \in E, \quad t \in R^+ \quad \text{and} \quad q \in \{0, 1, \ldots\} .$$

But (15) and (18) prove (A).
The proof is complete.

11. Remark. The proof of the implication (B) $\Rightarrow$ (A) in the preceding theorem uses an idea of Yosida [6] – see the beginning of the proof of his Theorem 1.

12. Proposition. *Let $\mathcal{T}$ be a semigroup of linear operators in E. Further, let M, ω, ϱ be nonnegative constants. Then the following two statements* (A) *and* (B) *are equivalent:*

(A) *for every $x \in E$ for which the function $\mathcal{T}(\cdot)\,x$ is infinitely differentiable on R^+ we have*

$$\left\| \frac{\mathrm{d}^q}{\mathrm{d}t^q} \mathcal{T}(t)\,x \right\| \leqq M e^{\omega t} \frac{q!\,\varrho^q}{t^q} \|x\|$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$,

(B) *the function $\mathcal{T}$ is infinitely differentiable on R^+ (as a function on R^+ into the Banach space of bounded linear operators on E into E) and*

$$\|\mathcal{T}^{(q)}(t)\| = M e^{\omega t} \frac{q!\,\varrho^q}{t^q}$$

for every $t \in R^+$ and $q \in \{0, 1, \ldots\}$.

Proof. The implication (B) $\Rightarrow$ (A) is trivial. In the proof of (A) $\Rightarrow$ (B) we use the properties (1), (2) from the proof of Theorem 9.

13. Remark. The semigroups satisfying the condition 12 (B) are usually called holomorphic or analytic. Mizohata [9] calls them parabolic which name seems to be the most specific and adequate as an abstract extension of the meaning of this term in the theory of partial differential equations. See also Remark 8.

The usual essentially equivalent approach to the parabolic semigroups consists in their characterization by means of an analytic continuation into a wedge-shaped domain around the positive half-axis (cf., e.g., [7] and [8]).

14. Theorem. *Let A be a linear operator from E into E. Then the following two statements* (A), (B) *are equivalent:*

(A) *the operator A is the generator of a semigroup $\mathcal{T}$ of linear operators in E (i.e., x belongs to the domain of A if and only if there is $y \in E$ such that*

$$\frac{1}{t}\left(\mathcal{T}(t)\,x - x\right) \xrightarrow[t \to 0_+]{} y\,;$$

in that case $Ax = y$), satisfying the condition (A) *from Theorem 9,*

(B) *there exist nonnegative constants M, ω, ϱ such that*

(I) $(\omega, \infty) \subseteq \varrho(A)$ (the resolvent set of A),