Werk

Label: Article

Jahr: 1979

PURL: https://resolver.sub.uni-goettingen.de/purl?31311157X_0104|log38

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Časopis pro pěstování matematiky, roč. 104 (1979), Praha

SOME REMARKS ON HAVLÍČEK-TIETZE CONFIGURATION

ANDRZEJ LEWANDOWSKI and HANNA MAKOWIECKA, Toruń

(Received December 27, 1976)

It is well known that in some projective planes there exist pairs of manifold homologic triangles. A particular case is a six-fold homology, when the triangles are homologic in each permutation of their vertices. Pairs of six-fold homologic triangles exist for example in the projective plane over the field of complex numbers.

In [5] K. HAVLÍČEK and J. TIETZE have proved the following theorem:

In the finite projective plane of order 4 there exists a set of four triangles, no two of them with a common vertex, and each two of them in six-fold homology. The centres of homologies of any two triangles are the vertices of the other two, the axes being the corresponding opposite sides.

The configuration described in this theorem will be called the configuration of Havlíček-Tietze, shortly (H-T).

Moreover, the authors of the cited paper proved that, in the projective plane of order 4, the lines of any configuration (H-T) meet altogether in the nine points which complete the plane.

Obviously, the projective plane of order 4 is not the only plane containing (H-T). Every projective plane with a finite subplane of order 4 contains this configuration.

In this paper we consider the problem of existence of the (H-T) configuration. In detail, we present a necessary and sufficient condition of existence of the above configuration in desarguesian planes.

Definition. We say that a pair of triangles T_1, T_2 is *special six-fold homologic* if:

1. T_1, T_2 have no common vertex,
2. T_1, T_2 are in six-fold homology,
3. the centres of homologies are vertices of a new pair of triangles T_3, T_4,
4. if a centre of any arbitrary homology of triangles T_1, T_2 is one of the vertices of a triangle T_i, $i = 3, 4$, then the axis of this homology is incident with the remaining vertices of T_i.

Theorem 1. *If in an arbitrary projective plane π there exists a pair of special six-fold homologic triangles, then π contains an* (H-T) *configuration.*

Proof. Let us assume that a pair of triangles $T_1 = \{A_1, A_2, A_3\}$ and $T_2 = \{B_1, B_2, B_3\}$ satisfies the conditions of the above definition. We define successively six points $C_1, C_2, C_3, D_1, D_2, D_3$ as centres of homologies of the following triples

$$(1)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_3 & B_2 & B_1 \end{pmatrix}, \quad (2)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_2 & B_1 & B_3 \end{pmatrix}, \quad (3)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_3 & B_2 \end{pmatrix}$$

$$(4)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_2 & B_3 & B_1 \end{pmatrix}, \quad (5)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_3 & B_1 & B_2 \end{pmatrix}, \quad (6)\ \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{pmatrix}.$$

Let $T_3 = \{C_1, C_2, C_3\}$, $T_4 = \{D_1, D_2, D_3\}$. It is easy to show that no three points from $A_1, A_2, A_3, B_1, B_2, B_3$ are collinear, and hence no pair of triangles from $T_1, \ldots, T_4$ has a common vertex. The vertices of triangles $T_1, \ldots, T_4$ determine 21 lines presented below:

$$\begin{array}{llll} a_1 = A_2A_3 & b_1 = B_2B_3 & c_1 = C_2C_3 & d_1 = D_2D_3 \\ a_2 = A_1A_3 & b_2 = B_1B_3 & c_2 = C_1C_3 & d_2 = D_1D_3 \\ a_3 = A_1A_2 & b_3 = B_1B_2 & c_3 = C_1C_2 & d_3 = D_1D_2 \end{array}$$

$$\begin{array}{lll} w_1 = A_1B_1 & w_4 = A_2B_1 & w_7 = A_3B_1 \\ w_2 = A_1B_3 & w_5 = A_2B_3 & w_8 = A_3B_3 \\ w_3 = A_1B_2 & w_6 = A_2B_2 & w_9 = A_3B_2 . \end{array}$$

Considering successively the homologies of triples (1), (2), …, (6) we obtain the following incidence conditions:

$$\begin{array}{lllll} C_3, D_3 \in w_1 , & C_1, D_2 \in w_2 , & C_2, D_1 \in w_3 , & C_2, D_2 \in w_4 , & C_3, D_1 \in w_5 \\ C_1, D_3 \in w_6 , & C_1, D_1 \in w_7 , & C_2, D_3 \in w_8 , & C_3, D_2 \in w_9 . & \end{array}$$

From the conditions considered above we obtain that the lines a_i, b_i, c_i, d_i, w_i, $i = 1, 2, 3$, are all different.

We define nine points:

$$\begin{array}{lll} W_1 = a_1 \cap b_1 & W_4 = a_2 \cap b_1 & W_7 = a_3 \cap b_1 \\ W_2 = a_1 \cap b_2 & W_5 = a_2 \cap b_2 & W_8 = a_3 \cap b_2 \\ W_3 = a_1 \cap b_3 & W_6 = a_2 \cap b_3 & W_9 = a_3 \cap b_3 . \end{array}$$

The centre of homology of the triples (1) is C_1 and $c_1 = C_2C_3$ is its axis. Thus $W_5, W_7, W_3 \in c_1$.

Similarly, from the homologies of triples (2), (3), …, (6) we have: $W_2, W_4, W_9 \in c_2$; $W_1, W_6, W_8 \in c_3$; $W_2, W_6, W_7 \in d_1$; $W_3, W_4, W_8 \in d_2$; $W_1, W_5, W_9 \in d_3$.

Therefore, from the incidences described above we obtain the following scheme (S):

$$a_1 = \{A_2, A_3, W_1, W_2, W_3, \ldots\} \quad b_1 = \{B_2, B_3, W_1, W_4, W_7, \ldots\}$$
$$a_2 = \{A_1, A_3, W_4, W_5, W_6, \ldots\} \quad b_2 = \{B_1, B_3, W_2, W_5, W_8, \ldots\}$$
$$a_3 = \{A_1, A_2, W_7, W_8, W_9, \ldots\} \quad b_3 = \{B_1, B_2, W_3, W_6, W_9, \ldots\}$$
$$c_1 = \{C_2, C_3, W_3, W_5, W_7, \ldots\} \quad d_1 = \{D_2, D_3, W_2, W_6, W_7, \ldots\}$$
$$c_2 = \{C_1, C_3, W_2, W_4, W_9, \ldots\} \quad d_2 = \{D_1, D_3, W_3, W_4, W_8, \ldots\}$$
$$c_3 = \{C_1, C_2, W_1, W_6, W_8, \ldots\} \quad d_3 = \{D_1, D_2, W_1, W_5, W_9, \ldots\}$$
$$w_1 = \{A_1, B_1, C_3, D_3, \ldots\} \quad w_4 = \{A_2, B_1, C_2, D_2, \ldots\} \quad w_7 = \{A_3, B_1, C_1, D_1, \ldots\}$$
$$w_2 = \{A_1, B_3, C_1, D_2, \ldots\} \quad w_5 = \{A_2, B_3, C_3, D_1, \ldots\} \quad w_8 = \{A_3, B_3, C_2, D_3, \ldots\}$$
$$w_3 = \{A_1, B_2, C_2, D_1, \ldots\} \quad w_6 = \{A_2, B_2, C_1, D_3, \ldots\} \quad w_9 = \{A_3, B_2, C_3, D_2, \ldots\}.$$

All lines of (S) are different, hence all the points of (S) are also different. From (S) we can easily verify that the set of triangles $T_1, \ldots, T_4$ realizes an (H-T) configuration (see [5]).

Theorem 2. *If in a projective Fano plane π there exsits an* (H-T) *configuration, then π has a finite subplane of order* 4.

Proof. By examining quadrangles (W_4, W_5, W_7, W_9), (W_4, W_5, W_8, W_9), (W_4, W_5, W_7, W_8), (W_1, W_2, W_7, W_9), (W_1, W_2, W_8, W_9), (W_1, W_2, W_7, W_8), (W_1, W_2, W_4, W_6), (W_1, W_2, W_5, W_6), (W_1, W_2, W_4, W_5), we obtain successively $W_1 \in w_1$; $W_2 \in w_3$; $W_3 \in w_2$; $W_4 \in w_4$; $W_5 \in w_6$; $W_6 \in w_5$; $W_7 \in w_7$; $W_8 \in w_9$; $W_9 \in w_8$. It follows from (S) that the vertices of triangles $T_1, \ldots, T_4$ with the points $W_1, W_2, \ldots W_9$ and appropriate 5-point lines form the set of all points and lines of the projective plane of order 4 (cf. [4], [5]).

Remark. In the projective Fano plane, the points $W_1, W_2, \ldots, W_9$ and the respective three-point lines determined by an (H-T) configuration from the affine plane of order 3.

Now, we shall consider the problem of existence of an (H-T) configuration on desarguesian projective planes.

It is well known that an arbitrary projective plane is desarguesian iff it is isomorphic with a projective plane over a field F, not necessarily commutative.

Theorem 3. *Let π be an arbitrary desarguesian projective plane. Then in π there exist a pair of triangles with no common vertex, and in six-fold homology if and only if π is isomorphic with a plane over a field F containing a root of the polynomial $x^2 + x + 1$ different from* 1.[1])

[1]) This implies that in a projective plane over any field with the characteristic 3 there exist no pairs of six-fold homologic triangles.

Proof. Let π be a projective plane over a field F containing a root of the polynomial $x^2 + x + 1$ different from 1. By means of an element $a \in F$ such that $a \neq 0, 1$ and $a^2 + a + 1 = 0$ we define the pair of triangles $T_1 = \{A_1, A_2, A_3\}$, $T_2 = \{B_1, B_2, B_3\}$ where

$$A_1 = (1, 0, 0) \quad A_2 = (0, 1, 0) \quad A_3 = (0, 0, 1)$$
$$B_1 = (a, 1, 1) \quad B_2 = (1, a, 1) \quad B_3 = (1, 1, a).$$

One can easily show that T_1 and T_2 have no common vertex and are in six-fold homology. The centres and axes of these six homologies are the points

$$C_1 = (a, 1, a) \quad D_1 = (1, a^2, a)$$
$$C_2 = (a, a, 1) \quad D_2 = (a^2, 1, a)$$
$$C_3 = (1, a, a) \quad D_3 = (1, 1, 1)$$

and lines

$$c_1 : x_1 + ax_2 + x_3 = 0 \quad d_1 : a^2x_1 + x_2 + ax_3 = 0$$
$$c_2 : x_1 + x_2 + ax_3 = 0 \quad d_2 : a^2x_1 + ax_2 + x_3 = 0$$
$$c_3 : ax_1 + x_2 + x_3 = 0 \quad d_3 : x_1 + x_2 + x_3 = 0.$$

Conversely, let $T_1' = \{A_1', A_2', A_3'\}$, $T_2' = \{B_1', B_2', B_3'\}$ be two triangles with no common vertex in the projective desarguesian plane over a field F. Assume that T_1', T_2' are in six-fold homology, and let P denote the centre of homology of the triples (A_1', A_2', A_3') and (B_1', B_2', B_3'). No three of the points A_1', A_2', A_3', P are collinear and thus there exists an automorphism of the plane, which transforms these points onto the points $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$, $D_3 = (1, 1, 1)$, respectively (cf. [3]).

Let B_1^*, B_2^*, B_3^* be the images of the points B_1', B_2', B_3'. Obviously the triangles $T_1 = \{A_1, A_2, A_3\}$ and $T_2^* = \{B_1^*, B_2^*, B_3^*\}$ are also in six-fold homology and have no common vertex.

A necessary condition for the homology of the triples (A_1, A_2, A_3) and (B_1^*, B_2^*, B_3^*) with the point D_3 as its centre can be expressed in the following form of homogeneous coordinates of the points

$$B_1 = (a, 1, 1) \quad B_2 = (1, b, 1) \quad B_3 = (1, 1, c)$$

where $a, b, c \neq 0, 1$. The remaining homologies hold if the following systems of equations have non-trivial solutions:

$$\left.\begin{aligned} x_2 - bx_3 &= 0 \\ cx_1 - x_3 &= 0 \\ x_1 - ax_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} cx_2 - x_3 &= 0 \\ x_1 - ax_3 &= 0 \\ bx_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_2 - bx_3 &= 0 \\ x_1 - ax_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_2 - x_3 &= 0 \\ cx_1 - x_3 &= 0 \\ bx_1 - x_2 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} cx_2 - x_3 &= 0 \\ x_1 - x_3 &= 0 \\ x_1 - ax_2 &= 0 \end{aligned}\right\}.$$