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LOCAL DETERMINACY OF ABSTRACT RIGHT PSEUDOPROCESSES

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1. INTRODUCTION

In the paper [1] devoted to the axiomatic theory of differential equations, O. HÁJEK introduced the notion of the process p in an abstract set P over a subset T of the reals as a relation p in the cartesian product $P \times T$, satisfying the following three conditions:

(i) $((y, v), (x, u)) \in \mathsf{p}$ implies $v \geqq u$;

(ii) $((y, u), (x, u)) \in \mathsf{p}$ implies $y = x$;

(iii) $((y, v), (x, u)) \in \mathsf{p}$ iff for each $w \in \langle u, v \rangle$ there exists $z \in P$ such that $((y, v), (z, w)) \in \mathsf{p}$, $((z, w), (x, u)) \in \mathsf{p}$.

In [2], omitting the group property (iii), Hájek introduced the notion of a pre-process and conjectured that many properties of the processes could be proved for the pre-processes as well. This idea inspired the authors to develop the theory of pre-processes (called pseudoprocesses in our terminology) and to prove some results concerning local determinacy of these objects. For the motivation of both the notions introduced and the investigations performed in the paper see [1], [2] and [4].

First of all we shall recall several notions and the notation concerning relations between two sets, which will be used throughout the paper.

1.1. Relations. Let X, Y be arbitrary sets. Any subset r of the cartesian product $X \times Y$ (in this order) is called a *relation between* X *and* Y. If $X = Y$, then a relation $\mathsf{r} \subset X \times X$ is called a *relation in* X. When a pair (x, y) belongs to a relation r, then we write either $(x, y) \in \mathsf{r}$ or $x\mathsf{r}y$.

Since relations are sets, the meaning of the symbols $\mathsf{r} \subset \mathsf{s}$, $\mathsf{r} \cup \mathsf{s}$, $\mathsf{r} \cap \mathsf{s}$ etc. for any two relations r, s between X and Y is obvious. Recall that the relation *inverse* to $\mathsf{r} \subset X \times Y$ is the relation $\mathsf{r}^{-1} \subset Y \times X$ defined as follows:

$$x\mathsf{r}^{-1}y \quad \text{iff} \quad y\mathsf{r}x\,. \tag{1.1.1}$$

The *identical relation* in X will be denoted by 1_X and defined by

(1.1.2) $$x 1_X y \quad \text{iff} \quad x = y \in X .$$

If $r \subset X \times Y$, $s \subset Y \times Z$, then the *composition* $r \circ s$ of the relations r and s (in this order) is the relation $r \circ s \subset X \times Z$ satisfying the following condition:

(1.1.3) $$x r \circ s z \text{ iff there exists } y \in Y \text{ such that } xry \text{ and } yrz .$$

For each $r \subset X \times Y$ we define the *domain* of the relation r as the set

(1.1.4) $$D_r = \{y \in Y \mid xry \text{ for some } x \in X\} .$$

Given $r \subset X \times Y$, we set

(1.1.5) $$ry = \{x \in X \mid (x, y) \in D_r\} ,$$

(1.1.6) $$rA = \{x \in X \mid (x, y) \in D_r \text{ for some } y \in A\}$$

for each $y \in Y$ and $A \subset Y$. Analogously, we set

(1.1.7) $$xr = \{y \in Y \mid (x, y) \in D_r\} ,$$

(1.1.8) $$Br = \{y \in Y \mid (x, y) \in D_r \text{ for some } x \in B\}$$

for each $x \in X$ and $B \subset X$.

The *partialization* of a relation $r \subset X \times Y$ to a subset $A \subset Y$ is the relation

(1.1.9) $$r|_A = r \cap (X \times A) .$$

Recall that $r \subset X \times Y$ is said to be *reflexive, symmetric* or *transitive* iff $1_X \subset r$, $r = r^{-1}$ or $r \circ r \subset r$, respectively.

It is evident that if $r, s \subset X \times X$, then

(1.1.10) $$r \subset s \quad \text{iff} \quad r^{-1} \subset s^{-1} ,$$

(1.1.11) $$(r \circ s)^{-1} = s^{-1} \circ r^{-1} .$$

In what follows, P will denote an arbitrary fixed set and T will denote a subset of the set R of all reals, ordered by the natural order relation $\leqq$ inherited from R. The symbol R^* will be used to denote the extended real line $R \cup \{-\infty, +\infty\}$ with the ordering extended from R to R^* in such a way that $-\infty < x < +\infty$ for each $x \in R$. If M is a subset of any one of the sets T, R, R^*, then sup M will denote the least upper bound (l.u.b.) of the set M in the set R^*.

The main objects investigated in this paper are relations r in the cartesian product $P \times T$. Each such relation determines uniquely a system of relations ${}_v r_u$ in P, $u, v \in T$, such that

(1.1.12) $$(y, v)\, r (x, u) \quad \text{iff} \quad y\, {}_v r_u\, x .$$

Conversely, each relation $\mathsf{r} \subset (P \times T) \times (P \times T)$ is uniquely determined by such a system of relations ${}_v\mathsf{r}_u \subset P$, $u, v \in T$ according to (1.1.12).

Evidently, (1.1.5) and (1.1.7) imply

$$(1.1.13) \qquad {}_v\mathsf{r}_u x = \{y \in P \mid y_v \mathsf{r}_u x\},$$

$$(1.1.14) \qquad y_v\mathsf{r}_u = \{x \in P \mid y_v \mathsf{r}_u x\}.$$

1.2. Lemma. *Let* r, s *be relations in* $P \times T$. *Then the following assertions are equivalent:*

(i) $\mathsf{r} \subset \mathsf{s}$.

(ii) $(y, v)\,\mathsf{r}(x, u)$ *implies* $(y, v)\,\mathsf{s}(x, u)$ *for all* $(x, u), (y, v) \in P \times T$.

(iii) $y_v\mathsf{r}_u x$ *implies* $y_v\mathsf{s}_u x$ *for all* $x, y \in P$, $u, v \in T$.

(iv) ${}_v\mathsf{r}_u x \subset {}_v\mathsf{s}_u x$ *for all* $x \in P$, $u, v \in T$.

(v) ${}_v\mathsf{r}_u \subset {}_v\mathsf{s}_u$ *for all* $u, v \in T$.

1.3. Lemma. *Let* p, r, s *be relations in* $P \times T$. *Then the following assertions hold:*

(i) ${}_v(\mathsf{r} \cup \mathsf{s})_u = {}_v\mathsf{r}_u \cup {}_v\mathsf{s}_u$ *for all* $u, v \in T$.

(ii) ${}_v(\mathsf{r} \cap \mathsf{s})_u = {}_v\mathsf{r}_u \cap {}_v\mathsf{s}_u$ *for all* $u, v \in T$.

(iii) $({}_v\mathsf{p}_u)^{-1} = {}_u\mathsf{p}_v^{-1}$ *for all* $u, v \in T$.

2. RIGHT PSEUDOPROCESSES

2.1. Definition. Let P be an arbitrary set, $T \subset R^{\#}$, $\mathsf{p} \subset (P \times T) \times (P \times T)$. The relation p is said to be a *pseudoprocess in P over T* iff it satisfies the condition

$$(I) \qquad {}_u\mathsf{p}_u \subset 1_P \quad \text{for all} \quad u \in T.$$

A pseudoprocess p in P over T is said to be *right* iff

$$(R) \qquad {}_v\mathsf{p}_u \neq \emptyset \quad \text{implies} \quad u \leqq v.$$

The set of all right pseudoprocesses in P over T will be denoted by $\mathsf{Ps}(P, T)$.

2.2. Remark. In accordance with (1.1.4), the domain of a pseudoprocess p in P over T is the set

$$D_\mathsf{p} = \{(x, u) \in P \times T \mid (y, v)\,\mathsf{p}(x, u) \quad \text{for some} \quad (y, v) \in P \times T\}.$$

Given a pseudoprocess p in P over T, we shall denote by I_p the set

$$(2.2.1) \qquad I_\mathsf{p} = \{(x, u) \in D_\mathsf{p} \mid (x, u)\,\mathsf{p}(x, u)\}.$$

Quite illustrative examples of pseudoprocesses may be found e.g. in [5].

The behaviour of many physical systems may be analysed by means of pseudoprocesses. From this point of view the following question is very natural: Given $\mathsf{p} \in \mathsf{Ps}(P, T)$, $(x, u) \in D_{\mathsf{p}}$, what is the set of such $t \in T$ that ${}_t\mathsf{p}_u x \neq \emptyset$? To answer this question, we need first to introduce several further notions.

2.3. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$. The map

$$e : D_{\mathsf{p}} \to R^{\#}\,, \tag{2.3.1}$$

defined by

$$e(x, u) = \sup \{t \in T \mid {}_t\mathsf{p}_u x \neq \emptyset\}\,, \tag{2.3.2}$$

is called the *extent of existence* of the right pseudoprocess p.

A right pseudoprocess p is said to have *local* (*global*) *existence at a point* $(x, u) \in D_{\mathsf{p}}$ iff $(x, u) \in I_{\mathsf{p}}$ and $e(x, u) > u$ ($e(x, u) = \sup T$).

A right pseudoprocess p is said to have *local* (*global*) *existence* iff it has local (global) existence at each point $(x, u) \in D_{\mathsf{p}}$.

A point $(x, u) \in I_{\mathsf{p}}$ is said to be a *start point* of the right pseudoprocess p iff $x_u\mathsf{p}_t = \emptyset$ for each $t \in T \cap \langle -\infty, u)$.

A point $(x, u) \in I_{\mathsf{p}}$ is said to be an *end point* of the right pseudoprocess p iff ${}_t\mathsf{p}_u x = \emptyset$ for each $t \in T \cap (u, +\infty\rangle$.

2.4. Remark. Evidently, if ${}_t\mathsf{p}_u x \neq \emptyset$, then necessarily $u \leqq t \leqq e(x, u)$.

A point $(x, u) \in I_{\mathsf{p}}$ is an end point of a right pseudoprocess p iff $y_t\mathsf{p}_u x$, $t \geqq u$, $t \in T$ implies $(y, t) = (x, u)$, i.e. iff $e(x, u) = u$.

2.5. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$. The map

$$d : D_{\mathsf{p}} \to R^{\#}\,, \tag{2.5.1}$$

defined by

$$d(x, u) = \sup \{w \in R \mid \operatorname{card}({}_t\mathsf{p}_u x) \leqq 1 \text{ for all } t \in T \cap \langle u, w\rangle\}\,, \tag{2.5.2}$$

is called the *extent of unicity* of the right pseudoprocess p.

A right pseudoprocess p is said to have *local* (*global*) *unicity at a point* $(x, u) \in D_{\mathsf{p}}$ iff $d(x, u) > u$ ($d(x, u) = +\infty$).

A right pseudoprocess p is said to have *local* (*global*) *unicity* iff it has local (global) unicity at each point $(x, u) \in D_{\mathsf{p}}$.

2.6. Lemma. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$. *Then the following assertions hold:*

(i) *If* $(x, u) \in D_{\mathsf{p}}$, *then* $u \leqq d(x, u) \leqq +\infty$.
(ii) *If* $d(x, u) < +\infty$, *then to any* $v > d(x, u)$ *there exists* $t \in T$ *such that* $d(x, u) \leqq t < v$ *and* $\operatorname{card}({}_t\mathsf{p}_u x) \geqq 2$; *especially,* $u \leqq d(x, u) \leqq e(x, u) \leqq \sup T$.
(iii) *If* $d(x, u) < +\infty$, *then* $e(x, u) > u$.
(iv) *If* $e(x, u) = u$, *then* $d(x, u) = +\infty$.

2.7. Remark. Let $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$, $\mathsf{p}' \subset \mathsf{p}$. Then evidently

$$(2.7.1) \qquad {}_v\mathsf{p}'_u \subset {}_v\mathsf{p}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad T\,,$$

$$(2.7.2) \qquad D_{\mathsf{p}'} \subset D_{\mathsf{p}}\,, \quad I_{\mathsf{p}'} \subset I_{\mathsf{p}}\,.$$

If ${}_v\mathsf{p}'_u x \neq \emptyset$, then also ${}_v\mathsf{p}_u x \neq \emptyset$ so that the corresponding extents of existence e' and e of the pseudoprocesses p' and p fulfil

$$(2.7.3) \qquad e'(x, u) \leqq e(x, u) \quad \text{for all} \quad (x, u) \in D_{\mathsf{p}'}\,.$$

Hence, if p' has local existence at a point $(x, u) \in I_{\mathsf{p}}$, then also p has local existence at the point (x, u). If a point $(x, u) \in D_{\mathsf{p}'}$ is an end point of p, then also (x, u) is an end point of p'. Finally, if $D_{\mathsf{p}'} = D_{\mathsf{p}}$ and if p' has global existence, then also p has global existence.

If card $({}_t\mathsf{p}_u x) \leqq 1$, then card $({}_t\mathsf{p}'_u x) \leqq 1$ so that the corresponding extents of unicity d' and d of pseudoprocesses p' and p fulfil

$$(2.7.4) \qquad d(x, u) \leqq d'(x, u) \quad \text{for all} \quad (x, u) \in D_{\mathsf{p}'}\,.$$

Hence, if p has local or global unicity at (x, u), then also p' has the same property. If $d'(x, u) < +\infty$, then (2.7.3), (2.7.4) and 2.6. (ii) yield

$$(2.7.5) \qquad d(x, u) \leqq d'(x, u) \leqq e'(x, u) \leqq e(x, u) \quad \text{for all} \quad (x, u) \in D_{\mathsf{p}'}\,.$$

2.8. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$, $s \subset P \times T$. The relation s is called a *solution* of the right pseudoprocess p iff the following three conditions are satisfied:

(i) the domain D_s od s is an interval in T;
(ii) s is a map of D_s into P;
(iii) $s(v)\, {}_v\mathsf{p}_u\, s(u)$ holds for all $u, v \in D_s$, $u \leqq v$.

The set of all solutions of p will be denoted by S_{p}.

2.10. Remark. According to the preceding definition, each $\mathsf{p} \in \mathsf{Ps}(P, T)$ is assigned a set S_{p} of maps from T into P — the set of solutions of p. A question arises whether, given an arbitrary set S of maps from T into P such that for each $s \in S$ its domain D_s is an interval in T, there exists a pseudoprocess p in P over T such that $S \subset S_{\mathsf{p}}$. Let us show that the answer to this question is affirmative.

Given a set P, $T \subset R^*$ and S a set of maps $s : T \to P$ with D_s an interval in T, let us define a relation p^S in the set $P \times T$ as follows:

(2.10.1) $(y, v)\, \mathsf{p}^S(x, u)$ iff there exists $s \in S$ such that $u, v \in D_s$, $u \leqq v$, $s(u) = x$, $s(v) = y$.

Clearly $\mathsf{p}^S \in \mathsf{Ps}(P, T)$ and ${}_v\mathsf{p}^S_u \neq \emptyset$ iff $u \leqq v$ in T and there exists $s \in S$ such that

$$(2.10.2) \qquad u, v \in D_s\,, \quad s(v)\, {}_v\mathsf{p}^S_u\, s(u)\,.$$

Hence

(2.10.3) $$S \subset S_{p^S}.$$

Let now $p \in Ps(P, T)$ be given and let us apply the preceding construction to the sets P, T and S_p. We obtain a right pseudoprocess p^{S_p}. According to (2.10.1), for each $(y, x) \in {}_vp_u^{S_p}$ there exists $s \in S_p$ such that $s(u) = x$, $s(v) = y$, $s(v)\, {}_vp_u\, s(u)$, i.e. $(y, x) \in {}_vp_u$. Hence

(2.10.4) $$p^{S_p} \subset p.$$

The converse icnclusion does not hold in general. (Take, for example, any $p \in Ps(P, T)$ with $D_p - I_p \neq \emptyset$.) This result leads to the following definition.

2.11. Definition. Let $p \in Ps(P, T)$. The pseudoprocess p is said to be *solution complete* iff for each pair $((y, v), (x, u)) \in p$ there exists $s \in S_p$ such that $s(u) = x$, $s(v) = y$.

2.12. Remark. Let $p \in Ps(P, T)$ be solution complete and let $(y, x) \in {}_vp_u$. Then there exists $s \in S_p$ such that

$$s(u) = x, \quad s(v) = y$$

and

$$(y, x) = (s(v), s(u)) \in {}_vp_u^{S_p}.$$

Hence

(2.12.1) $$p \subset p^{S_p}.$$

From (2.12.1) and (2.10.4) we obtain the following assertion.

2.13. Lemma. *Let* $p \in Ps(P, T)$. *Then* p *is solution complete iff* $p = p^{S_p}$.

2.14. Remark. Let $p, p' \in Ps(P, T)$, $p' \subset p$. If $s \in S_{p'}$, then all $u, v \in D_s$, $u \leqq v$ satisfy

$$(s(v), s(u)) \in {}_vp'_u \subset {}_vp_u$$

so that $s \in S_p$. Hence

(2.14.1) $$S_{p'} \subset S_p.$$

3. COMPOSITIVE RIGHT PSEUDOPROCESSES

3.1. In the theory of processes one of the axioms for an abstract process in P over T is the equality

(3.1.1) $${}_vp_u = {}_vp_t \circ {}_tp_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T.$$

This equality is clearly equivalent to the following two inclusions

(3.1.2) $${}_v\mathsf{p}_u \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T$$

and

(3.1.3) $${}_v\mathsf{p}_u \supset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T\,.$$

Now we shall investigate in some detail the right pseudoprocesses satisfying the condition (3.1.2).

3.2. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$. The right pseudoprocess p is said to be *compositive* iff the condition

(RC) $${}_v\mathsf{p}_u \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T$$

is satisfied.

The set of all compositive right pseudoprocesses in P over T will be denoted by $\mathsf{Psc}(P, T)$.

3.3. Remark. Given $\mathsf{p} \in \mathsf{Ps}(P, T)$, $u, v \in T$, then ${}_v\mathsf{p}_u \neq \emptyset$ need not imply ${}_u\mathsf{p}_u \neq \emptyset$ nor ${}_v\mathsf{p}_v \neq \emptyset$. Nonetheless, if p is compositive, then from the condition (RC) and the assumption ${}_v\mathsf{p}_u \neq \emptyset$ one obtains for $t = u$, $t = v$ the inclusions

$$\emptyset \neq {}_v\mathsf{p}_u \subset {}_v\mathsf{p}_u \circ {}_u\mathsf{p}_u\,, \quad \emptyset \neq {}_v\mathsf{p}_u \subset {}_v\mathsf{p}_v \circ {}_v\mathsf{p}_u\,,$$

respectively; hence it follows ${}_u\mathsf{p}_u \neq \emptyset$, ${}_v\mathsf{p}_v \neq \emptyset$, respectively. Thus for $\mathsf{p} \in \mathsf{Psc}(P, T)$ the sets D_p and I_p coincide and may be characterized as follows:

$$D_\mathsf{p} = I_\mathsf{p} = \{(x, u) \in P \times T \mid {}_u\mathsf{p}_u x \neq \emptyset\}\,.$$

3.4. Lemma. *Let* $\mathsf{p} \in \mathsf{Psc}(P, T)$, $(x, u) \in D_\mathsf{p}$. *Then the following assertions hold*:

(i) *If* $u < e(x, u)$, *then* ${}_t\mathsf{p}_u x \neq \emptyset$ *for all* $t \in T \cap \langle u, e(x, u))$.
(ii) If $u \leqq t \leqq v < d(x, u)$ *in* T *and if* $z\,{}_v\mathsf{p}_u x$, $y\,{}_t\mathsf{p}_u x$, *then also* $z\,{}_v\mathsf{p}_t y$.
(iii) *If* $u \leqq v < d(x, u)$ *in* T *and* $y\,{}_v\mathsf{p}_u x$, *then*

(3.4.1) $$d(x, u) \geqq d(y, v)\,, \quad e(x, u) \leqq e(y, v)\,.$$

Proof. The assertion (i) follows easily from (RC).

The assertion (ii) follows from (2.5.2) and (RC). Indeed, $\{z\} = {}_v\mathsf{p}_u x$, $\{y\} = {}_t\mathsf{p}_u x$ so that

$$\{z\} = {}_v\mathsf{p}_u x \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u x = {}_v\mathsf{p}_t y\,,$$

i.e. $z\,{}_v\mathsf{p}_t y$.

We shall prove the assertion (iii). Let $u \leqq v < d(x, u)$, let $y \in P$ be such that $\{y\} = {}_v\mathsf{p}_u x$. Then ($RC$) implies

(3.4.2) $${}_t\mathsf{p}_u x \subset {}_t\mathsf{p}_v \circ {}_v\mathsf{p}_u x = {}_t\mathsf{p}_v y \quad \text{for all} \quad u \leqq v \leqq t \quad \text{in} \quad T\,.$$

Thus

$$\operatorname{card}({}_t\mathsf{p}_u x) \leqq \operatorname{card}({}_t\mathsf{p}_v y) .$$

Hence and from (2.5.2) one easily obtains the first inequality in the assertion (iii). The second inequality follows from the inclusion

$$\{t \in T \mid {}_t\mathsf{p}_u x \neq \emptyset\} \subset \{t \in T \mid {}_t\mathsf{p}_v y \neq \emptyset\} ,$$

which is a direct consequence of 3.4.(i) and 3.4.(ii).

3.5. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$, $(x, u) \in D_\mathsf{p}$, $s \subset P \times T$. The relation s is called a *characteristic solution of* p *through the point* (x, u) iff it satisfies the following two conditions:

(i) $D_s = \{v \in T \mid \operatorname{card}({}_t\mathsf{p}_u x) = 1 \text{ for all } t \in T \cap \langle u, v\rangle\}$;
(ii) $s(v)\, {}_v\mathsf{p}_u x$ holds for all $v \in D_s$.

3.6. Remark. It follows immediately from the condition 3.5.(i) that D_s is an interval in T and that s is a map of D_s into P. Observe that a characteristic solution need not be a solution.

3.7. Lemma. *Let* $\mathsf{p} \in \mathsf{Psc}(P, T)$, $(x, u) \in D_\mathsf{p}$, $u < d(x, u)$, *let* s *be a characteristic solution of* p *through* (x, u). *Then* $s \in S_\mathsf{p}$ *with* D_s *an interval of the form* $\langle u, u')$ *or* $\langle u, u'\rangle$, *where*

$$u' = \min\{e(x, u),\ d(x, u)\} = \begin{cases} e(x, u) & \text{if } d(x, u) = +\infty , \\ d(x, u) & \text{if } d(x, u) < +\infty . \end{cases}$$

Proof follows easily from 3.5, 3.4.(ii) and (3.4.1).

3.8. Theorem. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$. *Then the following assertions hold*:

(i) *If* p *is solution complete, then it is compositive.*
(ii) *If* p *is compositive and has global unicity, then it is solution complete.*

Proof. Ad (i). We have to prove the inclusion

$$(3.8.1) \qquad {}_v\mathsf{p}_u \subset {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T .$$

Let $(y, x) \in {}_v\mathsf{p}_u$. Then there exists $s \in S_\mathsf{p}$ such that $s(u) = x$, $s(v) = y$ and

$$s(v)\, {}_v\mathsf{p}_t\, s(t) , \quad s(t)\, {}_t\mathsf{p}_u\, s(u) \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad D_s .$$

Thus

$$(3.8.2) \qquad s(v)\, {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u\, s(u) \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad D_s .$$

Since $s(u) = x$, $s(v) = y$, it follows from (3.8.2) that $(y, x) \in {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u$. The pair (y, x) was taken arbitrarily from ${}_v\mathsf{p}_u$, hence (3.8.1) holds.

Ad (ii). The assertion follows from 3.7 and 3.5.

3.9. Remark. In [2] Hájek applies some methods of the lattice theory in order to construct least upper bounds and greatest lower bounds of sets of processes. This idea motivates also the following definition.

3.10. Definition. Let p, $\mathsf{p}^\wedge \in \mathsf{Ps}(P, T)$. The pseudoprocess $\mathsf{p}^\wedge$ is called a *lower modification* of the pseudoprocess p iff

(i) $\mathsf{p}^\wedge \subset \mathsf{p}$;
(ii) $\mathsf{p}^\wedge \in \mathsf{Psc}(P, T)$;
(iii) $\mathsf{p}' \in \mathsf{Psc}(P, T)$, $\mathsf{p}' \subset \mathsf{p}$ implies $\mathsf{p}' \subset \mathsf{p}^\wedge$.

3.11. Lemma. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$. *Define* $\mathsf{p}^\sim \in \mathsf{Ps}(P, T)$ *as follows*:

(3.11.1) $$_v\mathsf{P}^\sim_u = \bigcap\{_v\mathsf{P}_t \circ {_t\mathsf{P}_u} \mid u \leqq t \leqq v \text{ in } T\}\,.$$

Then $\mathsf{p}^\sim$ *has the following properties*:

(i) $\mathsf{p}^\sim \subset \mathsf{p}$;
(ii) $\mathsf{p}^\sim = \mathsf{p}$ *iff* $\mathsf{p} \in \mathsf{Psc}(P, T)$;
(iii) $\mathsf{p}' \in \mathsf{Psc}(P, T)$, $\mathsf{p}' \subset \mathsf{p}$ *implies* $\mathsf{p}' \subset \mathsf{p}^\sim$.

Proof. Ad (i). Setting $t = v$ in (3.11.1) one obtains easily

(3.11.2) $$_v\mathsf{P}^\sim_u \subset {_v\mathsf{P}_u} \quad \text{for all} \quad u \leqq v \quad \text{in} \quad T\,,$$

whence $\mathsf{p}^\sim \subset \mathsf{p}$.

Ad (ii). Let $\mathsf{p}^\sim = \mathsf{p}$. Then

$$_v\mathsf{P}^\sim_u = {_v\mathsf{P}_u} = \bigcap\{_v\mathsf{P}_t \circ {_t\mathsf{P}_u} \mid u \leqq t \leqq v \quad \text{in} \quad T\}$$

so that

(3.11.3) $$_v\mathsf{P}_u \subset {_v\mathsf{P}_t} \circ {_t\mathsf{P}_u} \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T\,.$$

Thus p is compositive.

Conversely, if p is compositive, i.e. if (3.11.3) holds, then

$$_v\mathsf{P}_u \subset \bigcap\{_v\mathsf{P}_t \circ {_t\mathsf{P}_u} \mid u \leqq t \leqq v \quad \text{in} \quad T\} = {_v\mathsf{P}^\sim_u}\,.$$

Hence and from (3.11.2) we obtain easily that $\mathsf{p}^\sim = \mathsf{p}$.

Ad (iii). If $\mathsf{p}' \subset \mathsf{p}$, then (2.7.1) implies

$$_v\mathsf{P}'_t \circ {_t\mathsf{P}'_u} \subset {_v\mathsf{P}_t} \circ {_t\mathsf{P}_u} \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T\,.$$

Since p' is compositive, the inclusions

$$_v\mathsf{P}'_u \subset {_v\mathsf{P}'_t} \circ {_t\mathsf{P}'_u} \subset {_v\mathsf{P}_t} \circ {_t\mathsf{P}_u}$$

hold for all $u \leqq t \leqq v$ in T. Hence

$$ {}_v\mathsf{P}'_u \subset \bigcap\{{}_v\mathsf{P}_t \circ {}_t\mathsf{P}_u \mid u \leqq t \leqq v \text{ in } T\} = {}_v\mathsf{P}^{\sim}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad T, $$

which yields $\mathsf{p}' \subset \mathsf{p}^{\sim}$.

3.12. Construction of a lower modification. Let us define, by transfinite induction, pseudoprocesses p_n as follows:

$$ \mathsf{p}_0 = \mathsf{p}, \quad \mathsf{p}_{n+1} = \mathsf{p}^{\sim}_n, \quad \mathsf{p}_\omega = \bigcap_{n<\omega} \mathsf{p}_n, $$

where $\mathsf{p}^{\sim}_n$ denotes the pseudoprocess connected with p_n according to the preceding lemma. The sequence $(\mathsf{p}_n)_{n \geqq 0}$ is then constant starting at least from a certain ordinal, say m; thus $\mathsf{p}^{\sim}_m = \mathsf{p}_m$. Let us show that this p_m is the lower modification of the pseudoprocess p.

Clearly, pseudoprocesses p_n and $\mathsf{p}^{\sim}_n$ for each n satisfy the assumptions of Lemma 3.11. Hence all $\mathsf{p}^{\sim}_n$ satisfy conditions 3.10.(i) and 3.10.(iii) with $\mathsf{p}^{\sim}_n$ instead of $\mathsf{p}^{\wedge}$. Since $\mathsf{p}^{\sim}_m = \mathsf{p}_m$, 3.11.(ii) yields $\mathsf{p}_m \in \mathsf{Psc}(P, T)$ so that p_m satisfies all three conditions of Definition 3.10.

3.13. Remark. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$ and let $\mathsf{p}^{\wedge}$ be its lower modification. Since $\mathsf{p}^{\wedge} \subset \mathsf{p}$, (2.7.2) implies

$$ D_{\mathsf{p}^\wedge} \subset D_{\mathsf{p}}. \tag{3.13.1} $$

The inclusion in (3.13.1) cannot be in general replaced by the equality. However, if p is such that

$$ D_{\mathsf{p}} = I_{\mathsf{p}}, \tag{3.13.2} $$

and $(x, u) \in D_{\mathsf{p}}$ is arbitrary, then according to (3.11.2)

$$ {}_u\mathsf{P}^{\wedge}{}_u x = {}_u\mathsf{P}_u x = \{x\}, $$

hence $(x, u) \in D_{\mathsf{p}}$. Thus $D_{\mathsf{p}} \subset D_{\mathsf{p}^\wedge}$. This together with (3.13.1) gives the assertion

$$ D_{\mathsf{p}} = D_{\mathsf{p}^\wedge} \quad \text{whenever (3.13.2) holds}. \tag{3.13.3} $$

From 3.3 there follows that the equality $D_{\mathsf{p}} = D_{\mathsf{p}^\wedge}$ holds for all $\mathsf{p} \in \mathsf{Psc}(P, T)$.

Now, let I be an arbitrary set and let $\mathsf{p}_i \in \mathsf{Ps}(P, T)$ for all $i \in I$. Set

$$ \mathsf{p} = \bigcap_{i \in I} \mathsf{p}_i . $$

Clearly, $\mathsf{p} \in \mathsf{Ps}(P, T)$. However, if all p_i are compositive, p need not be compositive. Let $\bigwedge \mathsf{p}_i$ denote the lower modification of $\bigcap \mathsf{p}_i$. According to Definition 3.10 it holds $\bigwedge \mathsf{p}_i \in \mathsf{Psc}(P, T)$. From $\bigwedge \mathsf{p}_i \subset \mathsf{p}_i$ for all $i \in I$ one obtains

$$D_{\wedge \mathsf{p}_i} \subset D_{\mathsf{p}_j} \quad \text{for all} \quad j \in I\,, \tag{3.13.4}$$

hence

$$D_{\wedge \mathsf{p}_i} \subset \bigcap_{j \in I} D_{\mathsf{p}_j}\,. \tag{3.13.5}$$

If all p_i satisfy the condition (3.13.2), then the same holds also for their intersection and the equality

$$D_{\wedge \mathsf{p}_i} = \bigcap_{j \in I} D_{\mathsf{p}_j}$$

takes place. The condition (3.13.2) is fulfilled e.g. if all p_i are compositive.

3.14. Theorem. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$ *and let* $\mathsf{p}^\wedge$ *be the lower modification of* p. *Then* $S_{\mathsf{p}} = S_{\mathsf{p}^\wedge}$.

Proof. Let $s \in S_{\mathsf{p}}$. Then D_s is an interval in T and

$$(s(v), s(u)) \in {}_v\mathsf{p}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad D_s\,. \tag{3.14.1}$$

Let us show that

$$(s(v), s(u)) \in {}_v\mathsf{p}^\sim{}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad D_s\,, \tag{3.14.2}$$

where, according to (3.11.1),

$${}_v\mathsf{p}^\sim{}_u = \bigcap \{ {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \mid u \leqq t \leqq v \text{ in } T \}\,. \tag{3.14.3}$$

From (3.14.1) it follows

$$(s(v), s(t)) \in {}_v\mathsf{p}_t\,, \quad (s(t), s(u)) \in {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad D_s\,,$$

hence

$$(s(v), s(u)) \in {}_v\mathsf{p}_t \circ {}_t\mathsf{p}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad D_s\,.$$

This and (3.14.3) imply immediately (3.14.2). From (3.14.2) and from the construction of the lower modification $\mathsf{p}^\wedge$ described in 3.12 it follows easily that

$$(s(v), s(u)) \in {}_v\mathsf{p}^\wedge{}_u \quad \text{for all} \quad u \leqq v \quad \text{in} \quad D_s\,,$$

i.e. $s \in S_{\mathsf{p}^\wedge}$. We have proved the inclusion $S_{\mathsf{p}} \subset S_{\mathsf{p}^\wedge}$. Since $S_{\mathsf{p}^\wedge} \subset S_{\mathsf{p}}$ according to (2.14.1), the theorem is proved.

3.15. Corollary. *Let* $\mathsf{p}_i \in \mathsf{Ps}(P, T)$ *for* i *from an arbitrary index set* I. *Then*

$$S_{\wedge \mathsf{p}_i} = \bigcap_{j \in I} S_{\mathsf{p}_j}\,.$$

3.16. Remark. Let e_i and d_i denote the extent of existence and the extent of unicity, respectively, for $\mathsf{p}_i \in \mathsf{Ps}(P, T)$, $i \in I$, and let e and d denote the extent of existence and the extent of unicity, respectively, of $\mathsf{p} = \bigwedge \mathsf{p}_i$. Then

$$(3.16.1) \qquad e(x, u) \leqq \inf \{e_i(x, u) \mid i \in I\}, \quad d(x, u) \geqq \sup \{d_i(x, u) \mid i \in I\}$$

for all $(x, u) \in D_{\mathsf{p}}$. Hence it follows immediately that each start or end point of some p_i is the point of the same type of p. The converse does not hold. If some p_i has local or global unicity at (x, u), then p has the same property. The analogous assertion for local or global existence is not valid.

4. TRANSITIVE RIGHT PSEUDOPROCESSES

4.1. In 3.1 we decomposed the equality (3.1.1) into two inclusions (3.1.2) and (3.1.3). In the preceding section we have investigated right pseudoprocesses fulfilling the condition (3.1.2). Now we shall investigate the right pseudoprocesses which fulfil the condition (3.1.3) and also those fulfilling both the conditions (3.1.2) and (3.1.3).

4.2. Definition. Let $\mathsf{p} \in \mathsf{Ps}(P, T)$. The right pseudoprocess p is said to be *transitive* iff the condition

$$(RT) \qquad {}_v\mathsf{P}_u \supset {}_v\mathsf{P}_t \circ {}_t\mathsf{P}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T$$

is satisfied.

The right pseudoprocess p is called a *right process in P over T* iff it is compositive and transitive.

The set of all transitive right pseudoprocesses in P over T will be denoted by $\mathsf{Pst}(P, T)$ and the set of all right processes in P over T will be denoted by $\mathsf{P}(P, T)$.

4.3. Lemma. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$. *Then* $\mathsf{p} \in \mathsf{P}(P, T)$ *iff* $\mathsf{p} \circ \mathsf{p} = \mathsf{p}$.

Proof. Indeed, the equality $\mathsf{p} = \mathsf{p} \circ \mathsf{p}$ is equivalent with

$${}_v\mathsf{P}_u = {}_v\mathsf{P}_t \circ {}_t\mathsf{P}_u \quad \text{for all} \quad u \leqq t \leqq v \quad \text{in} \quad T .$$

4.4. Lemma. *Let* $\mathsf{p} \in \mathsf{Ps}(P, T)$, *let J be an interval in T and I an arbitrary set. Then the following assertions hold*:

(i) *If* $s \in S_{\mathsf{p}}$, *then* $s|_J \in S_{\mathsf{p}}$.
(ii) *If* $s_i \in S_{\mathsf{p}}$ *for* $i \in I$ *are such that* $D_{\bigcap s_i}$ *is an interval in T, then* $\bigcap s_i \in S_{\mathsf{p}}$.
(iii) *If* p *is transitive and* $s_i \in S_{\mathsf{p}}$ *for* $i \in I$ *are such that* $D_{s_i} \cap D_{s_j} \neq \emptyset$ *and* $s_i \cup s_j$ *is a map for all* $i, j \in I$, *then* $\bigcup s_i \in S_{\mathsf{p}}$.

4.5. Lemma. *Let* $\mathsf{p} \in \mathsf{Pst}(P, T)$, $u \leqq v$ *in* T, $y_v \mathsf{p}_u x$. *Then the following assertions hold*:

(i) *If* $v = e(x, u)$, *then* (y, v) *is an end point of* p.
(ii) *If* $v \leqq w$ *in* T, $z_w \mathsf{p}_v y$, *then also* $z_w \mathsf{p}_u x$.
(iii) $d(x, u) \leqq d(y, v)$, $e(x, u) \geqq e(y, v)$.
(iv) *If* $\mathsf{p} \in \mathsf{P}(P, T)$, $v < d(x, u)$ *in* T, *then*

$$d(x, u) = d(y, v), \quad e(x, u) = e(y, v).$$

4.6. Example. In the assertions 3.4.(iii) and 4.5.(iv) the local unicity of p at the point (x, u) is assumed. As will be shown, this assumption may not be omitted.

Define the right pseudoprocess p in $P = \{x, y, z, u\}$ over $T = \{0, 1, 2\}$ as follows:

$$
\begin{aligned}
&{}_0\mathsf{p}_0 = \{(x, x)\}, \\
&{}_1\mathsf{p}_0 = \{(y, x), (u, x)\}, \quad {}_1\mathsf{p}_1 = \{(y, y), (u, u)\}, \\
&{}_2\mathsf{p}_0 = \{(z, x)\}, \quad {}_2\mathsf{p}_1 = \{(z, u)\}, \quad {}_2\mathsf{p}_2 = \{(z, z)\}.
\end{aligned}
$$

Clearly $y_1 \mathsf{p}_0 x$, $z_2 \mathsf{p}_0 x$, $2 = e(x, 0) > e(y, 1) = 1$, $0 = d(x, 0) < d(y, 1) = +\infty$.

4.7. Remark. If $\mathsf{p} \in \mathsf{Psc}(P, T)$, it may happen that $y_v \mathsf{p}_u x$, $d(x, u) = +\infty$ and $d(y, v) < +\infty$. Lemma 4.5.(iii) yields that this situation is impossible if p is a process.

4.8. Remark. Let us show what is the advantage of passing to a lower modification of p in the case when p is transitive. From the construction of a lower modification $\mathsf{p}^\wedge$ of p, described in 3.12, one easily obtains that if $\mathsf{p} \in \mathsf{Pst}(P, T)$, then $\mathsf{p}^\wedge \in \mathsf{Pst}(P, T)$. Since $\mathsf{p}^\wedge \in \mathsf{Psc}(P, T)$ holds for each $\mathsf{p} \in \mathsf{Ps}(P, T)$, it is clear that if $\mathsf{p} \in \mathsf{Pst}(P, T)$, then $\mathsf{p}^\wedge \in \mathsf{P}(P, T)$.

If p_i for i from an arbitrary index set are right transitive pseudoprocesses, their intersection is again a transitive right pseudoprocess. If p_i are right processes, their intersection need not be a right process, because the intersection of compositive right pseudoprocesses need not be compositive. However, the lower modification $\bigwedge \mathsf{p}_i$ of the intersection of transitive right pseudoprocesses or right processes is a right process.

5. LOCAL DETERMINACY OF RIGHT PSEUDOPROCESSES

5.1. In this last section we are going to investigate the question, when the knowledge of the local behaviour of a right pseudoprocess enables us to make conclusions concerning the local behaviour of another right pseudoprocess. In this context we shall study also the problem of the local equivalence of right pseudoprocesses.

Given $\mathsf{p} \in \mathsf{Ps}(P, T)$, let us denote

$$L_{\mathsf{p}} = \{(s, u) \in S_{\mathsf{p}} \times T \mid u \in D_s\},$$

where, as usual, D_s denotes the domain of the solution $s \in S_p$, and let $\mathcal{S}_p$ or $\mathcal{E}_p$ stand for the set of all start or end points of p, respectively.

5.2. Definition. Let $p \in Ps(P, T)$, $(x, u) \in D_p$. The right pseudoprocess p is said to have *right* (or *left*) *local existence of solutions at a point* (x, u) iff the following conditions are fulfilled:

(i) $(x, u) \notin \mathcal{E}_p$ (or $(x, u) \notin \mathcal{S}_p$);

(ii) there exist $\varepsilon > 0$ and $s \in S_p$ such that

$$\langle u, u + \varepsilon \rangle \cap T \subset D_s \quad (\text{or } \langle u - \varepsilon, u \rangle \cap T \subset D_s),$$

respectively. The right pseudoprocess p is said to have *right* (or *left*) *local existence of solutions* iff it has this property at each point $(x, u) \in (D_p - \mathcal{E}_p)$, (or $(x, u) \in (D_p - \mathcal{S}_p)$), respectively.

5.3. Definition. Let $p \in Ps(P, T)$, $(x, u) \in D_p$. The right pseudoprocess p is said to have *bilateral local existence of solutions at a point* (x, u) iff it has

(i) right local existence of solutions at the point (x, u) if $(x, u) \in (D_p - \mathcal{E}_p)$;

(ii) left local existence of solutions at the point (x, u) if $(x, u) \in (D_p - \mathcal{S}_p)$.

The right pseudoprocess p is said to have *bilateral local existence of solutions* iff it has this property at each point $(x, u) \in D_p$.

5.4. Remark. Observe that if $p \in Ps(P, T)$ is solution complete then it has bilateral local existence of solutions. If p is compositive and has local unicity at a point (x, u) with $u < d(x, u) < +\infty$, then p has right local existence of solutions at the point (x, u).

5.5. Definition. Let $p, p' \in Ps(P, T)$. The right pseudoprocess p' is said to *determine the local behaviour* of the right pseudoprocess p (which is shortly written as $p' \prec p$) iff the following conditions are fulfilled:

(i) $p' \subset p$;

(ii) there exists a map

(5.5.1) $$k : L_p \to R^*$$

such that $k(s, u) > u$ for $u < \sup D_s$, $k(s, u) = u$ for $u = \max D_s$ and

(5.5.2) $$s|_{\langle u, k(s,u) \rangle} \in S_{p'} .$$

5.6. Definition. Let $p, p' \in Ps(P, T)$. The right pseudoprocess p is said to *determine the bilateral local behaviour* of the right pseudoprocess p (which is shortly written as $\mathsf{p}' \preccurlyeq \mathsf{p}$) iff the following conditions are fulfilled:

(i) $\mathsf{p}' \subset \mathsf{p}$;

(ii) there exist maps

$$k_1, k_2 : L_{\mathsf{p}} \to R^{*'} \tag{5.6.1}$$

such that

$$\begin{aligned} k_1(s, u) < u \quad &\text{for} \quad \inf D_s < u\,, \quad k_1(s, u) = u \quad \text{for} \quad \min D_s = u\,, \\ k_2(s, u) > u \quad &\text{for} \quad \sup D_s > u\,, \quad k_2(s, u) = u \quad \text{for} \quad \max D_s = u \end{aligned}$$

and

$$s\big|_{\langle k_1(s,u), k_2(s,u)\rangle} \in S_{\mathsf{p}'}\,. \tag{5.6.2}$$

5.7. Remark. Clearly the property $\preccurlyeq$ is stronger than $\prec$, i.e. if $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$, $\mathsf{p}' \preccurlyeq \mathsf{p}$, then $\mathsf{p}' \prec \mathsf{p}$. Since many assertions concerning the relations $\prec$ and $\preccurlyeq$ may be formulated simultaneously, we shall do it using the symbol $<$. In these assertions the symbol $<$ has to be replaced either by $\prec$ or by $\preccurlyeq$.

5.8. Lemma. *Let* $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$.

(i) *If* $\mathsf{p}' < \mathsf{p}$, *then* $S_{\mathsf{p}'} \subset S_{\mathsf{p}}$, *and* $D_{\mathsf{p}'} \subset D_{\mathsf{p}}$.

(ii) *If* $\mathsf{p}' \prec \mathsf{p}$, *then* $\mathscr{E}_{\mathsf{p}'} = \mathscr{E}_{\mathsf{p}} \cap D_{\mathsf{p}'}$.

(iii) *If* $\mathsf{p}' \preccurlyeq \mathsf{p}$, *then* $\mathscr{E}_{\mathsf{p}'} = \mathscr{E}_{\mathsf{p}} \cap D_{\mathsf{p}'}$ *and* $\mathscr{S}_{\mathsf{p}'} = \mathscr{S}_{\mathsf{p}} \cap D_{\mathsf{p}'}$.

(iv) *If* $\mathsf{p}' \prec \mathsf{p}$ *and* p *has right or left local existence of solutions, then* $D_{\mathsf{p}'} = D_{\mathsf{p}}$, $\mathscr{E}_{\mathsf{p}'} = \mathscr{E}_{\mathsf{p}}$; *if, in addition,* $\mathsf{p}' \preccurlyeq \mathsf{p}$, *then also* $\mathscr{S}_{\mathsf{p}'} = \mathscr{S}_{\mathsf{p}}$.

5.9. Lemma. *Let* $\mathsf{p}, \mathsf{p}', \mathsf{p}'' \in \mathsf{Ps}(P, T)$, $\mathsf{p}'' \subset \mathsf{p}' \subset \mathsf{p}$. *Then the following assertions hold*:

(i) *If* $\mathsf{p}'' < \mathsf{p}$, *then* $\mathsf{p}' < \mathsf{p}$.

(ii) *If* $\mathsf{p}'' < \mathsf{p}$, *then* $\mathsf{p}'' < \mathsf{p}'$.

(iii) *If* $\mathsf{p}'' < \mathsf{p}'$, $\mathsf{p}' < \mathsf{p}$, *then* $\mathsf{p}'' < \mathsf{p}$.

5.10. Lemma. *Let* $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$, $\mathsf{p} \subset \mathsf{p}'$. *Then* $\mathsf{p} < \mathsf{p}'$ *iff* $\mathsf{p}^\wedge < \mathsf{p}'$.

5.11. Lemma. *Let* $\mathsf{p}, \mathsf{p}', \mathsf{p}'' \in \mathsf{Ps}(P, T)$. *Then the following three assertions are equivalent.*

(i) $\mathsf{p}' < \mathsf{p}$, $\mathsf{p}'' < \mathsf{p}$.

(ii) $\mathsf{p}' \cap \mathsf{p}'' < \mathsf{p}$.

(iii) $\mathsf{p}' \wedge \mathsf{p}'' < \mathsf{p}$.

Proof. Let us prove the equivalence of the assertions (i) and (ii). According to Definitions 5.5 and 5.6 $p' < p$, $p'' < p$ iff $p' \subset p$, $p'' \subset p$ and for each $(s, u) \in L_p$ there exist intervals $I_1, I_2 \subset D_s$ such that

$$(s(t), s(v)) \in {}_tp'_v \quad \text{for all} \quad v \leqq t \quad \text{in} \quad I_1 ,$$
$$(s(t), s(v)) \in {}_tp''_v \quad \text{for all} \quad v \leqq t \quad \text{in} \quad I_2 .$$

This occurs iff $p' \cap p'' \subset p$ and

$$(s(t), s(v)) \in {}_tp'_v \cap {}_tp''_v = {}_t(p' \cap p'')_v \quad \text{for all} \quad v \leqq t \quad \text{in} \quad I_1 \cap I_2 ,$$

i.e. iff $p' \cap p'' < p$.

The equivalence of the assertions (ii) and (iii) follows directly from 5.10.

5.12. Definition. Let $p, p' \in Ps(P, T)$. The right pseudoprocesses p and p' are said to be *locally equivalent* or *bilaterally locally equivalent* (which is shortly written as $p \prec\!\succ p'$ or $p \preccurlyeq\!\succcurlyeq p'$) iff there exists $p'' \in Ps(P, T)$ such that $p'' \prec p$, $p'' \prec p'$ or $p'' \preccurlyeq p$, $p'' \preccurlyeq p'$.

5.13. Remark. Clerly, if $p \preccurlyeq\!\succcurlyeq p'$, then $p \prec\!\succ p'$. In what follows, the symbol $<>$ has to be replaced either by $\prec\!\succ$ or by $\preccurlyeq\!\succcurlyeq$.

5.14. Lemma. *Let* $p, p' \in Ps(P, T)$. *Then the following assertions are equivalent:*

(i) $p <> p'$.

(ii) $p \cap p' < p$, $p \cap p' < p'$.

(iii) $p \wedge p' < p$, $p \wedge p' < p'$.

5.15. Theorem. *The local equivalence and the bilateral local equivalence of right pseudoprocesses in P over T are equivalence relations on* $Ps(P, T)$.

Proof. The relation $<>$ is clearly reflexive and symmetric. So it remains to prove that it is also transitive.

Let $p, p', p'' \in Ps(P, T)$ be such that $p <> p'$, $p' <> p''$. Then 5.14. (ii) yields $p \cap p' < p'$, $p' \cap p'' < p'$, hence $p \cap p' \cap p'' < p'$ according to 5.11. This together with the inclusions

$$p \cap p' \cap p'' \subset p \cap p' \subset p' , \quad p \cap p' \cap p'' \subset p' \cap p'' \subset p'$$

guarantees that the assumptions of Lemma 5.11 are fulfilled. Applying this lemma, one obtains

$$p \cap p' \cap p'' < p \cap p' < p , \quad p \cap p' \cap p'' < p' \cap p'' < p'' ,$$

which means that $p <> p''$.

5.16. Theorem. *Let* $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$ *have right local existence of solutions. Then* $\mathsf{p} \prec\!\succ \mathsf{p}'$ *iff the following conditions are fulfilled*:

(i) $D_{\mathsf{p}} = D_{\mathsf{p}'}$, $\mathscr{E}_{\mathsf{p}} = \mathscr{E}_{\mathsf{p}'}$;

(ii) *there exists a map*

$$r : L_{\mathsf{p}} \to \mathsf{R}^* \tag{5.16.1}$$

such that $r(s, u) > u$ *for* $u < \sup D_s$, $r(s, u) = u$ *for* $u = \max D_s$ *and*

$$s|_{\langle u, r(s,u) \rangle} \in S_{\mathsf{p}'} . \tag{5.16.2}$$

Proof. Let $\mathsf{p} \prec\!\succ \mathsf{p}'$. There exists $\mathsf{p}'' \in \mathsf{Ps}(P, T)$ such that $\mathsf{p}'' \prec \mathsf{p}$, $\mathsf{p}'' \prec \mathsf{p}'$. According to 5.8.(iv) we have $D_{\mathsf{p}} = D_{\mathsf{p}''} = D_{\mathsf{p}'}$, $\mathscr{E}_{\mathsf{p}} = \mathscr{E}_{\mathsf{p}''} = \mathscr{E}_{\mathsf{p}'}$ so that (i) holds. Take any $(s, u) \in L_{\mathsf{p}}$. Since $\mathsf{p}'' \prec \mathsf{p}$, there exists a real $k(s, u)$ such that

$$s|_{\langle u, k(s,u) \rangle} \in S_{\mathsf{p}''} \subset S_{\mathsf{p}'} .$$

Hence the condition (ii) is satisfied with $r = k$.

Now we shall prove the second part of the theorem. Let $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$ satisfy the conditions (i) and (ii). We have to prove that $\mathsf{p} \prec\!\succ \mathsf{p}'$. According to 5.14.(ii) it is sufficient to show that $\mathsf{p} \cap \mathsf{p}' \prec \mathsf{p}$, $\mathsf{p} \cap \mathsf{p}' \prec \mathsf{p}'$.

Clearly $\mathsf{p} \cap \mathsf{p}' \subset \mathsf{p}$, $\mathsf{p} \cap \mathsf{p}' \subset \mathsf{p}'$, hence 5.5.(i) holds.

Take any $(s, u) \in L_{\mathsf{p}}$. Then, according to (5.16.2), there exists $r(s, u)$ such that

$$s|_{\langle u, r(s,u) \rangle} \in S_{\mathsf{p}'} .$$

Since $s \in S_{\mathsf{p}}$, it holds also

$$s|_{\langle u, r(s,u) \rangle} \in S_{\mathsf{p}} .$$

Hence

$$s|_{\langle u, r(s,u) \rangle} \in S_{\mathsf{p}'} \cap S_{\mathsf{p}} = S_{\mathsf{p}' \cap \mathsf{p}} ,$$

which is the condition 5.5.(ii). Thus $\mathsf{p} \prec\!\succ \mathsf{p}'$.

5.17. Example. Let us show that locally equivalent pseudoprocesses may have different start points.

Define a right pseudoprocess p in R over R as follows:

$$y_t \mathsf{p}_u x \quad \text{iff} \quad y = x - u + t \quad \text{for} \quad u \leqq t \quad \text{in} \quad R , \quad x, y \in R .$$

Define a right pseudoprocess p' in R over R as follows:

$$y_t \mathsf{p}_u x \quad \text{iff} \quad y = x - u + t$$

for $u \leqq t$ in R and $x, y \in (-\infty, 0)$ or $x, y \in \langle 0, +\infty)$. Clearly $D_{\mathsf{p}} = D_{\mathsf{p}'}$. For any $(s, u) \in L_{\mathsf{p}}$ there exists a real $r(s, u)$ such that $r(s, u) \in (u, +\infty)$ if $s(u) \geqq 0$, $r(s, u) \in$

$\in (u, u - s(u))$ if $s(u) < 0$ and it holds

$$s|_{\langle u, r(s,u)\rangle} \in S_{p'}\,.$$

Finally, neither p nor p′ has an end point, hence p $\prec\!\!\succ$ p′ according to Theorem 5.16. However, p has no start point, while each point $(0, u)$ with $u \in R$ is a start point of p′.

5.18. Theorem. *Let* p, p′ $\in$ Psc(P, T) *have local unicity. Then* p $\prec\!\!\succ$ p′ *iff there exists a map*

$$h : D = D_p \cup D_{p'} \to R^* \tag{5.18.1}$$

such that

$$h(x, u) > u \quad \textit{for all} \quad (x, u) \in D \tag{5.18.2}$$

and

$${}_t\mathsf{P}_u x = {}_t\mathsf{P}'_u x \quad \textit{for all} \quad u \leqq t \leqq h(x, u) \quad \textit{in} \quad T\,. \tag{5.18.3}$$

Proof. Let p $\prec\!\!\succ$ p′ so that the conditions (i) and (ii) of Theorem 5.16 are satisfied. Denote $D = D_p = D_{p'}$ and construct the map h from (5.18.1) having the properties (5.18.2) and (5.18.3).

Take $(x, u) \in D$. If a point (x, u) is an end point of p, then it is also an end point of p′ and Definition 2.3 yields

$${}_t\mathsf{P}_u x = {}_t\mathsf{P}'_u x = \emptyset \quad \text{for all} \quad t > u \quad \text{in} \quad T\,.$$

In this case we may define $h(x, u) > u$ arbitrarily. If the point (x, u) is not an end point of p, then p has local existence at (x, u). Since p is compositive and has local unicity at (x, u), there exists a solution $s \in S_p$ with a nondegenerate D_s and satisfying $s(u) = x$. From 5.18.(ii) we have a real $r(s, u)$ such that

$$s|_{\langle u, r(s,u)\rangle} \in S_{p'}\,,$$

whence the equality (5.18.3) follows with $h(x, u) = r(s, u)$.

Suppose now that there exists a map h from (5.18.1) having the properties (5.18.2) and (5.18.3). Then one verifies easily that the condition (5.18.3) together with 3.3, 2.3, 3.5 and 3.7 yields the conditions 5.16.(i) and 5.16.(ii), hence p $\prec\!\!\succ$ p′.

5.19. Remark. Example 5.17 shows that locally equivalent right pseudoprocesses need not coincide. A similar situation described in terms of local semi-dynamical systems may be found in [3], chap. III, items 2.12 and 2.13 and in [7]. As will be shown in Theorem 21, if local equivalence is replaced by bilateral local equivalence, we are able to obtain a more definite result. First we shall formulate the counterpart of Theorem 5.16 for bilaterally locally equivalent right pseudoprocesses.

5.20. Theorem. *Let* $\mathsf{p}, \mathsf{p}' \in \mathsf{Ps}(P, T)$ *have bilateral local existence of solutions. Then* $\mathsf{p} \lesseqgtr \mathsf{p}'$ *iff the following conditions are fulfilled*:

(i) $D_\mathsf{p} = D_{\mathsf{p}'}$, $\mathscr{E}_\mathsf{p} = \mathscr{E}_{\mathsf{p}'}$, $\mathscr{S}_\mathsf{p} = \mathscr{S}_{\mathsf{p}'}$;
(ii) *there exist maps*

(5.20.1) $$r_1, r_2 : L_\mathsf{p} \to \mathsf{R}^\#$$

such that

$$r_1(s, u) < u \quad \textit{for} \quad \inf D_s < u\,, \quad r_1(s, u) = u \quad \textit{for} \quad \min D_s = u\,,$$
$$r_2(s, u) > u \quad \textit{for} \quad \sup D_s > u\,, \quad r_2(s, u) = u \quad \textit{for} \quad \max D_s = u$$

and

(5.20.2) $$s|_{\langle r_1(s,u), r_2(s,u)\rangle} \in S_{\mathsf{p}'}\,.$$

Proof is an easy modification of the proof of Theorem 5.16.

5.21. Theorem. *Let* T *be a closed subset of* R *and let* $\mathsf{p}, \mathsf{p}' \in \mathsf{P}(P, T)$ *be solution complete processes. Then* $\mathsf{p} \lesseqgtr \mathsf{p}'$ *iff* $\mathsf{p} = \mathsf{p}'$.

Proof. If $\mathsf{p} = \mathsf{p}'$, then evidently $\mathsf{p} \lesseqgtr \mathsf{p}'$. So the only nontrivial part of the proof is that $\mathsf{p} \lesseqgtr \mathsf{p}'$ implies $\mathsf{p} = \mathsf{p}'$.

Let $\mathsf{p} \lesseqgtr \mathsf{p}'$. We shall prove that $\mathsf{p} \subset \mathsf{p}'$.

Take arbitrary $((y, v), (x, u)) \in \mathsf{p}$. Since p and p' are solution complete, they have bilateral local existence of solutions so that we can use Theorem 5.20. If $v = u$, then necessarily $y = x$ and using the equality $D_\mathsf{p} = D_{\mathsf{p}'}$ from 5.20.(i) one obtains $(x, u) \in D_{\mathsf{p}'}$, so that

(5.21.1) $$((y, v), (x, u)) \in \mathsf{p}'\,.$$

Thus, in what follows it will be supposed that $v > u$.

First, let us suppose that $(x, u) \notin \mathscr{S}_\mathsf{p}$ and $(y, v) \notin \mathscr{E}_\mathsf{p}$. Then there exist points (x', u'), (y', v') such that $u' < u < v < v'$ and

$$((x, u), (x', u')) \in \mathsf{p}\,, \quad ((y', v'), (y, v)) \in \mathsf{p}\,.$$

Since p is solution complete, there exists a solution $s \in S_\mathsf{p}$ such that

$$s(u') = x'\,, \quad s(u) = x\,, \quad s(v) = y\,, \quad s(v') = y'\,.$$

Clearly $\langle u, v\rangle \subset (u', v') \subset D_s$. The assumption $\mathsf{p} \lesseqgtr \mathsf{p}'$ and Lemma 5.14.(ii) imply $\mathsf{p} \cap \mathsf{p}' \preccurlyeq \mathsf{p}$, $\mathsf{p} \cap \mathsf{p}' \preccurlyeq \mathsf{p}'$ so that there exist maps k_1, k_2 from (5.6.1) such that

$$s|_{\langle k_1(s,t), k_2(s,t)\rangle} \in S_{\mathsf{p}\cap\mathsf{p}'} \subset S_{\mathsf{p}'} \quad \text{for each} \quad t \in \langle u', v'\rangle \quad \text{in} \quad T\,.$$

Thus for each $t \in \langle u', v'\rangle \cap T$ there holds

$$s_t = s|_{(k_1(s,t), k_2(s,t))} \in S_{\mathsf{p}'}\,.$$

The set $\langle u, v\rangle \cap T$, being an intersection of the compact set $\langle u, v\rangle$ and the closed set T is compact and

$$\langle u, v\rangle \cap T \subset \bigcup\{(k_1(s, t), k_2(s, t)) \mid t \in \langle u, v\rangle \cap T\}.$$

Thus there exists a finite number of $t_1, t_2, \ldots, t_n$ such that

$$\langle u, v\rangle \cap T \subset \bigcup_{i=1}^{n} (k_1(s, t_i), k_2(s, t_i)).$$

Since $s_{t_i} \subset s$ for each $i = 1, 2, \ldots, n$, it is also

$$s' = \bigcup_{i=1}^{n} s_{t_i} \subset s$$

with

$$\langle u, v\rangle \cap T \subset D_{s'} \subset \bigcup_{i=1}^{n} (k_1(s, t_i), k_2(s, t_i)).$$

Certainly $D_{s'}$ is an interval in T containing the set $\langle u, v\rangle \cap T$. From $s_{t_i} \in S_{p'}$ and from Lemma 4.4.(iii) one obtains $s' \in S_{p'}$. Hence and from $s' \subset s$, $u, v \in D_{s'}$ we conclude

$$s'(u) = s(u) = x\,, \quad s'(v) = s(v) = y\,, \quad s'(v)\, {}_v\mathrm{p}'_u\, s'(u)$$

so that (5.21.1) holds.

Now, let us suppose that $(x, u) \in \mathscr{S}_\mathrm{p}$. To each $s \in S_\mathrm{p}$ with $s(u) = x$, $s(v) = y$ there exists a real $k_2(s, u) > u$ such that

$$s|_{\langle u, k_2(s,u)\rangle} \in S_{\mathrm{p}\cap\mathrm{p}'} \subset S_{\mathrm{p}'}\,.$$

Denote

$$w = k_2(s, u)\,, \quad s'' = s|_{\langle u, w\rangle}\,, \quad z = s''(w)\,.$$

Clearly $((y, v), (z, w)) \in \mathrm{p}'$, $(z, w) \notin (\mathscr{S}_\mathrm{p} \cup \mathscr{E}_\mathrm{p})$ so that according to the first part of the proof $((y, v), (z, w)) \in \mathrm{p}'$. Since $((z, w), (x, u)) \in \mathrm{p}'$ as well and p' is transitive, one has (5.21.1) also in this case.

Finally, let us suppose that $(y, v) \in \mathscr{E}_\mathrm{p}$. Corresponding to $s \in S_\mathrm{p}$ with $s(u) = x$, $s(v) = y$ there exists a real $k_1(s, v) < v$ such that

$$s|_{\langle k_1(s,v), v\rangle} \in S_{\mathrm{p}\cap\mathrm{p}'} \subset S_{\mathrm{p}'}\,.$$

Now applying the same argument as in the preceding part of the proof one obtains again (5.21.1).

Since the pair $((y, v), (x, u))$ was taken arbitrarily in p, we have proved the inclusion $\mathrm{p} \subset \mathrm{p}'$.

The assumptions concerning p and p' are the same. So interchanging the role of p and p' in the above reasoning we obtain the opposite inclusion $\mathrm{p}' \subset \mathrm{p}$. Thus $\mathrm{p} = \mathrm{p}'$, which completes the proof.