Časopis pro pěstování matematiky, roč. 103 (1978), Praha

THE INSERTION OF REGULAR SETS IN POTENTIAL THEORY

EVA ČERMÁKOVÁ, Praha

(Received October 29, 1976)

Introduction. In 1924, N. WIENER [8] proposed a new construction of the generalized solution of the Dirichlet problem for the Laplace equation. His method essentially uses the following fact: Any couple (K, U) consisting of a compact set K and an open set U with $K \subset U$ is admissible in the sense that there is a set V regular for the Dirichlet problem such that

$$K \subset V \subset \overline{V} \subset U .$$

It is known that each couple (K, U) is also admissible for a wide class of more general second order elliptic partial differential equations than the Laplace equation. In fact, this follows from a result of R.-M. HERVÉ [4] (Proposition 7.1) established in the context of Brelot harmonic spaces. A related question in the same context is also investigated in [6]. On the other hand, a similar result is no longer valid e.g. for the heat equation as observed by H. BAUER in [1], p. 147. Consequently, the original Wiener's procedure is not directly applicable. (Note that the Wiener type solution has recently been investigated in [7] in the frame work of the axiomatic potential theory.)

The aim of this paper is to study in terms of Bauer's axiomatics necessary and sufficient conditions guaranteeing that a couple (K, U) is admissible. To this end, a special hull $r(K)$ of K is introduced in a suitable way so that the main result reads then as follows: The couple (K, U) is admissible, if and only if $r(K) \subset U$. For the case of the heat equation, several characterizations of $r(K)$ in terms of absorbent sets and balayage are given.

1. Terminology and notation. In what follows, X will denote a strong harmonic space in the sense of H. Bauer's axiomatics. For all notions we refer to [1]. For any set M we shall denote by M^*, $\operatorname{int} M$ and $\overline{M}$ its boundary, interior and closure, respectively.

Let U be an open subset of X and K a compact subset of U. The couple (K, U) is called *admissible* if there exists a regular set W such that $K \subset W \subset \overline{W} \subset U$. For a compact set $K \subset X$, we put

$$r(K) = \bigcap\{V;\ K \subset V \subset X;\ V \text{ regular}\}.$$

If there is no regular set V such that $K \subset V$, put $r(K) = X$.

2. Lemma. *If $r(K) \neq X$, then*

$$r(K) = \bigcap\{\overline{V};\ K \subset V \subset X;\ V \textit{ regular}\};$$

in particular, $r(K)$ is compact.

Proof. According to Theorem 4.3.5 of [1] to each regular set W such that $K \subset W$, there exists a regular set W_0 such that $K \subset W_0 \subset \overline{W}_0 \subset W$.

3. Theorem. *The following statements are equivalent:*

(i) *a couple (K, U) is admissible;*

(ii) $r(K) \neq X$, $r(K) \subset U$.

Proof. Implication (i) $\Rightarrow$ (ii) is obvious. Assume (ii) and let W be a regular set such that $K \subset W$. We can limit ourselves to the case $\overline{W} \cap (X \setminus U) \neq \emptyset$. Then $\overline{W} \cap (X \setminus U)$ is compact and $r(K) \cap (\overline{W} \cap (X \setminus U)) = \emptyset$, i.e. $[\overline{W} \cap (X \setminus U)] \subset [X \setminus \bigcap\{\overline{V}; K \subset V, V \text{reg.}\}]$, thus

$$\overline{W} \cap (X \setminus U) \subset \bigcup_{\substack{V \text{reg},\\ K \subset V}} (X \setminus \overline{V}).$$

We can therefore choose regular sets $V_1, \ldots, V_n$ such that

$$\overline{W} \cap (X \setminus U) \subset [X \setminus \bigcap_{i=1}^{n} \overline{V}_i].$$

By Corollary 4.2.7 of [1], $\bigcap_{i=1}^{n} V_i$ is a regular set. Obviously,

$$K \subset \bigcap_{i=1}^{n} V_i$$

and thus applying Theorem 4.3.5 of [1] we can find a regular set V_0,

$$K \subset V_0 \subset \overline{V}_0 \subset \bigcap_{i=1}^{n} V_i.$$

Put $W_0 = V_0 \cap W$. Then $K \subset W_0$, W_0 is (according to Corollary 4.2.7 of [1] again) regular. Moreover, $\overline{W}_0 \subset U$.

4. Notation. *For $E \subset X$, let $A(E, X)$ be the smallest absorbent set in X containing E. We shall write $A(x, X)$ instead of $A(\{x\}, X)$.*

5. Lemma. *The components of an absorbent set are absorbent sets.*

Proof. For S connected, $A(S, X)$ is always connected. (See Exercise 6.1.2 in [3].) Let B be a component of A. Then $A(B, X)$ is a connected absorbent set containing B. Consequently, $B = A(B, X)$ and B is absorbent.

In what follows, X will denote *the harmonic space corresponding to the heat equation on a Euclidean space* R^{n+1} ($n \geqq 1$) (see [1], Standard-Beispiel 2, p. 20).

6. Notation. Given a compact set $K \subset X$, the *parabolic hull* M_K of K is the union of K and the set of all $x \in X \setminus K$ for which $A(x, X \setminus K)$ is relatively compact. Denote by T_K the union of K and the set of all $x \in X \setminus K$ for which there exists no absorbent set B in X such that $\emptyset \neq B \subset A(x, X \setminus K)$.

Further put $L_K = \{x \in X;\ R_1^K(x) = 1\}$.

7. Theorem. *For a compact subset* $K \subset X$,

$$r(K) = M_K = T_K = L_K\,.$$

Thus, together with Theorem 3 we obtained a characterization of admissible couples (K, U) in terms of the parabolic hull of K.

The proof of this theorem will be divided into the following steps.

8. Proposition. *Let Y be an open subset of X and A a closed set in Y. Then the following assertions are equivalent:*

(i) *The set A is absorbent in the harmonic space Y.*

(ii) *For each $x \in A$ there exists a neighborhood U_x and an absorbent set B in X such that $U_x \cap A = U_x \cap B$.*

Proof. Suppose (i). For $x \in \operatorname{int} A$, choose a neighborhood U_x of x such that $U_x \subset A$, and put $B = X$. If $x \in Y$ is a boundary point of A, then we choose $a > 0$ in such a way that the set

$$U_x = \{y \in R^{n+1};\ \sum_{i=1}^{n} (y_i - x_i)^2 - (a + x_{n+1} - y_{n+1})^2 < 0\,;$$
$$x_{n+1} - a < y_{n+1} < x_{n+1} + a\}$$

is contained in Y. (The sets of this form will be called standard cones. Recall that each standard cone is a regular set – see [1], p. 21). For each $y \in U_x \cap A(x, X)$, $y \neq x$, there is a standard cone S such that $x \in S \subset \bar{S} \subset U_x$, $y \in S^*$. Then $y \in \operatorname{spt} \mu_x^S$, where μ_x^S denotes the harmonic measure corresponding to x and the regular set S (see [1], p. 21). Obviously, $\operatorname{spt} \mu_x^S \subset A$ and hence

$$U_x \cap A(x, X) \subset U_x \cap A\,.$$

Suppose now that there exists $z \in (U_x \cap A) \setminus A(x, X)$. The supports of harmonic measures μ_z^V corresponding to regular sets V, $V \subset U_x$ (consider e.g. standard cones) for which $z \in V$, cover the set $[A(z, X) \cap U_x] \setminus \{z\}$. Thus

$$x \in \operatorname{int}\left[U_x \cap A(z, X)\right] \subset U_x \cap A\,,$$

which yields a contradiction with the assumption that x is a boundary point of A. So we obtain $U_x \cap A(x, X) = U_x \cap A$ and we can put $B = A(x, X)$.

Now suppose (ii). By [2] absorbent sets in X are exactly those which are closed and finely open. It follows that there is a fine neighborhood V_x of x, contained in B. Since $U_x \cap V_x$ is a fine neighborhood of x contained in A, A is finely open, and (using [2] again) A is an absorbent set in Y.

9. Corollary. *Let Y be an open subset of X. For each component Q of the boundary of an absorbent set in Y there exists $c \in R$ such that $Q \subset \{x \in X;\ x_{n+1} = c\}$.*

10. Lemma. *For a compact $K \subset X$, $M_K \subset r(K)$.*

Proof. Assume that $K \neq \emptyset$ and choose $x^0 \in M_K \setminus K$. The standard cones are regular, hence $r(K) \neq X$. Suppose that there is a regular neighborhood V of K, such that $x^0 \notin V$. Putting

$$L = \{x \in X;\ x_i = x_i^0 \text{ for all } 1 \leqq i \leqq n,\ x_{n+1} \leqq x_{n+1}^0\}\,,$$

there exists $y \in L$ such that

$$y_{n+1} = \sup \{x_{n+1};\ x \in L \setminus A(x^0, X \setminus K)\}\,.$$

According to Proposition 8, $y_{n+1} < x_{n+1}^0$. Denote

$$L_0 = \{x \in L;\ x_{n+1} > y_{n+1}\}\,.$$

By Proposition 8, $y \notin A(x^0, X \setminus K)$. Simultaneously $y \in \overline{A(x^0, X \setminus K)}$ and hence $y \in K$. It follows $L_0 \cap V^* \neq \emptyset$ and using the fact that $L_0 \subset A(x^0, X \setminus K)$, we have

$$\emptyset \neq L_0 \cap V^* \subset A(x^0, X \setminus K) \cap V^*\,.$$

Let $y^0 \in A(x^0, X \setminus K)$ be chosen such that

$$y_{n+1}^0 = \min \{x_{n+1};\ x \in A(x^0, X \setminus K) \cap V^*\}\,.$$

First, consider the case when y^0 is a boundary point of $A(x^0, X \setminus K)$ relatively to the set $X \setminus K$. Using Proposition 8, there is a neighborhood U_{y^0} of y^0 such that

$$U_{y^0} \cap (X \setminus V) \subset \{x \in X;\ y_{n+1}^0 \leqq x_{n+1}\}\,.$$

It follows (cf. [1], Theorem 4.3.1. and p. 108) that y^0 is an irregular boundary point of V, which is a contradiction. Using a similar argument, y^0 cannot be in the interior of $A(x^0, X \setminus K)$. Thus, $M_K \setminus K \subset V$ and since V is an arbitrary regular set containing K, we have $M_K \setminus K \subset r(K)$. Obviously, $K \subset r(K)$.

The proof of the inclusion $r(K) \subset M_K$ will be more complicated.

11. Lemma. *For a compact set K in X, the set $\{x \in X;\ \hat{R}_1^K(x) = 1\}$ is bounded.*

Proof. Obviously it is sufficient to prove that $\{x \in X;\ \hat{R}_1^K(x) = 1\}$ is bounded for

$$K = \{x \in X;\ |x_i| \leqq a_i,\ i = 1, \ldots, n + 1\} \quad (a_i \geqq 0)\,.$$

(a) If $y \in X$ is such that $y_{n+1} < -a_{n+1}$, then

$$\hat{R}_1^K(y) = R_1^K(y) = 0\,.$$

We can take the superharmonic function (see [1], p. 34.)

$$u = \begin{cases} 0 & \text{on} \quad A(y, X)\,, \\ 1 & \text{on} \quad X \setminus A(y, X)\,. \end{cases}$$

(b) If $y \in X$ is such that $|y_i| \leqq a_i$ for $i = 1, \ldots, n$, $y_{n+1} > a_{n+1}$ consider the set

$$D = \{x \in X \setminus K;\ |x_i| < a_i + 1 \quad \text{for} \quad i = 1, \ldots, n,\ |x_{n+1}| < |y_{n+1}| + 1\}\,.$$

Obviously, $y \in D$. Choose $z \in D$, $z_i = -a_i - \frac{1}{2}$. Using (a), $\hat{R}_1^K(z) = 0$. Applying the maximum principle for the heat equation (e.g. Theorem 2.3 in [5] – note that $\hat{R}_1^K$ is a harmonic function on D, $\hat{R}_1^K \leqq 1$) we obtain $\hat{R}_1^K(y) < 1$.

(c) In the case that for $y \in X$, $y_{n+1} \geqq -a_{n+1}$ and there exists i $(i = 1, \ldots, n)$ such that $|y_i| > a_i$ we can proceed analogously.

12. Notation. For a compact set $\emptyset \neq K \subset X$, we define a sequence $\{K_n\}$:

$$K_n = \{x \in X;\ \operatorname{dist}(x, K) \leqq 1/n\}\,.$$

13. Lemma. $L_K = M_K$.

Proof. Let $K \neq \emptyset$ and consider $x^0 \in X \setminus M_K$. The set $A(x^0, X \setminus K)$ is unbounded, thus using the preceding lemma and Proposition 8, there is $y \in \operatorname{int} A(x^0, X \setminus K)$ such that $R_1^K(y) < 1$. The function $1 - R_1^K$ is harmonic on $X \setminus K$. By the Harnack inequality (see [1], Theorem 1.4.4) applied to $X \setminus K$ and to the Dirac measure at x^0 there is $\alpha \geqq 0$ such that

$$0 < 1 - R_1^K(y) \leqq \alpha(1 - R_1^K(x^0))\,.$$

It follows that $R_1^K(x^0) < 1$.

Thus we proved that $L_K \subset M_K$. Let $y^0 \in M_K \setminus K$, choose n_0 such that $y^0 \notin K_{n_0}$. Let $n \geqq n_0$ be a natural number. According to Proposition 8 we obtain that the "parabolic boundary" (see [5] Chap. 3) of int $A(y^0, X \setminus K)$ in X is contained in K. Using the fact that $\hat{R}_1^{K_n}(y) = 1$ for all $y \in K$ together with the minimum principle for superharmonic functions for the heat equation (see Theorem 2.1 in [5]), we have

$$\inf\{\hat{R}_1^{K_n}(y);\ y \in \operatorname{int} A(y^0, X \setminus K)\} = 1\,.$$

Since $y^0 \notin K_n$, $\hat{R}_1^{K_n}$ is continuous at y^0 (compare with Corollary 2.3.5 in [1]) and $\hat{R}_1^{K_n}(y^0) = R_1^{K_n}(y^0) = 1$. Now, applying the assertion of Appendix 3.2.1 of [1] we have

$$R_1^K = \inf_{n \in N} R_1^{K_n}\,,$$

and hence $R_1^K(y^0) = 1$ (note that $K_n \supset K_{n_0}$ for $n < n_0$ and $R_1^{K_n} \geqq R_1^{K_{n_0}}$). This means $y^0 \in L_K$. Obviously, $K \subset L_K$.

14. Remark. In the course of the preceding proof we used the equality

$$R_1^K = \inf_{n \in N} R_1^{K_n}\,.$$

It is an easy consequence that

$$\{x \in X;\ R_1^K(x) = 1\} = \bigcap_{n=1}^{\infty} \{x \in X;\ R_1^{K_n}(x) = 1\}\,.$$

Obviously, $\{x \in X;\ \hat{R}_1^K(x) = 1\} \cup K = \{x \in X;\ R_1^K(x) = 1\}$, so that

$$\bigcap_{n=1}^{\infty} \{x \in X;\ R_1^{K_n}(x) = 1\} = \bigcap_{n=1}^{\infty} \{x \in X;\ \hat{R}_1^{K_n}(x) = 1\}\,.$$

15. Lemma. *For a compact $K \subset X$, $r(K) \subset M_K$.*

Proof. Assume that $K \neq \emptyset$. Consider $x^0 \notin M_K$. Using Lemma 13 and the preceding remark, there exists a natural number n such that $\hat{R}_1^{K_m}(x^0) < 1$ for all $m \geqq n$. Simultaneously,

$$\inf_{x \in M_K} \hat{R}_1^{K_m}(x) = 1\,.$$

The set M_K is a closed subset of the compact set $r(K)$. Hence, using Proposition 3.1.2 of [3] there is a fundamental system of regular neighborhoods of M_K not containing the point x^0. Thus, $x^0 \notin r(K)$.

16. Lemma. $T_K = M_K$.

Proof. Suppose first that $x \in M_K \setminus T_K$. If B is an absorbent set in X such that $B \subset A(x, X \setminus K)$, then B is a compact absorbent set and hence (see [1], p. 31) must be empty. It follows that $M_K \subset T_K$. Suppose now that the set $A(x, X \setminus K)$ is unbounded. Let $D \supset K$ be an $(n + 1)$-dimensional cube in X such that its faces are