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SEVERAL THEOREMS CONCERNING EXTENSIONS OF MEROMORPHIC AND CONFORMAL MAPPINGS

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The main goal of the present paper is the proof of certain theorems concerning extensions of meromorphic and conformal mappings which are stronger than the well known ones (cf. [1], [2], [3], [5], [6], [7]). We prove the existence of extensions across more general parts V of the boundary of the definition domain of the corresponding mapping, instead of holomorphic functions we consider the meromorphic ones. While, as a rule, the results concern only local conformness of the extension at points of the corresponding part V of the boundary, we establish, among others, sufficient conditions for conformness on a region containing the whole V.

As for definitions, conventions, and notation we refer the reader to [8]. In addition we shall use the following definitions and notation:

$\mathbf{E}_1$ will stand for *the set of all finite real numbers*. Further, we put $^*\mathbf{E}_1 = \mathbf{E}_1 \cup \cup \{\infty\}$. By a *real number* we understand any number $z \in {}^*\mathbf{E}_1$. *The open upper* (*lower*) *half-plane* will be denoted by $\mathbf{E}^+(\mathbf{E}^-)$.

1. Definition 1. Let Ω be a region and let $V \subset \partial\Omega$. We say that V is a *free part* of $\partial\Omega$, iff there is a one-one continuous mapping λ of an interval (α, β) (where $-\infty \leqq \leqq \alpha < \beta \leqq +\infty$) onto V such that for each $t \in (\alpha, \beta)$ there are points $t' \in (\alpha, t)$, $t'' \in (t, \beta)$ and a Jordan region G such that

(1) $\lambda \mid \langle t', t'' \rangle$ is a cut in G;

(2) one component of $G - \lambda((t', t''))$ is contained in Ω, the other one in $\mathbf{S} - \bar{\Omega}$.

Remark 1. If V is a free part of $\partial\Omega$, then each one-one continuous mapping λ of (α, β) onto V satisfies the above mentioned conditions.

Notation. For each continuous mapping $\lambda : (\alpha, \beta) \to \mathbf{S}$ denote

(3_1) $(\lambda) = \lambda((\alpha, \beta))$,

(3_2) $\mathscr{P}(\lambda) = \{z \in \mathbf{S};$ there are $t_n \in (\alpha, \beta)$ with $t_n \to \alpha, \lambda(t_n) \to z\}$,

(3_3) $\mathscr{K}(\lambda) = \{z \in \mathbf{S};$ there are $t_n \in (\alpha, \beta)$ with $t_n \to \beta, \lambda(t_n) \to z\}$.

Remark 2. Obviously, we have

$$(4) \qquad \mathscr{P}(\lambda) = \bigcap_{n=1}^{\infty} \overline{\lambda((\alpha, \alpha_n))} = \operatorname{ls} \lambda((\alpha, \alpha_n))$$

for any decreasing sequence of points $\alpha_n \in (\alpha, \beta)$, $\alpha_n \to \alpha$.

This implies $\mathscr{P}(\lambda)$ is a non-empty continuum. The equality $\mathscr{P}(\lambda) = \{a\}$ (where $a \in \mathbf{S}$) holds iff the limit $\lambda(\alpha+)$ exists and equals a.

Similarly for $\mathscr{K}(\lambda)$.

Lemma 1. *Let Ω be a region, V a free part of $\partial\Omega$. Then the following two assertions hold*:

(5) *For each $z \in V$ and for each sequence of points $z_n \in \Omega$ with $z_n \to z$ there is a curve φ from the point z into Ω such that $z_n \in \langle\varphi\rangle$ for all n.*

(6) *For each $z \in V$ there is one and only one bundle $\mathscr{S}_z \in \mathfrak{S}(\Omega)$ with $o(\mathscr{S}_z) = z$.*

Proof. Let λ be the same as in Definition 1. If $z \in V$, then there is a $t \in (\alpha, \beta)$ such that $\lambda(t) = z$. Let G be a Jordan region satisfying (1) and (2).

If $z_n \in \Omega$, $z_n \to z$, then there is an n_0 such that $z_n \in G$ for all $n > n_0$. Obviously, for the unit circle $\boldsymbol{U}$ the following assertion holds:

(7) If $w_n \in \boldsymbol{U}$, $w_n \to w \in \partial\boldsymbol{U}$, then there is a curve ψ from the point w into $\boldsymbol{U}$ such that $w_n \in \langle\psi\rangle$ for all n.

By a well known theorem (see [4]), a homeomorphism of $\overline{\boldsymbol{G}}$ onto $\overline{\boldsymbol{U}}$ exists. This, obviously, implies that an assertion similar to (7) holds for the region G. Hence there is a curve $\varphi^* : \langle\alpha, \beta\rangle \to \mathbf{S}$ from z into G such that $z_n \in \langle\varphi^*\rangle$ for each $n > n_0$. As Ω is a region, there is an extension $\varphi : \langle\alpha, \gamma\rangle \to \mathbf{S}$ of φ^* with $(\varphi\rangle \subset \Omega$ and $z_n \in \langle\varphi\rangle$ for all n. This proves (5).

Obviously,

(8) if $w \in \partial\boldsymbol{U}$, then there is one and only one bundle $\mathscr{S} \in \mathfrak{S}(\boldsymbol{U})$ with $o(\mathscr{S}) = w$.

Consequently, an analogous assertion holds for each Jordan region. Since for each curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ from z into Ω there is a $\gamma \in (\alpha, \beta)$ such that $\varphi \mid \langle\alpha, \gamma\rangle$ is a curve from z into G, all curves from z into Ω belong to the same bundle of $\mathfrak{S}(\Omega)$. This proves (6).

Lemma 2. *Suppose that Ω is a region, $\lambda : (\alpha, \beta) \to \partial\Omega$ a one-one continuous mapping, (λ) a free part of $\partial\Omega$. Then for each $t \in (\alpha, \beta)$ and for each $\delta > 0$ there are numbers $t', t'' \in (\alpha, \beta)$ and a Jordan region G satisfying conditions* (1) *and* (2) *such that*

$$(9) \qquad t - \delta < t' < t < t'' < t + \delta\,,$$

$$(10) \qquad \operatorname{diam}^* G < \delta\,,$$

(11) $\partial G = \langle\varphi_1\rangle \cup \langle\varphi_2\rangle$, *where φ_j $(j = 1, 2)$ are simple curves with i.p. $\varphi_j = \lambda(t')$, e.p. $\varphi_j = \lambda(t'')$, $(\varphi_1) \subset \Omega$, $(\varphi_2) \subset \mathbf{S} - \overline{\Omega}$.*

Proof. Let $t \in (\alpha, \beta)$ and $\delta > 0$ be fixed. Then there are numbers $T' \in (\alpha, t)$, $T'' \in (t, \beta)$ and a Jordan region G_0 such that

(12) $\lambda \mid \langle T', T'' \rangle$ is a cut in G_0, $G_0 - \lambda((T', T'')) = G_1 \cup G_2$, where $G_1 \subset \Omega$ and $G_2 \subset \mathbf{S} - \bar{\Omega}$ are components of $G_0 - \lambda((T', T''))$.

Let h_j $(j = 1, 2)$ be a homeomorphic mapping of $\bar{G}_j$ onto $\mathbf{U}$ which maps G_j conformally onto $\mathbf{U}$ [1]). Obviously, there exist linear curves ψ_j such that

$$(13_1) \qquad i.p.\ \psi_j,\ e.p.\ \psi_j \in \partial \mathbf{U}\,, \quad (\psi_j) \subset \mathbf{U}\,,$$

$$(13_2) \qquad i.p.\ \psi_j \neq h_j(\lambda(t)) \neq e.p.\ \psi_j\,,$$

$$(13_3) \qquad t - \delta < (h_1)_{-1}\,(i.p.\ \psi_1) = (h_2)_{-1}\,(i.p.\ \psi_2) < t < (h_1)_{-1}\,(e.p.\ \psi_1) = = (h_2)_{-1}\,(e.p.\ \psi_2) < t + \delta\,,$$

(13_4) if M_j $(j = 1, 2)$ is the component of $\mathbf{U} - (\psi_j)$ containing $h_j(\lambda(t))$ on its boundary, then diam* $(h_j)_{-1}\,(M_j) < \frac{1}{2}\delta$.

Take $\varphi_j = (h_j)_{-1} \circ \psi_j$, $t' = (h_j)_{-1}\,(i.p.\ \psi_j)$, $t'' = (h_j)_{-1}\,(e.p.\ \psi_j)$, and let G be the component of $\mathbf{S} - (\langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle)$ containing $\lambda(t)$. Then all conditions required above are fulfilled.

Theorem 1,1. *Let F be a conformal mapping of Ω onto $\mathbf{U}$ and let $V \subset \partial\Omega$ be a free part of the boundary of a region $\Omega_1 \subset \Omega$.*

Then there is a mapping F^* of $\Omega_1 \cup V$ such that the following conditions hold:

(14) *$F^* = F$ on Ω_1;*

(15) *F^* is continuous and one-one on $\Omega_1 \cup V$;*

(16) *$C_1 = F^*(V)$ is either an open arc of the circumference $\mathbf{C} = \partial\mathbf{U}$ or a set of the form $\mathbf{C} - \{a\}$ where $a \in \mathbf{C}$;*

(17) *the function*

$$\Phi^* = \begin{cases} F_{-1} & on \quad \mathbf{U}\,, \\ (F^*)_{-1} & on \quad C_1 \end{cases}$$

is continuous and one-one on $\mathbf{U} \cup C_1$.

Proof. Let λ be a continuous and one-one mapping of (α, β) onto V. By Lemma 1 and by our assumptions, for each point $z \in V$ there is one and only one bundle $\mathscr{S}^1_z \in \mathfrak{S}(\Omega_1)$ with $o(\mathscr{S}^1_z) = z$. Let $\mathscr{S}_z \in \mathfrak{S}(\Omega)$ be the bundle containing $\mathscr{S}^1_z$. Take

$$(18) \qquad F^*(z) = \begin{cases} F(z) & \text{for} \quad z \in \Omega_1\,, \\ W_F(\mathscr{S}_z) & \text{for} \quad z \in V\,. \end{cases}$$

Then (14) holds and F^* is continuous on Ω_1. Let $z \in V$, $z_n \in \Omega_1$, $z_n \to z$. By Lemma 1 there is a curve $\varphi \in \langle 0, 1 \rangle \to \mathbf{S}$ from z into Ω_1 with $z_n \in \langle \varphi \rangle$ for all n. Then $\varphi \in \mathscr{S}_z$ and, obviously,

$$\lim F(z_n) = (F \circ \varphi)\,(0+) = W_F(\mathscr{S}_z) = F^*(z)\,.$$

[1]) The existence of such a mapping is proved e.g. in [9], p. 538.

This proves that for each $z \in V$, the function F^* is continuous at z with respect to $\Omega_1 \cup \{z\}$. By a well known theorem (cf. [9], p. 516), this implies the continuity of F^* on $\Omega_1 \cup V$.

Now, $F^* \mid \Omega_1 = F \mid \Omega_1$ is one-one, W_F is one-one on $\mathfrak{S}(\Omega)$ [2]) (which implies that $F^* \mid V$ is one-one), and the sets $F^*(\Omega_1) \subset \boldsymbol{U}$, $F^*(V) \subset \partial \boldsymbol{U}$ are disjoint. Thus F^* is one-one.

Since, by (15), $F^* \circ \lambda$ is one-one and continuous, the assertion (16) holds.

It remains to prove (17). The continuity of Φ^* on $\boldsymbol{U}$ is obvious, as the inverse of a conformal mapping is conformal. By proving that

$$(19) \qquad w_n \in \boldsymbol{U}, \quad w_n \to w \Rightarrow F_{-1}(w_n) \to (F^*)_{-1}(w)$$

for each $w \in C_1$ the proof of continuity of Φ^* on $\boldsymbol{U} \cup C_1$ will be completed.

Thus let $w_n \in \boldsymbol{U}$, $w_n \to w \in C_1$. Let $t \in (\alpha, \beta)$ be the point with $F^*(\lambda(t)) = w$. By Lemma 2, there are points $t' \in (\alpha, t)$, $t'' \in (t, \beta)$ and a Jordan region G satisfying (1) and (2) such that

(20) $\partial G = \langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle$, where φ_j $(j = 1, 2)$ are simple curves with *i.p.* $\varphi_j = \lambda(t')$, *e.p.* $\varphi_j = \lambda(t'')$, $(\varphi_1) \subset \Omega_1$, $(\varphi_2) \subset \boldsymbol{S} - \bar{\Omega}_1$.

Then

$$(21) \qquad G - \lambda((t', t'')) = G_1 \cup G_2 ,$$

where G_j $(j = 1, 2)$ are Jordan regions such that

$$(22) \qquad \partial G_j = \lambda(\langle t', t'' \rangle) \cup (\varphi_j) ,$$

$$(23) \qquad G_1 \cup (\varphi_1) \subset \Omega_1 , \quad G_2 \cup (\varphi_2) \subset \boldsymbol{S} - \bar{\Omega}_1 .$$

Denote by ψ_1 the F-image of φ_1. Then

$$(24) \qquad \boldsymbol{U} - (\psi_1) = U_1 \cup U_2 ,$$

where U_1, U_2 are disjoint Jordan regions. As φ_1 is a cut in Ω, G_1 is obviously a component of $\Omega - (\varphi_1)$. Choose the notation so that

$$(25) \qquad U_1 = F(G_1) .$$

Then, obviously, $w \in \partial U_1 - \bar{U}_2$, and the conditions $w_n \in \boldsymbol{U}$, $w_n \to w$ imply $w_n \in U_1$ for all n sufficiently large. Further, it follows that $z_n = F_{-1}(w_n) \in G_1$ for such n. Suppose $z_n \to (F^*)_{-1}(w)$ is not true. Then there is a subsequence $\{z_{n_k}\}$ with $z_{n_k} \to z' \neq (F^*)_{-1}(w)$. As obviously $z' \in \lambda(\langle t', t'' \rangle)$, we have by (15) $w_{n_k} = F(z_{n_k}) \to F^*(z') \neq w$. This contradiction proves our assertion.

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[2]) See [9], p. 535.

Obviously, Φ^* is one-one. This completes the proof of Theorem 1,1.

2. Definition 2. Let $\lambda : (\alpha, \beta) \to \mathbf{S}$ (where $-\infty \leqq \alpha < \beta \leqq +\infty$). Suppose there exists a function Λ meromorphic on a region X containing (α, β) and conformal at each point[3]) $z \in (\alpha, \beta)$ such that $\Lambda \mid (\alpha, \beta) = \lambda$. Then we say the mapping λ is *analytic*. We say the mapping $\lambda : (\alpha, \beta) \to \mathbf{S}$ is *strictly analytic* iff there is a conformal extension Λ of λ to a region X containing (α, β).

Remark 1. Obviously, any strictly analytic mapping is analytic and one-one. As the following example shows, the converse assertion is false.

Take

$$\lambda(t) = e^{2it} - ie^{it} - 1 \quad \text{for} \quad t \in \left(0, \frac{5\pi}{6}\right).$$

Then λ is analytic: The meromorphic extension

$$\Lambda(z) = e^{2iz} - ie^{iz} - 1 \quad (z \in \mathbf{E})$$

is conformal at each point $z \in \mathbf{E}$ with $\Lambda'(z) = 2ie^{2iz} + e^{iz} \neq 0$, i.e. at each point $z \in \mathbf{E}$ with $e^{iz} \neq \frac{1}{2}i$; none of the points z with $e^{iz} = \frac{1}{2}i$, however, lies in $(0, \frac{5}{6}\pi)$.

λ is one-one: If $F(z) = z^2 - iz - 1$ and $F(z_1) = F(z_2)$, $z_1 \neq z_2$, then $z_1 + z_2 = i$. If $t_1, t_2 \in (0, \frac{5}{6}\pi)$, $t_1 \neq t_2$, then, as we easily see, $e^{it_1} + e^{it_2} \neq i$. This implies that $\Lambda(t_1) \neq \Lambda(t_2)$ for each two distinct numbers $t_1, t_2 \in (0, \frac{5}{6}\pi)$.

λ is not strictly analytic: Since $\Lambda(\frac{1}{6}\pi) = \Lambda(\frac{5}{6}\pi)$, we have $\Lambda(U(\frac{5}{6}\pi)) \cap \Lambda(X^*) \neq \emptyset$ for any $U(\frac{5}{6}\pi)$ and for any region X^* containing $(\frac{1}{3}\pi, \frac{2}{3}\pi)$. Hence it follows easily that the mapping Λ is not one-one in any region X containing $(0, \frac{5}{6}\pi)$.

Theorem 2,1. *Let $\lambda : (\alpha, \beta) \to \mathbf{S}$ be a one-one analytic mapping. Then the following conditions are equivalent to each other*:

I. *λ is strictly analytic.*

II. $(\mathscr{P}(\lambda) \cup \mathscr{K}(\lambda)) \cap (\lambda) = \emptyset$.

III. *For each $t \in (\alpha, \beta)$ and for each $\delta > 0$ there are points $t', t'' \in (\alpha, \beta)$ and an open set G such that $t - \delta < t' < t < t'' < t + \delta$ and $G \cap (\lambda) = \lambda((t', t''))$.*

Proof. First we prove the implication I $\Rightarrow$ II. If condition I holds, there is a conformal mapping Λ of a region X containing (α, β) such that $\Lambda \mid (\alpha, \beta) = \lambda$. We may suppose that $X \cap {}^*\mathbf{E}_1 = (\alpha, \beta)$. Then $\alpha, \beta \in \partial X$ and for each sequence of points $t_n \in (\alpha, \beta)$ with either $t_n \to \alpha$ or $t_n \to \beta$ we have ls $\Lambda(t_n) \subset \partial\Lambda(X)$ (see [8], (3)). This proves the inclusion $\mathscr{P}(\lambda) \cup \mathscr{K}(\lambda) \subset \partial\Lambda(X)$. As $(\lambda) = \Lambda((\alpha, \beta)) \subset \Lambda(X) \subset \mathbf{S} - \partial\Lambda(X)$, condition II holds.

[3]) We say a meromorphic function is *conformal at a point z* iff it is locally one-one at z.

Now we prove the implication III $\Rightarrow$ I. It is easy to see that the following general assertion holds:

(26) If F is meromorphic on an open set Ω, one-one on a compact subset $K \subset \Omega$, and conformal at each point $z \in K$, then there is a $\delta > 0$ such that F is conformal on $U(K, \delta)$ [4]).

Suppose now that condition III holds and let Λ be a meromorphic extension of λ to a region X containing (α, β). We have to prove that there is a region X^* such that $(\alpha, \beta) \subset X^* \subset X$ and $\Lambda \mid X^*$ is one-one.

First we prove

(27) for each interval $\langle \alpha', \beta' \rangle \subset (\alpha, \beta)$ there is a $\delta > 0$ such that Λ is one-one on the rectangle $M = \{z \in \mathbf{E};\ \mathrm{Re}\, z \in \langle \alpha', \beta' \rangle,\ |\mathrm{Im}\, z| \leqq \delta\}$ and $\Lambda(M) \cap \lambda((\alpha, \alpha') \cup \cup (\beta', \beta)) = \emptyset$.

Choose points $\alpha^* \in (\alpha, \alpha')$, $\beta^* \in (\beta', \beta)$; by (26) there is a $\delta^* > 0$ such that Λ is one-one on the rectangle $M^* = \{z;\ \mathrm{Re}\, z \in \langle \alpha^*, \beta^* \rangle,\ |\mathrm{Im}\, z| \leqq \delta^*\}$. Let us show that

(28) $$\mathrm{dist}^* (\lambda(\langle \alpha', \beta' \rangle),\ \lambda((\alpha, \alpha^*\rangle \cup \langle \beta^*, \beta))) > 0\,.$$ [5])

Suppose (28) does not hold. Then there are points $t_n \in \langle \alpha', \beta' \rangle$, $t_n^* \in (\alpha, \alpha^*\rangle \cup \langle \beta^*, \beta)$ with $\varrho^*(\lambda(t_n), \lambda(t_n^*)) \to 0$. Since $\langle \alpha', \beta' \rangle$ is compact, we may suppose $\lim t_n = t$ exists. Then $t \in \langle \alpha', \beta' \rangle$ and, as λ is continuous, $\lambda(t_n) \to \lambda(t)$, $\lambda(t_n^*) \to \lambda(t)$. By III, there are points t', t'' with $\alpha^* < t' < t < t'' < \beta^*$ and an open set G with $\lambda(t) \in G$ $G \cap \lambda((\alpha, t'\rangle \cup \langle t'', \beta)) = \emptyset$. This, however, is impossible, since $\lambda(t_n^*) \in G$ for all n sufficiently large.

This completes the proof of (28). By (28), and since $\langle \alpha', \beta' \rangle$ is compact and Λ continuous, there is a $\delta \in (0, \delta^*)$ with

(29) $$\Lambda(M) \cap \lambda((\alpha, \alpha^*\rangle \cup \langle \beta^*, \beta)) = \emptyset$$

(where M is the same as in (27)). M and $(\alpha^*, \alpha') \cup (\beta', \beta^*)$ are disjoint subsets of M^*, Λ is one-one on M^*. This implies

(30) $$\Lambda(M) \cap \lambda((\alpha^*, \alpha') \cup (\beta', \beta^*)) = \emptyset\,.$$

By (29) and (30), we have

$$\Lambda(M) \cap \lambda((\alpha, \alpha') \cup (\beta', \beta)) = \emptyset\,.$$

This completes the proof of (27).

Choose numbers α_n (where n is an integer) such that $\alpha_m < \alpha_n$ for each pair $m < n$, and $\lim_{n \to \infty} \alpha_{-n} = \alpha$, $\lim_{n \to \infty} \alpha_n = \beta$. For each pair of integers $m < n$ and for each $\delta > 0$ we set

(31) $$A(m, n; \delta) = \{z;\ \mathrm{Re}\, z \in \langle \alpha_m, \alpha_n \rangle,\ |\mathrm{Im}\, z| \leqq \delta\}\,, \quad L_{m,n} = \lambda(\langle \alpha_m, \alpha_n \rangle)\,.$$

[4]) By definition, $U(K, \delta) = \bigcup_{z \in K} U(z, \delta)$.

[5]) By dist* we denote the distance measured with the aid of the metric ϱ^*.

We shall say *a set M has the property $W(m, n)$* (where $m < n$ are integers) iff the following four conditions hold:

1. M is a compact subset of X;
2. $A \mid M$ is one-one;
3. $M \cap (\alpha, \beta) = \langle \alpha_m, \alpha_n \rangle$;
4. $A(M) \cap (\lambda) = L_{m,n}$.

It is easy to see that the following two assertions hold:

(32) If M has the property $W(m, n)$, if $m \leqq m_1 < n_1 \leqq n$, and if N is a compact subset of M with $N \cap (\alpha, \beta) = \langle \alpha_{m_1}, \alpha_{n_1} \rangle$, then N has the property $W(m_1, n_1)$.

(33) If M has the property $W(m, n)$ and if either $p < q < m$ or $n < p < q$, then there is a $\delta > 0$ such that $A(M) \cap A(A(p, q; \delta)) = \emptyset$.

By (27) it also follows that

(34) for any two integers $m < n$ there is a $\delta > 0$ such that the rectangle $A(m, n; \delta)$ has the property $W(m, n)$.

Now we shall construct (by induction) rectangles $A_0, A_1, A_{-1}, \ldots, A_n, A_{-n}, \ldots$ such that

(35) $$X^* = \operatorname{int}\left(\bigcup_{n=-\infty}^{+\infty} A_n\right) \text{ is a subregion of } X\,,$$

(36) $$(\alpha, \beta) \subset X^*\,,$$

(37) $$A \Big| \bigcup_{n=-\infty}^{+\infty} A_n \text{ is one-one}\,.$$

Rectangles A_n^* which occur in the construction have auxiliary significance only.

By (34), there is a $\delta_0 > 0$ such that the rectangle $A_0^* = A(-1, 2; \delta_0)$ has the property $W(-1, 2)$; set $A_0 = A(0, 1; \delta_0)$. By (32), the rectangle A_0 has the property $W(0, 1)$, whence, by (33), there is a $\delta_1 > 0$ such that

(38) $$A(A_0) \cap A(A(2, 3; \delta_1)) = \emptyset\,.$$

By (34) and (32), we may obviously suppose that $\delta_1 \in (0, \delta)$ and that

(39) the rectangle $A_1^* = A(1, 3; \delta_1)$ has the property $W(1, 3)$.

Let us prove that

(40) the set $A_0 \cup A_1^*$ has the property $W(0, 3)$.

If $z_1, z_2 \in A_0 \cup A_1^*$, then either $z_1, z_2 \in A_0^*$ or $z_1, z_2 \in A_1^*$, or one of the points z_1, z_2 lies in A_0, the other one in $A(2, 3; \delta_1)$. The mapping A is one-one on A_0^*, one-one on A_1^*, and (38) holds. This implies A is one-one on $A_0 \cup A_1^*$. All the other conditions which together yield (40) are obvious.

Set $A_1 = A(1, 2; \delta_1)$. By (32)–(34), there is a $\delta_{-1} \in (0, \delta_0)$ such that

(41) $$A(A_0 \cup A_1^*) \cap A(A(-2, -1; \delta_{-1})) = \emptyset$$

and that

(42) the rectangle $A_{-1}^* = A(-2, 0; \delta_{-1})$ has the property $W(-2, 0)$.

Again, it is easy to see that

(43) the set $A_{-1}^* \cup A_0 \cup A_1^*$ has the property $W(-2, 3)$:

If $z_1, z_2 \in A_{-1}^* \cup A_0 \cup A_1^*$, then either $z_1, z_2 \in A_0^*$ or $z_1, z_2 \in A_0 \cup A_1^*$ or $z_1, z_2 \in A_{-1}^*$, or one of the points z_1, z_2 belongs to $A_0 \cup A_1^*$, the other one to $A(-2, -1; \delta_{-1})$. A is one-one on A_0^*, $A_0 \cup A_1^*$, A_{-1}^*, and (41) holds.

Set $A_{-1} = A(-1, 0; \delta_{-1})$. Suppose that for a certain $n \in \mathbf{N}$, positive numbers $\delta_n < \delta_{n-1} < \ldots < \delta_1 < \delta_0$, $\delta_{-n} < \delta_{-n+1} < \ldots < \delta_{-1} < \delta_0$ and rectangles $A_n^* = A(n, n + 2; \delta_n)$, $A_{-n}^* = A(-n - 1, -n + 1; \delta_{-n})$, $A_k = A(k, k + 1; \delta_k)$, where $-n \leqq k \leqq n$ are already constructed, and that

(44) the set $A_{-n}^* \cup \bigcup_{|k|<n} A_k \cup A_n^*$ has the property $W(-n - 1, n + 2)$.

Then the rectangles $A_{n+1}^*, A_{n+1}, A_{-n-1}^*, A_{-n-1}$ will be constructed as follows:

By (44) and (32), the set $A_{-n}^* \cup \bigcup_{|k|<n} A_k$ has the property $W(-n - 1, n + 1)$. Hence by (32)–(34), there is a $\delta_{n+1} \in (0, \delta_n)$ such that

(45) $$A(A_{-n}^* \cup \bigcup_{k=-n+1}^{n} A_k) \cap A(A(n + 2, n + 3; \delta_{n+1})) = \emptyset$$

and

(46) the rectangle $A_{n+1}^* = A(n + 1, n + 3; \delta_{n+1})$ has the property $W(n + 1, n + 3)$.

As above, it is easy to prove that

(47) the set $A_{-n}^* \cup \bigcup_{k=-n+1}^{n} A_k \cup A_{n+1}^*$ has the property $W(-n - 1, n + 3)$.

Denote $A_{n+1} = A(n + 1, n + 2; \delta_{n+1})$. By (47) and (32), the set $\bigcup_{k=-n}^{n} A_k \cup A_{n+1}^*$ has the property $W(-n, n + 3)$. Hence by (32)–(34), there is a number $\delta_{-n-1} \in (0, \delta_{-n})$ such that

(48) $$A(\bigcup_{k=-n}^{n} A_k \cup A_{n+1}^*) \cap A(A(-n - 2, -n - 1; \delta_{-n-1})) = \emptyset$$

and

(49) the rectangle $A_{-n-1}^* = A(-n - 2, -n, \delta_{-n-1})$ has the property $W(-n - 2, -n)$.

Again, it follows easily that

(50) the set $A^*_{-n-1} \cup \bigcup_{|k|<n+1} A_k \cup A^*_{n+1}$ has the property $W(-n-2, n+3)$.

Putting $A_{-n-1} = A(-n-1, -n; \delta_{-n-1})$ we complete the induction step.

Now, for each integer n we have sets A^*_n, A_n satisfying (44). By (44) and (32),

(51) the set $\bigcup_{k=-n}^{n} A_k$ has the property $W(-n, n+1)$

(for each natural number n). This implies the function Λ is one-one on $\bigcup_{k=-n}^{n} A_k$ for any natural number n; as a consequence, it is one-one on $\bigcup_{k=-\infty}^{+\infty} A_k$. Obviously, conditions (35), (36) hold as well. This completes the proof of the implication III $\Rightarrow$ I.

It remains to prove the implication II $\Rightarrow$ III. Let Λ be a meromorphic extension of λ to a region X containing (α, β). Choose α_n as in the proof of III $\Rightarrow$ I and use the same notation. By (26), for each $n \in \mathbf{N}$ there is a number $\Delta_n > 0$ such that Λ is one-one on $A(-n, n; \Delta_n)$. By II and since λ is one-one, the compact set $\mathscr{P}(\lambda) \cup \cup \lambda((\alpha, \alpha_{-n-1}\rangle \cup \langle\alpha_{n+1}, \beta)) \cup \mathscr{K}(\lambda)$ is disjoint with $\lambda(\langle\alpha_{-n}, \alpha_n\rangle)$. Thus we may suppose that Δ_n also satisfies the condition

$$(52) \qquad \Lambda(A(-n, n; \Delta_n)) \cap (\mathscr{P}(\lambda) \cup \lambda((\alpha, \alpha_{-n-1}\rangle \cup \langle\alpha_{n+1}, \beta)) \cup \mathscr{K}(\lambda)) = \emptyset\,.$$

Let $t \in (\alpha, \beta)$ and $\delta > 0$ be fixed numbers. Then there is a number $n \in \mathbf{N}$ with $t \in (\alpha_{-n}, \alpha_n)$. Further, there is a $\delta' \in (0, \delta)$ such that

$$(53) \qquad U(t, \delta') \subset A(-n, n; \Delta_n) \cap A(-n-1, n+1; \Delta_{n+1})\,.$$

Set $t' = t - \delta'$, $t'' = t + \delta'$, $G = \Lambda(U(t, \delta'))$. Since Λ is one-one on $A(-n-1, n+1; \Delta_{n+1})$ and $U(t, \delta') \cap ((\alpha_{-n-1}, t'\rangle \cup \langle t'', \alpha_{n+1})) = \emptyset$ we have

$$(54) \qquad G \cap \lambda((\alpha_{-n-1}, t'\rangle \cup \langle t'', \alpha_{n+1})) = \emptyset\,.$$

Conditions (52), (53) imply that

$$(55) \qquad G \cap (\mathscr{P}(\lambda) \cup \lambda((\alpha, \alpha_{-n-1}\rangle \cup \langle\alpha_{n+1}, \beta)) \cup \mathscr{K}(\lambda)) = \emptyset\,.$$

From (54), (55) and from the inclusion $(t', t'') \subset U(t, \delta')$ (which implies $\lambda((t', t'')) \subset G$) it follows that $G \cap (\lambda) = \lambda((t', t''))$. This completes the proof of Theorem 2,1.

Remark 2. As we can see at the end of the proof just completed, we have even $G \cap \overline{(\lambda)} = G \cap (\mathscr{P}(\lambda) \cup (\lambda) \cup \mathscr{K}(\lambda)) = \lambda((t', t''))$.

This implies, obviously, that (under the assumptions of Theorem 2,1) conditions I–III of Theorem 2,1 are equivalent to the following assertion:

III'. For each $t \in (\alpha, \beta)$ and each $\delta > 0$ there are points $t', t'' \in (\alpha, \beta)$ and a Jordan region G such that $t - \delta < t' < t < t'' < t + \delta$ and $G \cap \overline{(\lambda)} = \lambda((t', t''))$.

(As Λ is one-one on $\Lambda(-n, n; \Delta_n) \subset X$ and $\overline{U(t, \delta')} \subset \Lambda(-n, n; \Delta_n)$, the set $G = \Lambda(U(t, \delta'))$ is a Jordan region. The equality $\overline{(\lambda)} = \mathscr{P}(\lambda) \cup (\lambda) \cup \mathscr{K}(\lambda)$ is obvious.)

Remark 3. As in Theorem 2,1, let $\lambda : (\alpha, \beta) \to \boldsymbol{S}$ be one-one and analytic. It follows immediately that conditions I–III of Theorem 2,1 are equivalent to the following assertion:

IV. If Λ is a meromorphic extension of λ to a region X containing (α, β), then there is a subregion X^* of X containing (α, β) such that Λ is conformal on X^*.

3. Definition 3. We say that a free part V of the boundary of a region Ω is *analytic* iff there is a one-one analytic mapping λ of an interval (α, β) onto V.

Theorem 2,1 and Lemma 2 immediately imply the following assertion:

Theorem 3,1. *Let Ω be a region, $\lambda : (\alpha, \beta) \to \partial\Omega$ a one-one analytic mapping such that (λ) is a free part of $\partial\Omega$. Then λ is strictly analytic.*

The following theorem is one of the fundamental theorems concerning the extension of a meromorphic function across a free part of the boundary:

Theorem 3,2. 1. *Let V be an analytic free part of the boundary of a region Ω, $\mu : (\gamma, \delta) \to \boldsymbol{S}$ a one-one analytic mapping. Suppose F is meromorphic on Ω, continuous on $\Omega \cup V$, and $F(V) \subset (\mu)$. Then there is a region Ω^* containing $\Omega \cup V$ and a function F^* meromorphic on Ω^* such that $F^* = F$ on $\Omega \cup V$.*

2. *Suppose, moreover, that F is one-one on $\Omega \cup V$. If F^* is a meromorphic extension of F to a region Ω^* containing $\Omega \cup V$, then F^* is conformal at each point $z \in V$. More generally: For each compact subset K of V there is a $\Delta > 0$ such that $F^* \mid U(K, \Delta)$ is conformal.*

Proof. 1. Let the assumptions of the first part of the theorem be fulfilled. By Theorem 3,1 (and Remark 3, Section 2), there is an interval (α, β), a region $X \supset (\alpha, \beta)$, and a conformal mapping $\Lambda : X \to \boldsymbol{S}$ such that $\lambda = \Lambda \mid (\alpha, \beta)$ maps (α, β) onto V. Besides, there is a region $Y \supset (\gamma, \delta)$ and a function M meromorphic on Y, conformal at each point of (γ, δ) with $M \mid (\gamma, \delta) = \mu$.

If F is constant, the assertion of the first part of Theorem 3,2 is obvious. Thus, let us suppose F is not constant.

If $z \in V$, then $F(z) \in (\mu)$ and $\mu_{-1}(F(z)) \in (\gamma, \delta)$. Since M is conformal at $\mu_{-1}(F(z))$, there is an $\eta_z > 0$ such that

(56) the points γ, δ do not lie in the set $A_z = U(\mu_{-1}(F(z)), \eta_z)$

and

(57) the mapping $M^z = M \mid A_z$ is one-one.

The domain $M(A_z)$ of M^z_{-1} [6]) is a region containing $F(z)$. Since F is continuous at z with respect to $\Omega \cup V$, there is, by Lemma 2, a Jordan region G_z such that

(58) $$z \in G_z\,,$$

(59) $G_z - (\lambda) = G_z^1 \cup G_z^2$, where $G_z^1 \subset \Omega$, $G_z^2 \subset \mathbf{S} - \bar{\Omega}$ are Jordan regions with $z \in \partial G_z^1 \cap \partial G_z^2$,

(60) $$F(G_z \cap (\Omega \cup V)) \subset M(A_z)\,.$$

As $z \in V = (\lambda)$, we have $\lambda_{-1}(z) \in (\alpha, \beta)$. Since Λ is continuous, there is a $\Delta_z > 0$ such that

(61) $B_z = U(\lambda_{-1}(z), \Delta_z)$ is a subset of X and does not contain any one of the points α, β,

(62) $$\Lambda(B_z) \subset G_z\,.$$

Then obviously

(63) $$B_z - (\alpha, \beta) = B_z^1 \cup B_z^2\,,$$

where B_z^1, B_z^2 are disjoint open half-circles. Since Λ is one-one on X, the regions $\Lambda(B_z^j)$ $(j = 1, 2)$ are disjoint with the set (λ). Hence by (62), (59), each of the regions $\Lambda(B_z^j)$ is a subset either of Ω of or $\mathbf{S} - \bar{\Omega}$. Since the region $\Lambda(B_z)$ (containing the point $z \in \bar{\Omega} \cap \overline{(\mathbf{S} - \bar{\Omega})}$) intersects both Ω and $\mathbf{S} - \bar{\Omega}$, one of the regions $\Lambda(B_z^j)$ must be a subset of Ω, the other one a subset of $\mathbf{S} - \bar{\Omega}$. Hence one of the regions $\Lambda(B_z^j)$ is contained in G_z^1, the other one in G_z^2. Choose the notation so that

(64) $$\Lambda(B_z^1) \subset G_z^1\,(\subset \Omega)\,, \quad \Lambda(B_z^2) \subset G_z^2\,(\subset \mathbf{S} - \bar{\Omega})\,.$$

The function $M^z_{-1} \circ F \circ \Lambda$ is holomorphic on B_z^1, continuous on $B_z^1 \cup (B_z \cap \mathbf{E}_1)$, and maps the interval $B_z \cap \mathbf{E}_1$ into the interval (γ, δ). According to the Schwarz reflection principle there is a function g_z holomorphic on B_z such that

(65) $$g_z = M^z_{-1} \circ F \circ \Lambda \quad \text{on} \quad B_z^1 \cup (B_z \cap \mathbf{E}_1)\,.$$

Take

(66) $$F_z = M \circ g_z \circ \Lambda_{-1} \quad \text{on} \quad \Lambda(B_z)\,;$$

then F_z is obviously meromorphic on its definition domain and

(67) $$F_z = F \quad \text{on} \quad \Lambda(B_z) \cap (\Omega \cup V) = \Lambda(B_z^1 \cup (B_z \cap \mathbf{E}_1))\,.$$

Suppose $z, \zeta \in V$ are two points with

(68) $$\Lambda(B_z) \cap \Lambda(B_\zeta) \neq \emptyset\,.$$

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[6]) We write M^z_{-1} instead of the more correct $(M^z)_{-1}$.

As Λ is one-one, it follows that $B_z \cap B_\zeta \neq \emptyset$. As B_z, B_ζ are circles with centres in $\mathbf{E}_1$, we have $B_z \cap B_\zeta \cap \mathbf{E}_1 \neq \emptyset$. As the set $B_z \cap B_\zeta \cap \mathbf{E}_1$ has accumulation points in $B_z \cap B_\zeta$, the set $\Lambda(B_z \cap B_\zeta \cap \mathbf{E}_1)$ has accumulation points in the set $\Lambda(B_z) \cap \Lambda(B_\zeta) = \Lambda(B_z \cap B_\zeta)$, which is (as a conformal image of the region $B_z \cap B_\zeta$) a region. By (67) and by an analogous condition for B_ζ we have $F_z = F_\zeta = F$ on $\Lambda(B_z \cap B_\zeta \cap \mathbf{E}_1)$. By a well known „unicity theorem" this implies

$$F_z = F_\zeta \quad \text{on} \quad \Lambda(B_z) \cap \Lambda(B_\zeta) . \tag{69}$$

As F is continuous on $\Omega \cup V$, we have

$$F_z = F_\zeta = F \quad \text{on} \quad \Lambda(B_z) \cap \Lambda(B_\zeta) \cap (\Omega \cup V) . \tag{70}$$

This implies that on the set

$$\Omega^* = \Omega \cup \bigcup_{z \in V} \Lambda(B_z) , \tag{71}$$

it is consistent to define a function F^* as follows:

$$F^* = \begin{cases} F & \text{on} \quad \Omega \cup V, \\ F_z & \text{on} \quad \Lambda(B_z) \quad \text{where} \quad z \in V . \end{cases} \tag{72}$$

It is evident that Ω^* is a region containing $\Omega \cup V$ and that F^* is a meromorphic extension of F to Ω^*.

This completes the proof of the first part of the theorem.

2. In the proof of the second part we shall use the following assertion (which is important by itself):

Lemma 3. *Let F be meromorphic on a region Z symmetric with respect to the real axis $^*\mathbf{E}_1$ and let $F(Z \cap {}^*\mathbf{E}_1) \subset {}^*\mathbf{E}_1$. Then:*

1. *F is one-one on Z iff it is one-one on $Z \cap \overline{\mathbf{E}^+}$ and $F(Z \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 = \emptyset$.*
2. *If F is one-one on $Z \cap \overline{\mathbf{E}^+}$, then it is conformal at each point $z \in Z \cap {}^*\mathbf{E}_1$.*

First we prove the second part of Theorem 3,2 by means of Lemma 3: If F is one-one on $\Omega \cup V$, then for each $z \in V$ the function g_z is one-one on $B_z^1 \cup (B_z \cap \mathbf{E}_1)$. Lemma 3 implies g_z is conformal at $\lambda_{-1}(z)$. Further, it follows that F_z is conformal at z. The same is true for any extension F^*.

The rest of the second part of Theorem 3,2 is a consequence of what has just been proved, and of (26).

Proof of Lemma 3. Suppose the conditions for F and Z from Lemma 3 are satisfied.

1. Suppose first $F(Z \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 \neq \emptyset$; this means that F assumes a real value at a certain point $z \in Z \cap \mathbf{E}^+$. According to the Schwarz reflexion principle, this implies $F(\bar{z}) = \overline{F(z)} = F(z)$; we have, of course, $\bar{z} \in Z$, $\bar{z} \neq z$. Hence F is not one-one on Z.

Suppose now F is not one-one on Z; we have to show that the following implication holds: If $F \mid Z \cap \overline{\mathbf{E}^+}$ is one-one, then $F(Z \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 \neq \emptyset$. If F is one-one on $Z \cap \overline{\mathbf{E}^+}$, then by the Schwarz reflexion principle, it is one-one on $Z \cap \overline{\mathbf{E}^-}$ as well. Since F is not one-one on Z, there are points $z_1 \in Z \cap \mathbf{E}^+$, $z_2 \in Z \cap \mathbf{E}^-$ with $F(z_1) = F(z_2)$. Taking $z_1^* = \bar{z}_2$ we have $z_1^* \in Z \cap \mathbf{E}^+$, and by the Schwarz principle, $F(z_1^*) = \overline{F(z_1)}$. If $F(z_1) \in {}^*\mathbf{E}_1$, there is nothing more to prove. If $F(z_1) \notin {}^*\mathbf{E}_1$, then one of the numbers $F(z_1)$, $F(z_1^*)$ lies in $\mathbf{E}^+$, the other one in $\mathbf{E}^-$. Hence the set $F(Z \cap \mathbf{E}^+)$ intersects both $\mathbf{E}^+$ and $\mathbf{E}^-$. As we prove easily, the set $Z \cap \mathbf{E}^+$ is a region[7]. This implies that $F(Z \cap \mathbf{E}^+)$ is a region as well. Hence $F(Z \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 \neq \emptyset$, which completes the proof.

2. Let F be one-one on $Z \cap \overline{\mathbf{E}^+}$. First, suppose $z_0 \in Z \cap \mathbf{E}_1$, $F(z_0) \in \mathbf{E}_1$. Choose $\delta > 0$ so that $U(z_0, \delta) \subset Z$ and that F is holomorphic on $U(z_0, \delta)$. Then the function $F \mid (z_0 - \delta, z_0 + \delta)$ is real, finite, one-one, and continuous. Thus it is strictly monotone, and $F((z_0 - \delta, z_0 + \delta))$ is a certain interval (α, β) (where $-\infty \leqq \alpha < \beta \leqq \leqq +\infty$). Let $\eta > 0$ be such that $(F(z_0) - \eta, F(z_0) + \eta) \subset (\alpha, \beta)$. Since F is continuous, there is a $\Delta \in (0, \delta)$ such that $F(U(z_0, \Delta)) \subset U(F(z_0), \eta)$. As F is one-one on $Z \cap \overline{\mathbf{E}^+}$, we have

$$F(U(z_0, \Delta) \cap \mathbf{E}^+) \cap F(U(z_0, \delta) \cap \mathbf{E}_1) = \emptyset . \tag{73}$$

Obviously, $F(U(z_0, \Delta) \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 \subset U(F(z_0), \eta) \cap {}^*\mathbf{E}_1 = (F(z_0) - \eta, F(z_0) + \eta) \subset \subset (\alpha, \beta)$ and $F(U(z_0, \delta) \cap \mathbf{E}_1) = F((z_0 - \delta, z_0 + \delta)) = (\alpha, \beta)$; this implies that

$$F(U(z_0, \Delta) \cap \mathbf{E}^+) \cap {}^*\mathbf{E}_1 = \emptyset .$$

By the first part of the present Lemma, $F \mid U(z_0, \Delta)$ is one-one. This completes the proof in the case $z_0 \in Z \cap \mathbf{E}_1$, $F(z_0) \in \mathbf{E}_1$. If $z_0 = \infty$, we investigate $F \circ Id^{-1}$ instead of F; if $F(z_0) = \infty$, we investigate $1/F$, and use what we have proved already.

Remark 1. The assumptions of the second part of Theorem 3,2 do not ensure that the extension F^* of F is one-one on a certain region $\Omega^{**} \subset \Omega^*$ containing $\Omega \cup V$. This will be obvious, if we take e.g.

$$\Omega = \{z; |\mathrm{Re}\, z| < 1,\ 0 < \mathrm{Im}\, z < 2\pi\} , \quad F = \exp , \quad V = (-1, 1) , \quad \mu = Id \text{ on } \mathbf{E}_1 .$$

Indeed, any region Ω^{**} containing the set $\Omega \cup V$ contains pairs of points z, $z + 2\pi i$ at which the exponential function assumes the same value.

Nonetheless, in this case there exists a region Ω_1 containing V such that the extension is one-one on Ω_1. However, take

$$\mu(t) = e^{2it} - ie^{it} - 1 \quad \text{for} \quad t \in \langle 0, \tfrac{5}{6}\pi \rangle .$$

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[7]) This is a consequence of the symmetry of the region Z with respect to the real axis.

Then $\mathbf{S} - \langle\mu\rangle$[8]) has precisely two components; one of them is bounded, the other one unbounded. For the unbounded component G of $\mathbf{S} - \langle\mu\rangle$ we have $\partial G = \langle\mu\rangle$ so that G is a simply connected region. It may be proved that for any conformal mapping F of $\mathbf{U}$ onto G there is an open arc C_1 of the circumference $\mathbf{C} = \partial\mathbf{U}$ such that F may be extended to a homeomorphic mapping of the set $\mathbf{U} \cup C_1$ so that $F(C_1) = (\mu)$ (denoting the extension by the same letter F).

Take $\lambda = F_{-1} \circ \mu$ on $(0, \frac{5}{6}\pi)$. Then $(\lambda) = C_1$ is an analytic free part of the boundary of $\mathbf{U}$, $\mu \mid (0, \frac{5}{6}\pi)$ is a one-one analytic mapping, $F((\lambda)) = (\mu)$ and F is one-one and continuous on $\mathbf{U} \cup (\lambda)$. By Theorem 3,1, F may be extended to a meromorphic function on a region U^* containing $\mathbf{U} \cup (\lambda)$. It is not too difficult to prove that the extension is not one-one on any region $U_1 \subset U^*$ containing (λ). (Cf. the example in Remark 1, Section 2.)

As the following theorem shows, the essential point in the example above is that the mapping μ is not strictly analytic.

Theorem 3,3. *Let V be an analytic free part of the boundary of a region Ω, λ a one-one analytic mapping of (α, β) onto V, $\mu : (\gamma, \delta) \to \mathbf{S}$ a strictly analytic mapping. Suppose F is meromorphic on Ω, continuous and one-one on $\Omega \cup V$, $F(V) \subset (\mu)$.*

Then there is a region Ω_1 containing V and a conformal mapping F_1 of Ω_1 such that $F_1 = F$ on $\Omega_1 \cap (\Omega \cup V)$; moreover, $F \circ \lambda$ is a strictly analytic mapping.

Remark 2. If the assumptions of Theorem 3,3 are satisfied, then by Theorem 3,2 there is a meromorphic extension F^* of F to a certain region Ω^* containing $\Omega \cup V$. For each extension F^* there exists by Theorem 3,3 a region $\Omega_1 \subset \Omega^*$ such that $V \subset \Omega_1$ and that the mapping $F^* \mid \Omega_1$ is conformal.

Proof of Theorem 3,3. By Theorem 3,2 there is an extension F^* of F to a region Ω^* containing $\Omega \cup V$. Then the mapping $\varphi = F \circ \lambda = F^* \circ \lambda$ is one-one and analytic. The function $\psi = \mu_{-1} \circ \varphi$ is a one-one continuous mapping of the interval (α, β) into the interval (γ, δ), hence a real strictly monotone continuous function.

Suppose ψ is increasing; the proof for a decreasing ψ is analogous. $\psi((\alpha, \beta))$ is a subinterval (γ', δ') of (γ, δ). As it is easy to see, the following assertions hold: If $\gamma' = \gamma$, then $\mathscr{P}(\varphi) = \mathscr{P}(\mu)$; if $\gamma' > \gamma$, then $\mathscr{P}(\varphi) = \{\mu(\gamma')\}$; if $\delta' = \delta$, then $\mathscr{K}(\varphi) = \mathscr{K}(\mu)$; if $\delta' < \delta$, then $\mathscr{K}(\varphi) = \{\mu(\delta')\}$.

By Theorem 2,1 we have

$$(\mathscr{P}(\mu) \cup \mathscr{K}(\mu)) \cap (\mu) = \emptyset\,; \tag{74}$$

hence, according to what we have just said,

$$(\mathscr{P}(\varphi) \cup \mathscr{K}(\varphi)) \cap (\varphi) = \emptyset\,. \tag{75}$$

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[8]) $\langle\mu\rangle$ is a part of a cardioid similar to the figure 9.

Thus by Theorem 2,1, the mapping $\varphi = F \circ \lambda$ is strictly analytic.

By Theorem 3,1 the mapping λ is strictly analytic as well. Hence there is a region $X \supset (\alpha, \beta)$ and a conformal mapping $\Lambda : X \to \mathbf{S}$ such that $\Lambda \mid (\alpha, \beta) = \lambda$. Evidently we may assume that $\Lambda(X) \subset \Omega^*$. Hence by Remark 3, Section 2, the mapping $F^* \circ \Lambda$ (which is a meromorphic extension of the strictly analytic mapping $\varphi = F \circ \lambda$) is conformal on a certain region $X_1 \subset X$ containing (α, β). This implies that $F^* = (F^* \circ \Lambda) \circ \Lambda_{-1}$ is conformal on the region $\Omega_1 = \Lambda(X_1)$ containing V. Thus by putting $F_1 = F^* \mid \Omega_1$ we complete the proof.

4. Definition 4. We say that a topological circumference[9]) T is *analytic* iff there is a conformal mapping f of a region X containing $\mathbf{C}$ such that $f(\mathbf{C}) = T$.

Theorem 4,1. *Suppose Ω is a Jordan region the boundary of which is an analytic topological circumference. Let F be meromorphic on Ω, continuous on $\bar{\Omega}$. Then the following two assertions hold:*

1. *Suppose that either there is a one-one analytic mapping $\mu : (\gamma, \delta) \to \mathbf{S}$ with $F(\partial\Omega) \subset (\mu)$, or $F(\partial\Omega)$ is an analytic topological circumference. Then there is a region Ω^* containing $\bar{\Omega}$ and a function F^* meromorphic on Ω^* such that $F^* = F$ on $\bar{\Omega}$.*

2. *Suppose that F is one-one on $\bar{\Omega}$ and that the topological circumference $F(\partial\Omega)$ is analytic. Then for each meromorphic extension F^* of F to a region Ω^* containing $\bar{\Omega}$ there is a $\Delta > 0$ such that F^* is one-one on $U(\partial\Omega, \Delta)$.*

Proof. Since $\partial\Omega$ is an analytic topological circumference, there is a conformal mapping f of a region $X \supset \mathbf{C}$ with $f(\mathbf{C}) = \partial\Omega$. By the compactness of the set $\mathbf{C}$ there is an $\eta \in (0, \pi)$ such that $G = \{z; e^{-\eta} < |z| < e^{\eta}\}$ is a subset of X. Of course, we may suppose that

$$X = \{z; e^{-\eta} < |z| < e^{\eta}\}. \tag{76}$$

For each $z \in \partial\Omega$ we have $f_{-1}(z) \in \mathbf{C}$. Hence there is an $\alpha_z \in \mathbf{E}_1$ such that $f_{-1}(z) = e^{i\alpha_z}$. If $\Delta_z \in (0, \eta)$, then $\exp \circ iId$ is a conformal mapping of the open rectangle

$$I_z = \{z; |\mathrm{Re}\, \zeta - \alpha_z| < \Delta_z, |\mathrm{Im}\, \zeta| < \Delta_z\}$$

into X. Hence for each $z \in \partial\Omega$ the function

$$\lambda_z(t) = f(e^{it}), \quad t \in (\alpha_z - \Delta_z, \alpha_z + \Delta_z), \tag{77}$$

is a one-one analytic mapping. Besides, the set (λ_z) contains the point z and, obviously, it is an analytic free part of $\partial\Omega$.

In the first part of the assertion of Theorem 4,1 we suppose that either there is a one-one analytic mapping $\mu : (\gamma, \delta) \to \mathbf{S}$ with $F(\partial\Omega) \subset (\mu)$ or $F(\partial\Omega)$ is an analytic

[9]) i.e. a homeomorphic image of $\mathbf{C}$.

topological circumference. In the former case put $\mu_z = \mu$ for each $z \in \partial\Omega$. Then obviously

$$(78) \qquad F((\lambda_z)) \subset (\mu_z) \quad \text{for each} \quad z \in \partial\Omega\,.$$

In the latter case choose a number $\Delta_z \in (0, \eta)$ small enough to ensure $F((\lambda_z)) \neq F(\partial\Omega)$. Then there is a point $w_z \in F(\partial\Omega) - F((\lambda_z))$. Since $F(\partial\Omega)$ is an analytic topological circumference, there is a conformal mapping g of a region $Y \supset \mathbf{C}$ with $g(\mathbf{C}) = F(\partial\Omega)$. If we choose $\beta_z \in \mathbf{E}_1$ with $g(e^{i\beta_z}) = w_z$ and put

$$(79) \qquad \mu_z(t) = g(e^{it}) \quad \text{for} \quad t \in (\beta_z, \beta_z + 2\pi)\,,$$

then μ_z is a one-one analytic mapping satisfying (78).

By the first part of Theorem 3,2, to each $z \in \partial\Omega$ there is a region Ω_z^* containing $\Omega \cup (\lambda_z)$ and a function F_z^* meromorphic on Ω_z^* such that $F_z^* = F$ on $\Omega \cup (\lambda_z)$. For each $z \in \partial\Omega$ we have $z \in (\lambda_z) \subset \Omega_z^*$. Hence there is a $\vartheta_z > 0$ such that, taking

$$(80) \qquad U_z = U(f_{-1}(z), \vartheta_z)\,,$$

we have

$$(81) \qquad U_z \subset X\,, \quad f(U_z) \subset \Omega_z^*\,.$$

Suppose that for certain two points $z, \zeta \in \partial\Omega$ we have $f(U_z) \cap f(U_\zeta) \neq \emptyset$. The region $U_z \cap U_\zeta$ intersects $\mathbf{C}$ and, therefore, also $\mathbf{U}$. Hence $f(U_z) \cap f(U_\zeta) = f(U_z \cap U_\zeta)$ is a region intersecting Ω. As

$$(82) \qquad F_z^* = F \quad \text{on} \quad \bar{\Omega} \cap f(U_z)\,, \quad F_\zeta^* = F \quad \text{on} \quad \bar{\Omega} \cap f(U_\zeta)\,,$$

we have

$$(83) \qquad F_z^* = F = F_\zeta^* \quad \text{on} \quad f(U_z) \cap f(U_\zeta) \cap \bar{\Omega}\,.$$

By the „unicity theorem" this implies that

$$(84) \qquad F_z^* = F_\zeta^* \quad \text{on} \quad f(U_z) \cap f(U_\zeta)\,.$$

Hence it is consistent to define a function F^* on the set

$$(85) \qquad \Omega^* = \bar{\Omega} \cup \bigcup_{z \in \partial\Omega} f(U_z)$$

(which is obviously a region containing $\bar{\Omega}$) as follows:

$$(86) \qquad F^* = \begin{cases} F & \text{on} \quad \bar{\Omega}\,, \\ F_z & \text{on} \quad f(U_z) \quad \text{where} \quad z \in \partial\Omega\,. \end{cases}$$

Obviously, this function is meromorphic on Ω^* and $F^* = F$ on $\bar{\Omega}$.