ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

*Vydává Matematický ústav ČSAV, Praha*

SVAZEK 103 * PRAHA 20. 11. 1978 * ČÍSLO 4

# ON THE TOLERANCE EXTENSION PROPERTY

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(Received September 19, 1975)

The Congruence Extension Property is one of the important properties of classes of algebras. Some conditions for classes of algebras to satisfy this property are studied in [1], [2], [3] and [4]. It is proved in [1] and [4] that a class of algebras closed under subalgebras satisfies the Congruence Extension Property if and only if it satisfies the so called Principal Congruence Extension Property. The aim of this paper is to give an analogous characterization for extensions of tolerances in the case of classes of commutative semigroups.

Let $A$ be a set. By a *tolerance* (or *tolerance relation*) *on* $A$ we mean a reflexive and symmetric binary relation on $A$. A tolerance $T$ on $A$ is said to be *compatible* (*with an algebra* $\mathfrak{A} = (A, F)$) provided $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in T$ for each $n$-ary $f \in F$ $(n > 0)$ and arbitrary $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ with $\langle a_i, b_i\rangle \in T$ for $i = 1, \ldots, n$. For the concept and properties of compatible tolerances see e.g. [5]–[15].

Denote by $LT(\mathfrak{A})$ the set of all tolerances compatible with an algebra $\mathfrak{A}$. Clearly every congruence on $\mathfrak{A}$ belongs to $LT(\mathfrak{A})$, thus $LT(\mathfrak{A}) \neq \emptyset$. As is proved in [6], $LT(\mathfrak{A})$ is an algebraic lattice (i.e., complete compactly generated lattice) with respect to the set inclusion. In the general case, $LT(\mathfrak{A})$ is not a sublattice of the congruence lattice (see [6], [9]). If $T_i \in LT(\mathfrak{A})$ for $i \in I$, denote by $\bigvee_A\{T_i; i \in I\}$ the supremum of $\{T_i, i \in I\}$ in $LT(\mathfrak{A})$. The infimum is clearly equal to the set-intersection.

**Definition 1.** Let $\mathfrak{A} = (A, F)$ be an algebra, $a, b \in A$. The compatible tolerance $T_A(a, b) = \bigcap\{T \in LT(\mathfrak{A});\ \langle a, b\rangle \in T\}$ is called the *principal tolerance on* $\mathfrak{A}$ *generated by* $a, b$.

The concept of principal tolerance is clearly an analogon of the *principal congruence* in the sense of [1], [4].

If $R$ is a binary relation on a set $M$ and $S \subseteq M$, denote by $R|_S$ the *restriction* of $R$ onto $S$, i.e. $R|_S = R \cap (S \times S)$. Evidently, the restriction of a compatible tolerance onto a subalgebra is also a tolerance compatible with this subalgebra.

**Definition 2.** A class $\mathscr{C}$ of algebras is said to *satisfy the (Principal) Tolerance Extension Property* if for each $\mathfrak{A} \in \mathscr{C}$ and each subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ every (principal) tolerance compatible with $\mathfrak{B}$ is the restriction of a tolerance compatible with $\mathfrak{A}$.

We abbreviate the Principal Tolerance Extension Property by (PTEP) and the Tolerance Extension Property by (TEP).

**Lemma 1.** *Let* $\mathfrak{B} = (B, F)$ *be a subalgebra of* $\mathfrak{A} = (A, F)$ *and* $T_\alpha \in LT(\mathfrak{A})$ *for* $\alpha \in I$. *Then* $\bigvee_B\{T_\alpha|_B;\ \alpha \in I\} \subseteq (\bigvee_A\{T_\alpha;\ \alpha \in I\})|_B$.

Proof. Let $\langle a, b\rangle \in \bigvee_B\{T_\alpha|_B;\ \alpha \in I\}$. Then $a, b \in B$ and, by Theorem 2 in [6], there exists a polynomial $p(x_1, \ldots, x_n)$ over $F$ and elements $a_1, \ldots, a_n, b_1, \ldots, b_n \in B$ such that $\langle a_i, b_i\rangle \in T_{\alpha_i}$ for some $\alpha_i \in I$ $(i = 1, \ldots, n)$ and $a = p(a_1, \ldots, a_n)$, $b = p(b_1, \ldots, b_n)$. Hence by the same argument $\langle a, b\rangle \in \bigvee_A\{T_\alpha;\ \alpha \in I\}$. As $a, b \in B$, the proof is complete.

**Lemma 2.** *Let* $\mathscr{C}$ *be a class of algebras closed under subalgebras satisfying (PTEP). Then* $T_B(a, b) = T_A(a, b)|_B$ *for each subalgebra* $\mathfrak{B} = (B, F)$ *of* $\mathfrak{A} \in \mathscr{C}$ *and every* $a, b \in B$.

Proof. If $T \in LT(\mathfrak{A})$ and $T_B(a, b) = T|_B$, then clearly $T_A(a, b) \subseteq T$, thus $T_A(a, b)|_B \subseteq T|_B$. Moreover, $\langle a, b\rangle \in T_A(a, b)|_B \in LT(\mathfrak{B})$ implies $T_B(a, b) \subseteq T_A(a, b)|_B$. Consequently,

$$T_B(a, b) \subseteq T_A(a, b)|_B \subseteq T|_B = T_B(a, b)$$

which proves the statement.

**Notation.** Let $(S, \circ)$ be a semigroup and $a \in S$. Put $a^1 = a$, $a^{n+1} = a \circ a^n$ for $n > 0$. Although $(S, \circ)$ need not contain the unit element, let us agree upon the following abbreviation: if $a, b \in S$ and $c = a^m \circ b$ for $m \geqq 0$, then $c = b$ is meant in the case $m = 0$. Analogously for $c = a \circ b^m$.

**Lemma 3.** *Let* $\mathfrak{S} = (S, \circ)$ *be a commutative semigroup and* $a, b \in S$. *Then*

$$T_S(a, b) = \{\langle x, y\rangle;\ x = a^i \circ b^n \circ z^k,\ y = a^j \circ b^m \circ z^k,\ \text{where } i, j, m, n \geqq 0,\ k \in \{0, 1\},\ z \in S,\ i + n + k > 0,\ i + n = j + m\}.$$

Proof. Put

$$R = \{\langle x, y\rangle;\ x = a^i \circ b^n \circ z^k,\ y = a^j \circ b^m \circ z^k,\ \text{where } i, j, n, m \geqq 0,\ k \in \{0, 1\},\ z \in S,\ i + n + k > 0,\ i + n = j + m\}.$$

Clearly $R \subseteq T_S(a, b)$. For $k = 0$, $i = 1$, $n = 0$, $j = 0$, $m = 1$ we have $\langle a, b\rangle \in R$, for $k = 1$, $i = j = n = m = 0$ we have $\langle z, z\rangle \in R$ for each $z \in S$; since $i + n =$

$= j + m$, $R$ is also symmetric, thus $R$ is a tolerance on $S$. We shall prove that $R$ is a tolerance compatible with $\mathfrak{S}$. If $\langle x, y\rangle \in R$, $\langle u, v\rangle \in R$, then $x = a^i \circ b^j \circ z^k$, $y = a^j \circ b^m \circ z^k$, $u = a^{i'} \circ b^{n'} \circ t^{k'}$, $v = a^{j'} \circ b^{m'} \circ t^{k'}$ for prescribed $i, j, n, m, i', j'$, $n', m', k, k'$, thus $x \circ u = a^{i+i'} \circ b^{n+n'} w^{k_1}$, $y \circ v = a^{j+j'} \circ b^{m+m'} \circ w^{k_1}$, where clearly $(i + i') + (n + n') = (j + j') + (m + m')$. Put $k_1 = 1$ and $w = z^k \circ t^{k'}$ for $k + k' > 0$, $k_1 = 0$ for $k + k' = 0$. Thus $k_1 \in \{0, 1\}$ and clearly $(i + i') + (n + n') + k_1 > 0$, hence also $\langle x \circ u, y \circ v\rangle \in R$. Hence $R \in LT(\mathfrak{S})$, thus $T_S(a, b) \subseteq R$ which proves the converse inclusion.

**Lemma 4.** *Let $\mathfrak{M} = (M, \circ)$ be a subsemigroup of the commutative semigroup $\mathfrak{S} = (S, \circ)$ and $T \in LT(\mathfrak{M})$. If*

$$\langle x, y\rangle \in (\bigvee_S\{T_S(a, b); \langle a, b\rangle \in T\})|_M ,$$

*then there exist $a_0, b_0 \in M$ with $\langle a_0, b_0\rangle \in T$ and $\langle x, y\rangle \in T_S(a_0, b_0)|_M$.*

Proof. Let $x, y \in M$ and $\langle x, y\rangle \in \bigvee_S\{T_S(a, b); \langle a, b\rangle \in T\}$. Then, by Theorem 2 in [6], there exist $x_p, y_p \in S$ $(p = 1, \ldots, r)$ and an $r$-ary $(r > 0)$ semigroup polynomial $q$ with $\langle x_p, y_p\rangle \in T_S(a_p, b_p)$ for some $\langle a_p, b_p\rangle \in T$ and $x = q(x_1, \ldots, x_r)$, $y = q(y_1, \ldots, y_r)$. As $T \subseteq M \times M$, clearly $a_p, b_p \in M$. Since $S$ is a commutative semigroup, $q(x_1, \ldots, x_r) = x_1^{s_1} \circ \ldots \circ x_r^{s_r}$, $q(y_1, \ldots, y_r) = y_1^{s_1} \circ \ldots \circ y_r^{s_r}$ for some $s_p \geqq 0$ (and $s_1 + \ldots + s_r > 0$). By Lemma 3, there exist $z_1, \ldots, z_r \in S$ and $i_p, n_p, j_p, m_p \geqq 0$, $k_p \in \{0, 1\}$ such that $x_p = a_p^{i_p} \circ b_p^{n_p} \circ z_p^{k_p}$, $y_p = a_p^{j_p} \circ b_p^{m_p} \circ z_p^{k_p}$, $i_p + n_p = j_p + m_p$, $i_p + n_p + k_p > 0$ for $p = 1, \ldots, r$. If $s_1(i_1 + n_1) + \ldots + s_r(i_r + n_r) \neq 0$ put $i = 1$, $j = 1$, $a_0^i = (a_1^{i_1} \circ b_1^{n_1})^{s_1} \circ \ldots \circ (a_r^{i_r} \circ b_r^{n_r})^{s_r}$, $b_0 = (a_1^{j_1} \circ b_1^{m_1})^{s_1} \circ \ldots \circ (a_r^{j_r} \circ b_r^{m_r})^{s_r}$. In the opposite case, put $i = 0$, $j = 0$. Put $k = 1$, $w^k = z_1^{k_1} \circ \ldots \circ z_r^{k_r}$ provided $k_1 + \ldots + k_r > 0$ and $k = 0$ in the opposite case. Thus $x = a_0^i \circ w^k$, $y = b_0^j \circ w^k$, $i = j$, $k \in \{0, 1\}$, i.e. $\langle x, y\rangle \in T_S(a_0, b_0)$ by Lemma 3. Hence $\langle x, y\rangle \in T_S(a_0, b_0)|_M$. Further, $\langle a_p, b_p\rangle \in T \in LT(\mathfrak{M})$ for $p = 1, \ldots, r$ imply $\langle a_0, b_0\rangle \in T$. As $a_p, b_p \in M$ and $M$ is a subsemigroup, we have $a_0, b_0 \in M$.

**Theorem 1.** *Let $\mathscr{C}$ be a class of commutative semigroups closed under subsemigroups. Then the following conditions are equivalent:*

(a) *$\mathscr{C}$ satisfies (PTEP);*

(b) *$\mathscr{C}$ satisfies (TEP).*

Proof. (b) $\Rightarrow$ ($a$) is trivial. Conversely, let $\mathscr{C}$ satisfy (PTEP), let $\mathfrak{B} = (B, \circ)$ be a subsemigroup of $\mathfrak{A} = (A, \circ) \in \mathscr{C}$ and $T \in LT(\mathfrak{B})$. By Theorem 14 in [6], $T = \bigvee_B\{T_B(a, b); \langle a, b\rangle \in T\}$. Put $T_A = \bigvee_A\{T_A(a, b); \langle a, b\rangle \in T\}$. Then by Lemma 1 and Lemma 2,

$$T_A|_B = (\bigvee_A\{T_A(a, b); \langle a, b\rangle \in T\})|_B \supseteq \bigvee_B\{T_A(a, b)|_B; \langle a, b\rangle \in T\} = \bigvee_B\{T_B(a, b); \langle a, b\rangle \in T\} = T.$$

Conversely, if $\langle x, y\rangle \in T_A|_B$, then $\langle x, y\rangle \in (\bigvee_A\{T_A(a, b);\ \langle a, b\rangle \in T\})|_B$ and, by Lemma 4, there exist $a_0, b_0 \in B$ with $\langle a_0, b_0\rangle \in T$, $\langle x, y\rangle \in T_A(a_0, b_0)|_B$. According to Lemma 2, $\langle x, y\rangle \in T_B(a_0, b_0) \subseteq T$ which proves the converse inclusion.

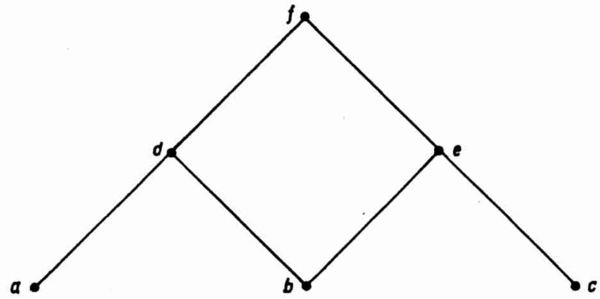


Fig. 1.

**Example.** *The class of all semilattices does not satisfy (TEP).* If e.g. $(S, \circ)$ is semilattice with the diagram in Fig. 1 and $(P, \circ)$ its subsemilattice for $P = \{a, d, f, e, c\}$, then $\langle d, e\rangle \notin T_P(a, c)$. However, $\langle a, c\rangle \in T_S(a, c)$, $\langle b, b\rangle \in T_S(a, c) \Rightarrow \langle d, e\rangle \in T_S(a, c)|_P$. According to Lemma 2, the class of all semilattices does not satisfy (TEP).

In [10], [11] compatible tolerances on semilattices with diagrams in the form of a tree are studied. Let $(S, \circ)$ be a semilattice. Call $(S, \circ)$ a *tree-semilattice*, if it satisfies

(*) $a, b, c \in S$, $a \circ b = b$, $a \circ c = c$ imply $b \circ c = b$ or $b \circ c = c$, which is equivalent to

(**) the Hasse diagram of $(S, \circ)$ (ordering induced by $b \leqq a$ iff $a \circ b = a$) is a *tree*.

Clearly, every subsemilattice of a tree-semilattice is also a tree-semilattice.

**Theorem 2.** *Every class of tree-semilattices closed under subsemilattices satisfies (TEP).*

Proof. Let $(P, \circ)$ be a subsemilattice of a tree-semilattice $(S, \circ)$. With respcct to the idempotency of semilattice operation, we obtain by Lemma 3:

$$T_S(a, b) = \{\langle x, y\rangle;\ x = a^i \circ b^n \circ z^k,\ y = a^j \circ b^m \circ z^k, \text{ where } z \in S \text{ and }$$
$$i, j, k, n, m \in \{0, 1\},\ i + n + k \neq 0,\ i + n = 0 \text{ iff } j + m = 0\}$$

for each $a, b \in P$. Clearly $T_S(a, b)|_P \supseteq T_P(a, b)$. Suppose $\langle c, d\rangle \in T_S(a, b)|_P$, $\langle c, d\rangle \notin T_P(a, b)$. Thus $c = a^i \circ b^n \circ z^k$, $d = a^j \circ b^m \circ z^k$ for $z \in S$ and $i, j, n, m$ determined in the above definition of $T_S(a, b)$. Since $c \circ z^k = c$, $d \circ z^k = d$, we have by (*) $c \circ d = c$ or $c \circ d = d$ or $k = 0$. If $k = 0$, then $c = a^i \circ b^n$, $d = a^j \circ b^m$, $a, b \in P$, $i + n \neq 0$, $j + m \neq 0$, thus clearly $\langle c, d\rangle \in T_P(a, b)$, which is a contradiction.

Suppose $c \circ d = c$.

1°. If $j = m = 0$, then $i = n = 0$, thus $k = 1$. Hence $c = z = d$. Moreover, $c, d \in T_S(a, b)|_P$ implies $z \in P$, i.e. $\langle c, d\rangle \in T_P(a, b)$.

2°. If $m = 1$ and $i = j = 0$, then $n = 1$. Hence $c = b \circ z^k$, $d = b \circ z^k$ and $\langle c, d\rangle \in T_P(a, b)$ analogously as in 1°.

If $m = 1$ and $i = 1$ or $j = 1$, then $d \circ b = d$ and, by $(*)$, $d \circ a = a \circ b \circ z^k = (a^i \circ b^n \circ z^k) \circ (a^j \circ b^m \circ z^k) = c \circ d = c$, hence $\langle c, d\rangle = \langle a \circ d,\ b \circ d\rangle \in T_P(a, b)$.

3°. If $j = 1$ and $n = m = 0$, then $i = 1$. Hence $c = a \circ z^k$, $d = a \circ z^k$ and $\langle c, d\rangle \in T_P(a, b)$ as in 2°.

If $j = 1$ and $n = 1$ or $m = 1$, then $d \circ a = d$ and by $(*)$, $d \circ b = a \circ b \circ z^k = (a^i \circ b^n \circ z^k) \circ (a^j \circ b^m \circ z^k) = c \circ d = c$; thus also $\langle c, d\rangle = \langle b \circ d,\ a \circ d\rangle \in T_P(a, b)$.

The contradiction is obtained in all cases for $c \circ d = c$. For $c \circ d = d$ the proof is analogous. Hence $T_S(a, b)|_P \subseteq T_P(a, b)$, thus $T_S(a, b)|_P = T_P(a, b)$ for every tree-semilattice $(S, \circ)$ and each of its subsemilattices $(P, \circ)$ and each $a, b \in P$. Consequently, the class of tree-semilattices closed under subsemilattices satisfies (PTEP), and, by Theorem 1, the statement is proved.

We can easily show that in the case of lattices the assertion analogous to Theorem 1 is not true.

**Proposition.** *Let $\mathcal{K}$ be a class of lattices closed under sublattices. Then the following conditions are equivalent:*

(a) *$\mathcal{K}$ satisfies (TEP);*

(b) *$L \in \mathcal{K}$ implies $L$ is a chain.*

Proof. The implication (b) ⇒ (a) is clear. Conversely, let $\mathcal{K}$ satisfy (TEP), $L \in \mathcal{K}$ and let us assume that $L$ is not a chain. Then there exist non-comparable $a, b \in L$. Consider the sublattice

$S = \{a \wedge b, a, a \vee b\}$ and the relation

$T_S = \{\langle a, a\rangle, \langle a \wedge b, a \wedge b\rangle, \langle a \vee b, a \vee b\rangle, \langle a \wedge b, a\rangle, \langle a, a \wedge b\rangle, \langle a \vee b, a\rangle, \langle a, a \vee b\rangle\}$.

Evidently, $T_S$ is a tolerance compatible with $S$ and $\langle a \wedge b, a \vee b\rangle \notin T_S$. Suppose that there exists a tolerance $T$ compatible with $L$ such that $T|_S = T_S$. Then $\langle a \wedge b, a\rangle \in T$, $\langle b, b\rangle \in T$, thus also $\langle b, a \vee b\rangle \in T$. As $\langle a, a \vee b\rangle \in T$, we obtain $\langle a \wedge b, a \vee b\rangle \in T$ which contradicts $T|_S = T_S$. Thus $\mathcal{K}$ does not satisfy (TEP) contrary to the assumption.

**Remark.** We can give an example of the class of algebras satisfying (PTEP) and does not satisfying (TEP). If $\mathcal{C}$ is a class of all distributive lattices, then by Proposition, $\mathcal{C}$ does not satisfy (TEP). On the other hand, $\mathcal{C}$ satisfies (PTEP) since $\mathcal{C}$ satisfies the Principal Congruence Extension Property (see [1]) and $T_L(a, b) = \Theta_L(a, b)$ for each $L \in \mathcal{C}$, $a, b \in L$ as it was shown in [16] ($\Theta_L(a, b)$ is the *principal congruence on L generated by a, b*).