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ON AN INTEGRAL FORMULA

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One of the main tools used in the global differential geometry is the integral formula (1.14.1) of [1] for the so-called Codazzi-tensors. In the following paper, I present a more (possibly the most) general integral formula; the above mentioned formula appears then as its simple corollary.

Given a Riemannian manifold (M, g), $\dim M = n$. Let ∇ be its associated Euclidean connection. In each coordinate neighborhood U of M, we may choose orthonormal sections $\{v_1 \ldots v_n\}$ of $T(U)$ such that ∇ is given by

$$\nabla m = \omega^i v_i, \quad \nabla v_i = \omega_i^j v_j ; \quad \omega_i^j + \omega_j^i = 0 \tag{1}$$

and we have

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad d\omega_i^j = \omega_i^k \wedge \omega_k^j - \tfrac{1}{2} R_{ikl}^j \omega^k \wedge \omega^l, \quad R_{ikl}^j + R_{ilk}^j = 0 . \tag{2}$$

The curvature tensor (at $m \in U$)

$$R : T_m(M) \times T_m(M) \times T_m(M) \to T_m(M), \tag{3}$$

$$R(y^i v_i, z^i v_i)(x^i v_i) = R_{ikl}^j x^i y^k z^l v_j$$

satisfies (2_3) and

$$R_{ikl}^j + R_{jkl}^i = 0, \quad R_{ikl}^j = R_{kij}^l, \quad R_{ikl}^j + R_{ilj}^k + R_{ijk}^l = 0, \tag{4}$$

i.e.,

$$R(y, z)x + R(z, y)x = 0, \quad g(x, R(z, t)y) + g(y, R(z, t)x) = 0, \tag{5}$$

$$g(y, R(z, t)x) = g(t, R(x, y)z),$$

$$g(y, R(z, t)x) + g(z, R(t, y)x) + g(t, R(y, z)x) = 0$$

for $x, y, z, t \in T_m(M)$.

Let (E, g^*), $\dim E = n + N$ be a Euclidean vector bundle over M; on E, let a metric connection ∇^* satisfying

$$\nabla^*_x g^*(\xi, \eta) = g^*(\nabla^*_x \xi, \eta) + g^*(\xi, \nabla^*_x \eta) \tag{6}$$

for each $x \in T_m(M)$, $m \in M$ and local sections $\xi, \eta : M \to E$ defined in a neighborhood of m be given. Suppose that E is trivial over U, and let $w_\alpha : U \to E$ $(\alpha, \beta, \ldots = 1, \ldots, N)$ be orthonormal sections; ∇^* is then given by

$$\nabla^* w_\alpha = \tau^\beta_\alpha w_\beta, \quad \tau^\beta_\alpha + \tau^\alpha_\beta = 0. \tag{7}$$

We have

$$d\tau^\beta_\alpha = \tau^\gamma_\alpha \wedge \tau^\beta_\gamma - \tfrac{1}{2} S^\beta_{\alpha ij} \omega^i \wedge \omega^j, \quad S^\beta_{\alpha ij} + S^\beta_{\alpha ji} = 0, \tag{8}$$

the curvature tensor of ∇^* being

$$\begin{gathered} S : T_m(M) \times T_m(M) \times E_m \to E_m, \\ S(x^i v_i, y^i v_i)(\xi^\alpha w_\alpha) = S^\beta_{\alpha ij} \xi^\alpha x^i y^j w_\beta. \end{gathered} \tag{9}$$

Evidently,

$$S^\beta_{\alpha ij} + S^\alpha_{\beta ij} = 0, \tag{10}$$

i.e., S satisfies

$$S(x, y)\,\xi + S(y, x)\,\xi = 0, \quad g^*(\xi, S(x, y)\,\eta) + g^*(\eta, S(x, y)\,\xi) = 0 \tag{11}$$

for each $\xi, \eta \in E_m$; $x, y \in T_m(M)$.

For each $m \in M$, let a p-linear mapping

$$F : \times^p T_m(M) \to E_m \tag{12}$$

be given. The mappings

$$F^{(1)} : \times^{p+1} T_m(M) \to E_m, \quad F^{(2)} : \times^{p+2} T_m(M) \to E_m \tag{13}$$

let be introduced by

$$\begin{gathered} F^{(1)}(x_{(1)}, \ldots, x_{(p)}, x) = \nabla^*_x F(x'_{(1)}, \ldots, x'_{(p)}) - \\ - \sum_{r=1}^{p} F(x_{(1)}, \ldots, x_{(r-1)}, \nabla_x x'_{(r)}, x_{(r+1)}, \ldots, x_{(p)}), \\ F^{(2)} = (F^{(1)})^{(1)}, \end{gathered} \tag{14}$$

where $x_{(1)}, \ldots, x_{(p)}, x \in T_m(M)$ and $x'_{(1)}, \ldots, x'_{(p)}$ are local sections of $T(M)$ such that $x'_{(1)}(m) = x_{(1)}, \ldots, x'_{(p)}(m) = x_{(p)}$.

Lemma. *The mappings $F^{(1)}$ and $F^{(2)}$ are well defined, i.e., they do not depend on the choice of the sections $x'_{(1)}, \ldots, x'_{(p)}$. Further,*

$$(15) \qquad F^{(2)}(x_{(1)}, \ldots, x_{(p)}, y, x) - F^{(2)}(x_{(1)}, \ldots, x_{(p)}, x, y) =$$
$$= \sum_{r=1}^{d} F(x_{(1)}, \ldots, x_{(r-1)}, R(x, y)\, x_{(r)}, x_{(r+1)}, \ldots, x_{(p)}) - S(x, y)\, F(x_{(1)}, \ldots, x_{(p)}) .$$

Proof. Let us write, in U,

$$(16) \qquad F(x^i_{(1)}v_i, \ldots, x^i_{(p)}v_i) = F^\alpha_{i_1 \ldots i_p} x^{i_1}_{(1)} \ldots x^{i_p}_{(p)} w_\alpha .$$

The components $F^\alpha_{i_1 \ldots i_p i}$ let be introduced by

$$(17) \qquad \mathrm{d}F^\alpha_{i_1 \ldots i_p} - \sum_{r=1}^{p} F^\alpha_{i_1 \ldots i_{r-1} i i_{r+1} \ldots i_p} \omega^i_{i_r} + F^\beta_{i_1 \ldots i_p} \tau^\alpha_\beta = F^\alpha_{i_1 \ldots i_p i} \omega^i .$$

Then it is easy to see that

$$(18) \qquad F^{(1)}(x^i_{(1)}v_i, \ldots, x^i_{(p)}v_i, x^i v_i) = F^\alpha_{i_1 \ldots i_p i} x^{i_1}_{(1)} \ldots x^{i_p}_{(p)} x^i \omega_\alpha .$$

The exterior differentiation of (17) yields

$$(19) \qquad (\mathrm{d}F^\alpha_{i_1 \ldots i_p i} - F^\alpha_{i_1 \ldots i_p j} \omega^j_i - \sum_{r=1}^{p} F^\alpha_{i_1 \ldots i_{r-1} j i_{r+1} \ldots i_p i} \omega^j_{i_r} + F^\beta_{i_1 \ldots i_p i} \tau^\alpha_\beta) \wedge \omega^i =$$
$$= \tfrac{1}{2}\Big(\sum_{r=1}^{p} F^\alpha_{i_1 \ldots i_{r-1} k i_{r+1} \ldots i_p} R^k_{i_r j i} - F^\beta_{i_1 \ldots i_p} S^\alpha_{\beta j i}\Big)\, \omega^j \wedge \omega^i .$$

The components $F^\alpha_{i_1 \ldots i_p i j}$ let be defined by

$$(20) \qquad \mathrm{d}F^\alpha_{i_1 \ldots i_p i} - F^\alpha_{i_1 \ldots i_p j} \omega^j_i - \sum_{r=1}^{p} F^\alpha_{i_1 \ldots i_{r-1} j i_{r+1} \ldots i_p i} \omega^j_{i_r} + F^\beta_{i_1 \ldots i_p i} \tau^\alpha_\beta = F^\alpha_{i_1 \ldots i_p i j} \omega^j .$$

Clearly,

$$(21) \qquad F^{(2)}(x^i_{(1)}v_i, \ldots, x^i_{(p)}v_i, x^i v_i, y^i v_i) = F^\alpha_{i_1 \ldots i_p i j} x^{i_1}_{(1)} \ldots x^{i_p}_{(p)} x^i y^j w_\alpha ,$$

and (19) implies

$$(22) \qquad F^\alpha_{i_1 \ldots i_p j i} - F^\alpha_{i_1 \ldots i_p i j} = \sum_{r=1}^{p} F^\alpha_{i_1 \ldots i_{r-1} k i_{r+1} \ldots i_p} R^k_{i_r i j} - F^\beta_{i_1 \ldots i_p} S^\alpha_{\beta i j} ,$$

i.e., (15). QED.

Let us introduce the following notation. For $H : \times^q T_m(M) \to \mathbb{R}$ a q-linear mapping, $q \geqq 2$, $1 \leqq q_1 < q_2 \leqq q$, write

$$(23) \qquad H(x_{(1)}, \ldots, x_{(q_1-1)}, A, x_{(q_1+1)}, \ldots, x_{(q_2-1)}, A, x_{(q_2+1)}, \ldots, x_{(q)}) =$$
$$= \sum_{i=1}^{n} H(x_{(1)}, \ldots, x_{(q_1-1)}, v_i, x_{(q_1+1)}, \ldots, x_{(q_2-1)}, v_i, x_{(q_2+1)}, \ldots, x_{(q)}) ,$$

where $\{v_1, \ldots, v_n\}$ is an arbitrary orthonormal basis of $T_m(M)$; (23) is well defined.

Theorem. *Let (M, g), (E, g^*, ∇^*) be as above. At each $m \in M$, let p-linear mappings $F, G : \times^p T_m(M) \to E_m$ be given. On M, consider the 1-form*

$$\varphi(x) = g^*(F(A_1, \ldots, A_{p-1}, x),\ G^{(1)}(A_1, \ldots, A_{p-1}, B, B)) - \tag{24}$$
$$- g^*(F(A_1, \ldots, A_{p-1}, B),\ G^{(1)}(A_1, \ldots, A_{p-1}, x, B)) .$$

Let ∂M be the boundary of M and $*$ the Hodge operator. Then

$$\int_{\partial M} * \varphi = \int_M \{g^*(F^{(1)}(A_1, \ldots, A_{p-1}, B, B),\ G^{(1)}(A_1, \ldots, A_{p-1}, C, C)) - \tag{25}$$
$$- g^*(F^{(1)}(A_1, \ldots, A_{p-1}, B, C),\ G^{(1)}(A_1, \ldots, A_{p-1}, C, B)) +$$
$$+ g^*(F(A_1, \ldots, A_{p-1}, B),\ \sum_{r=1}^{p-1} G(A_1, \ldots, A_{r-1}, R(B. C)\, A_r, A_{r+1}, \ldots, A_{p-1}, C) +$$
$$+ G(A_1, \ldots, A_{p-1}, R(B, C)\, C) - S(B, C)\, G(A_1, \ldots, A_{p-1}, C))\}\, \mathrm{d}o ,$$

$\mathrm{d}o$ being the volume element on M.

Proof. On U we get

$$\varphi = \delta_{\alpha\beta}\delta^{i_1 j_1} \ldots \delta^{i_{p-1} j_{p-1}} \delta^{kl} (F^\alpha_{i_1 \ldots i_{p-1} i} G^\beta_{j_1 \ldots j_{p-1} kl} - F^\alpha_{i_1 \ldots i_{p-1} k} G^\beta_{j_1 \ldots j_{p-1} il})\, \omega^i . \tag{26}$$

Hence

$$\mathrm{d} * \varphi = \delta_{\alpha\beta}\delta^{i_1 j_1} \ldots \delta^{i_{p-1} j_{p-1}} \delta^{kl} \delta^{ij} (F^\alpha_{i_1 \ldots i_{p-1} ij} G^\beta_{j_1 \ldots j_{p-1} kl} - F^\alpha_{i_1 \ldots i_{p-1} kj} G^\beta_{j_1 \ldots j_{p-1} il} + \tag{27}$$
$$+ F^\alpha_{i_1 \ldots i_{p-1} i} G^\beta_{j_1 \ldots j_{p-1} klj} - F^\alpha_{i_1 \ldots i_{p-1} k} G^\beta_{j_1 \ldots j_{p-1} ilj})\, \mathrm{d}o =$$
$$= \delta_{\alpha\beta}\delta^{i_1 j_1} \ldots \delta^{i_{p-1} j_{p-1}} \delta^{kl} \delta^{ij} \{F^\alpha_{i_1 \ldots i_{p-1} ij} G^\beta_{j_1 \ldots j_{p-1} kl} - F^\alpha_{i_1 \ldots i_{p-1} kj} G^\beta_{j_1 \ldots j_{p-1} il} +$$
$$+ F^\alpha_{i_1 \ldots i_{p-1} i} (G^\beta_{j_1 \ldots j_{p-1} klj} - G^\beta_{j_1 \ldots j_{p-1} kjl})\}\, \mathrm{d}o ;$$

using (22) for G, we get (25). QED.

Corollary. *Let (M, g) be a Riemannian manifold. Let $T(x, y)$ be a symmetric tensor on M, and let there be a function $\tau : \partial M \to \mathbb{R}$ such that $T(x, y) = \tau g(x, y)$ on ∂M. Further, let $T(x, y)$ have, at $m \in M$, orthonormal eigen-vectors $\{v_1, \ldots, v_n\}$ with the corresponding eigen-values $k_1, \ldots, k_n$, and let K_{ij} be the sectional curvature corresponding to the 2-direction $\{v_i, v_j\}$. Then*

$$\int_M \{T^{(1)}(A, B, B)\, T^{(1)}(A, C, C) - T^{(1)}(A, B, C)\, T^{(1)}(A, C, B) - \tag{28}$$
$$- \sum_{1 \leq i < j \leq n} K_{ij}(k_i - k_j)^2\}\, \mathrm{d}o = 0 .$$