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ON A PROBLEM OF R. HÄGGKVIST CONCERNING EDGE-COLOURINGS OF GRAPHS

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At the 5th Hungarian Colloquium on Combinatorics in Keszthely in 1976 R. HÄGGKVIST has proposed the following problem [1]:

Let $Q(n, G)$ be the set of n-line-colourings of G. Let $q \in Q(n, G)$. Define $L(q)$ ($l(q)$) as the maximal (minimal) length of a cycle with edges from two of q's line-colour classes. Put

$$L(n, G) = \min_{q \in Q(n,G)} L(q), \quad l(n, G) = \max_{q \in Q(n,Q)} l(q).$$

Give bounds on $L(n, G)$ and $l(n, G)$ for reasonable defined graphs G. Especially: Is $L(n, K_{n,n}) = 2n$?

In this paper we shall study $L(n, K_{n,n})$ for n which is a power of 2. Instead of "line" we shall say "edge".

Theorem. *Let $n = 2^m$, where m is a positive integer. Then $L(n, K_{n,n}) = 4$.*

Proof. For each positive integer m denote $G(m) = K_{n,n}$, where $n = 2^m$. Denote $N = \{1, 2, \ldots, n\}$, $P = \{n + 1, n + 2, \ldots, 2n\}$. The vertices of $G(m)$ are $u_1, \ldots, u_n$, $v_1, \ldots, v_n$, the edges are u_iv_j for each i and j from N. For each $G(m)$ we shall introduce an edge-colouring $q(m)$ by n colours such that no vertex of $G(m)$ is incident with any two edges of the same colour. We define it recurrently. In the graph $G(1)$ we colour the edges u_1v_1, u_2v_2 by the colour 1, the edges u_1v_2, u_2v_1 by the colour 2. Now let the colouring $q(m)$ of $G(m)$ by the colours from N be given for some m; we shall construct the colouring $Q(m + 1)$ of the edges of $G(m + 1)$. Consider four graphs H_1, H_2, H_3, H_4 which are all isomorphic to $G(m)$. The vertices of H_1 are denoted in the same way as in $G(m)$; we may consider $G(m)$ and H_1 as the same graph. The vertices of H_2 are $u_{n+1}, \ldots, u_{2n}, v_{n+1}, \ldots, v_{2n}$ and the edges are u_iv_j for all i and j from P. The vertices of H_3 are $u_1, \ldots, u_n, v_{n+1}, \ldots, v_{2n}$ and the edges are u_iv_j for each i from N and each j from P. The vertices of H_4 are $u_{n+1}, \ldots, u_{2n}$, $v_1, \ldots, v_n$ and the edges are u_iv_j for each i from P and each j from N. Now we shall colour the edges of the graphs H_1, H_2, H_3, H_4. The graph H_2 is considered the same as $G(m)$, therefore its edges will be coloured by the colours from N in the same way as the edges of $G(m)$. Also the edges of H_2 will be coloured by the colours from N;