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NOTES ON LATTICE CONGRUENCES

IVAN CHAJDA, Přerov

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It is well-known that each ideal of a lattice L is a kernel of at least one congruence relation on L if and only if L is distributive (see e.g. [1]), and that there exists a one-to-one correspondence between congruences and ideals for relatively complementary distributive lattices (see [2]). An approach adopted in [3] enables us to investigate the relationship between congruences and ideals also for modular lattices.

Definition 1. Let J be an ideal of a given lattice L. Denote $a \vee J = \{a \vee j; j \in J\}$. A binary relation T_J on L defined by the rule

$\langle x, y \rangle \in T_J$ if and only if there exists $u \in L$ with $x, y \in u \vee J$ is said to be *induced by the ideal J.*

It is clear that T_J is a symmetrical relation on L. Further, for each $a \in L$ and an arbitrary $j \in J$ we have $a = a \vee (a \wedge j)$; clearly $a \wedge j \in J$, thus $a \in a \vee J$, which implies the reflexivity of T_J. Thus T_J is a tolerance relation on L (see [3]). In [3], conditions of the compatibility of T_J are studied (for the compatibility, see [4]). We shall now investigate the conditions for T_J to be a congruence relation. By Definition 1, if T_J is a congruence relation, J is a kernel of T_J.

Theorem 1. *Let L be a lattice and J an ideal of L. If the relation T_J induced by J is compatible on L, then T_J is a congruence relation on L.*

Proof. As T_J is reflexive, symmetrical and compatible, we must prove only its transitivity. Suppose $a, b, c \in L$ and $\langle a, b \rangle \in T_J$, $\langle b, c \rangle \in T_J$. Then there exist $u, v \in L$ and $i, j, k, l \in J$ with $a = u \vee i$, $b = u \vee j$, $b = v \vee k$, $c = v \vee l$. As $i, l \in J$, we have

$$(1^\circ) \qquad \langle i, l \rangle \in T_J\,.$$

From $u \in u \vee J$, $a \in u \vee J$ it follows $\langle u, a \rangle \in T_J$. Analogously it can be proved that $\langle u, b \rangle \in T_J$, $\langle v, b \rangle \in T_J$, $\langle v, c \rangle \in T_J$. As T_J is symmetrical, also $\langle b, v \rangle \in T_J$. From the compatibility of T_J then

(2°) $$\langle u, b\rangle \in T_J,\ \langle b, v\rangle \in T_J \Rightarrow \langle (u \wedge b), (b \wedge v)\rangle \in T_J\,.$$

From $b = u \vee j$ we have $b \geqq u$, from $b = v \vee k$ then $b \geqq v$. Then (2°) implies $\langle u, v\rangle \in T_J$, which together with (1°) implies

$$\langle (u \vee i), (v \vee l)\rangle \in T_J\,,$$

thus $\langle a, c\rangle \in T_J$. Hence T_J is transitive.

Lemma 1. *Let L be a lattice and J its ideal. Let T_J be the relation induced by J. If $a, b, c, d \in L$ and $\langle a, b\rangle \in T_J$, $\langle c, d\rangle \in T_J$, then*

$$\langle (a \vee c), (b \vee d)\rangle \in T_J\,.$$

Proof. If $\langle a, b\rangle \in T_J$, $\langle c, d\rangle \in T_J$, then $a = u \vee i$, $b = u \vee j$, $c = v \vee k$, $d = v \vee l$ for some $u, v \in L$, $i, j, k, l \in J$. Hence $a \vee c = (u \vee v) \vee (i \vee k)$, $b \vee d = (u \vee v) \vee (j \vee l)$, thus $a \vee c \in (u \vee v) \vee J$ and $b \vee d \in (u \vee v) \vee J$, i.e. $\langle (a \vee c), (b \vee d)\rangle \in T_J$.

Lemma 2. *Let L be a lattice, J an ideal of L and T_J the relation induced by J. If $a, b \in L$ and $\langle a, b\rangle \in T_J$, then $a = (a \wedge b) \vee i$, $b = (a \wedge b) \vee j$ for some $i, j \in J$.*

Proof. If $\langle a, b\rangle \in T_J$, then by Definition 1, $a = u \vee i$, $b = u \vee j$ for some $u \in L$, $i, j \in J$. Hence $a \geqq a \wedge b \geqq u$, $a \geqq i$, thus $a = a \vee i \geqq (a \wedge b) \vee i \geqq u \vee i = a$, i.e. $a = (a \wedge b) \vee i$. Analogously it can be proved that $b = (a \wedge b) \vee j$.

Lemma 3. *Let L be a modular lattice, J an ideal of L and T_J the relation induced by J. Let $c, d \in L$ and $c \leqq d$. If $\langle c, d\rangle \in T_J$ and T_J is transitive, then $\langle (a \wedge d), (a \wedge c)\rangle \in T_J$ for each $a \in L$.*

Proof. Let $\langle c, d\rangle \in T_J$. Then there exist $u \in L$ and $i, j \in J$ with $c = u \vee j$, $d = u \vee i$. As $c \leqq d$ and L is modular, we have $j \leqq i$, thus $d = c \vee i$.

Put $x = a \wedge d$, $y = x \vee c$, $t = y \wedge i$. Then $y \geqq c$, $d \geqq x$. From these inequalities and by the modularity of L we obtain

$$c \vee t = c \vee (y \wedge i) = (c \vee i) \wedge y = d \wedge y = d \wedge (x \vee c) =$$
$$= (d \wedge x) \vee c = (d \wedge (a \wedge d)) \vee c = (a \wedge d) \vee c = y\,.$$

As $t \in J$, this implies $\langle y, c\rangle \in T_J$. From $y = c \vee t$, $t \leqq x \vee t$ and by the modularity of L it follows

$$((x \vee t) \wedge c) \vee t = (x \vee t) \wedge (c \vee t) = (x \vee t) \wedge (c \vee t \vee t) =$$
$$= (x \vee t) \wedge (y \vee t) = (x \vee t) \wedge (x \vee c \vee t) = x \vee t\,,$$

hence $\langle (x \vee t) \wedge c, (x \vee t) \rangle \in T_J$. Clearly also $\langle (x \vee t), x \rangle \in T_J$. By the transitivity of T_J, $\langle (x \vee t) \wedge c, x \rangle \in T_J$. By Lemma 2, there exists $q \in J$ with $x = (x \wedge ((x \vee t) \wedge c)) \vee q$. However, $x \wedge ((x \vee t) \wedge c) = x \wedge c$, thus $x = (x \wedge c) \vee q$, i.e. $\langle x, (x \wedge c) \rangle \in T_J$. As $x \wedge c = a \wedge d \wedge c = a \wedge c$, this implies $\langle (a \wedge d), (a \wedge c) \rangle \in T_J$.

Theorem 2. *Let L be a modular lattice, J its ideal and T_J the relation induced by J. If T_J is transitive, then it is a congruence relation on L.*

Proof. If T_J is transitive, it is an equivalence relation on L. It remains to prove the compatibility of T_J. Let $a, b, c, d \in L$ and $\langle a, b \rangle \in T_J$, $\langle c, d \rangle \in T_J$. By Lemma 1, we must prove only that T_J preserves the operation $\wedge$. By Lemma 2, there exist $i, j \in J$ with $a = (a \wedge b) \vee i$, $b = (a \wedge b) \vee j$. By Theorem 1 in [3] ,$(a \wedge b) \vee J$ is a convex sublattice of L, thus

$$a \in (a \wedge b) \vee J,\ b \in (a \wedge b) \vee J \Rightarrow a \vee b \in (a \wedge b) \vee J,$$

hence $\langle (a \wedge b), (a \vee b) \rangle \in T_J$. Analogously it can be proved that $\langle (c \wedge d), (c \vee d) \rangle \in T_J$. By Lemma 3, this implies

$$\langle (a \wedge c \wedge d), (a \wedge (c \vee d)) \rangle \in T_J .$$

Thus $a \wedge c \wedge d \in u_0 \vee J$, $a \wedge (c \wedge d) \in u_0 \vee J$ for some $u_0 \in L$. By Theorem 1 in [3], $u_0 \vee J$ is a convex sublattice of L; clearly

$$a \wedge c \wedge d \leqq a \wedge c \leqq a \wedge (c \vee d),\quad a \wedge c \wedge d \leqq a \wedge d \leqq a \wedge (c \vee d),$$

thus also $a \wedge c \in u_0 \vee J$ and $a \wedge d \in u_0 \vee J$, hence $\langle (a \wedge c), (a \wedge d) \rangle \in T_J$. Analogously also $\langle (a \wedge d), (b \wedge d) \rangle \in T_J$, thus the transitivity of T_J implies $\langle (a \wedge c), (b \wedge d) \rangle \in T_J$, i.e. T_J is a compatible relation.

Corollary. *Let L be a modular lattice, J an ideal of L and T_J the relation induced by J. Then the following assertions are equivalent:*

(a) T_J is a compatible relation on L.
(b) T_J is transitive.
(c) T_J is an equivalence relation on L.
(d) T_J is a congruence relation on L with the kernel J.

Proof. The implication (d) $\Rightarrow$ (a) is clear and (a) $\Leftrightarrow$ (d) follows by Theorem 1. The implication (d) $\Rightarrow$ (c) $\Rightarrow$ (b) is also clear and (b) $\Rightarrow$ (d) by Theorem 2.

The following concept is transferred from [3]: