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RELATIVE BICOMPLEMENTS AND TOLERANCE EXTENSION PROPERTY IN DISTRIBUTIVE LATTICES

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It is a well-known result of A. DAY that if L is a sublattice of a distributive lattice D then every congruence relation on L can be extended to a congruence relation on the whole D. The purpose of the present paper is to give a characterization of such pairs $[D, L]$ that every compatible tolerance relation on L can be extended on D.

By a tolerance relation we mean a reflexive and symmetric binary relation. It need not be transitive. The concept was introduced by ZEEMAN and compatible tolerance relations on algebras were for the first time studied by B. ZELINKA. Some properties of compatible tolerance relations on distributive lattices are important: Let D be a distributive lattice, T a compatible tolerance relation on D. Then

1) $\{x \in D \mid [x, a] \in T\}$ forms a convex sublattice of D for each fixed $a \in D$;

2) $[x, y] \in T \Leftrightarrow [x \wedge y, x \vee y] \in T$ for arbitrary $x, y \in D$;

3) the intersection of an arbitrary set of compatible tolerance relations on D is again a compatible tolerance relation on D;

4) to every binary relation R on D there exists a least compatible tolerance relation $T(R)$ on D containing R. $T(R)$ will be called the compatible tolerance relation generated by R. $T(R) = \{[a, b] \mid$ there exists an $(m + n)$-ary term t such that $a = t(a_1, \ldots, a_m, x_1, \ldots, x_n)$, $b = t(b_1, \ldots, b_m, x_1, \ldots, x_n)$ for some $[a_i, b_i] \in R \cup R^*\}$ where $[x, y] \in R^* :\Leftrightarrow [y, x] \in R$.

The following result is well-known:

Lemma 1. *Let D be a distributive lattice, J an ideal (dual ideal) in D, $a \in D \setminus J$. Then there exists an ideal (dual ideal) I, $J \subseteq I$, $a \notin I$, which is maximal with this property. I is prime.*

Lemma — Definition. *Let D be a distributive lattice, $a, b \in D$, $a < b$, I an ideal in D not containing b, F a dual ideal in D not containing a, which are both maximal* *with these properties and let $D = I \cup F$. Then $T = (I \times I) \cup (F \times F)$ is a compatible tolerance relation on D.*

Such tolerance relations will be called to be *of the type* τ.

Definition. By a *relative bicomplement* of a subinterval $\langle \bar{a}, \bar{b} \rangle$ of an interval $\langle a, b \rangle$ we mean an element x with the property $x \wedge \bar{a} = a$, $x \vee \bar{b} = b$.

A sublattice L of a lattice D is said to be *closed under relative bicomplements* if whenever $\langle \bar{a}, \bar{b} \rangle$ has a relative bicomplement in $\langle a, b \rangle$ in D, $a, b, \bar{a}, \bar{b} \in L$, then $\langle \bar{a}, \bar{b} \rangle$ has a relative bicomplement in $\langle a, b \rangle$ in L, too.

Let L be a sublattice of a lattice D, T a compatible tolerance relation on L. A compatible tolerance relation $\bar{T}$ on D is said to be an *extension* of T if $\bar{T}|_L = T$.

$[D, L]$ is said to have the *tolerance extension property* (TEP) (τ*-tolerance extension property* (τ TEP)) if every compatible tolerance relation on L(of the type τ) has an extension on D.

Lemma 2. *Let T be a compatible tolerance relation on the distributive lattice D. Let $a, b \in D$, $[a, b] \notin T$, $a < b$. Then there exists a compatible tolerance relation T_{ab} of the type τ containing T and not containing $[a, b]$.*

Proof. Suppose $a, b \in D$, $a < b$, T is a compatible tolerance relation on D, $[a, b] \notin T$. Let $A = \{x \in D \mid [x, a] \in T\}$, let J be the ideal in D generated by A. Clearly $b \notin J$, hence there exists (by Lemma 1) an ideal I_{ab} containing J and not containing b which is maximal with this property, I_{ab} prime. Let B denote the set $\{x \in D \mid [x, y] \in T \text{ for some } y \in D \setminus I_{ab}\}$. $B \neq \emptyset$ for $b \in B$. B is a dual ideal in D not containing a:

(i) $x, y \in B \Rightarrow \exists x', y' \in D \setminus I_{ab}$, $[x, x'], [y, y'] \in T \Rightarrow [x \wedge y, x' \wedge y'] \in T$, $x' \wedge \wedge y' \in D \setminus I_{ab}$ (for I_{ab} is prime) $\Rightarrow x \wedge y \in B$;

(ii) $x \in B$, $x \leqq y \Rightarrow \exists x' \in D \setminus I_{ab}$, $[x, x'] \in T \Rightarrow [y \vee x', y \vee x] \in T$, $y \vee x = y$, $y \vee x' \in D \setminus I_{ab} \Rightarrow y \in B$;

(iii) $a \notin B$ for $[x, a] \in T \Rightarrow x \in A \subseteq J \subseteq I_{ab}$.

By Lemma 1, there exists a dual ideal F_{ab} containing B and not containing a which is maximal with this property. $T \subseteq T_{ab} = (I_{ab} \times I_{ab}) \cup (F_{ab} \times F_{ab})$. Clearly $[a, b] \notin \notin T_{ab}$. Q.E.D.

Proposition 1. *Let T be a compatible tolerance relation on a distributive lattice L. Then T can be represented as an intersection of a set of compatible tolerance relations of the type τ.*

Proof. Let $C = (L \times L) \setminus T$. Clearly $T \subseteq \bigcap_{\substack{[a,b] \in C \\ a<b}} T_{ab}$. Conversely, if $[x, y] \notin T$, then by (2) $[x \wedge y, x \vee y] \notin T$, hence $[x \wedge y, x \vee y] \notin T_{x \wedge y, x \vee y}$, again by (2) $[x, y] \notin T_{x \wedge y, x \vee y}$ and therefore $[x, y] \notin \bigcap_{\substack{[a,b] \in C \\ a<b}} T_{ab}$. Consequently $T = \bigcap_{\substack{[a,b] \in C \\ a<b}} T_{ab}$. Q.E.D.

Proposition 2. $[D, L]$ *has* TEP *if and only if it has* τ TEP.

Proof. $\Rightarrow$ Clear.

$\Leftarrow$ Let T be a compatible tolerance relation on L. By Proposition 1 $T = \bigcap_{\substack{[a,b]\in C \\ a<b}} T_{ab}$, T_{ab} have extensions $\bar{T}_{ab}$. $\bigcap_{\substack{[a,b]\in C \\ a<b}} \bar{T}_{ab}$ is an extension of T: It is clearly a compatible tolerance relation on D, $T = \bigcap_{\substack{[a,b]\in C \\ a<b}} T_{ab} \subseteq \bigcap_{\substack{[a,b]\in C \\ a<b}} \bar{T}_{ab}$, $x, y \in L, [x, y] \in \bigcap_{\substack{[a,b]\in C \\ a<b}} \bar{T}_{ab} \Rightarrow [x, y] \in \bar{T}_{ab}\ \forall a, b \in L$, $a < b, [a, b] \notin T \Rightarrow [x, y] \in T_{ab}\ \forall a, b \in L, a < b, [a, b] \notin T \Rightarrow [x, y] \in \bigcap_{\substack{[a,b]\in C \\ a<b}} T_{ab} = T$.

Q.E.D.

Proposition 3. *Let L be a sublattice of a distributive lattice D not closed under relative bicomplements. Then $[D, L]$ has not* TEP.

Proof. Let $\langle \bar{a}, \bar{b}\rangle$ be a subinterval of $\langle a, b\rangle$ which has a relative bicomplement x in D but no relative bicomplement in L. A compatible tolerance relation T of the type τ on L which has no extension on D will be constructed. Let A denote the set $\{d \in L \mid d \wedge \bar{a} = a\}$, let J be the ideal in L generated by $A \cup \{\bar{b}\}$. $b \notin J$ (as $b \in J \Rightarrow$ $\Rightarrow \bar{b} \vee \bigvee_{i=1}^{n} d_i \geqq b,\ d_i \in A \Rightarrow \bar{b} \vee (b \wedge \bigvee_{i=1}^{n} d_i) = (\bar{b} \vee b) \wedge (\bar{b} \vee \bigvee_{i=1}^{n} d_i) = b,\ \bar{a} \wedge$ $\wedge (b \wedge \bigvee_{i=1}^{n} d_i) = a$, i.e. $b \wedge \bigvee_{i=1}^{n} d_i$ is a relative bicomplement of $\langle \bar{a}, \bar{b}\rangle$ in $\langle a, b\rangle$ in L), therefore there exists in L an ideal I containing J and not containing b which is maximal with this property. Let E be the dual ideal in L generated by the set $(L \setminus I) \cup$ $\cup \{\bar{a}\}$. $a \notin E$ (as $a \in E \Rightarrow \exists x \in L \setminus I, x \wedge \bar{a} \leqq a \Rightarrow x \vee a \in L \setminus I, (x \vee a) \wedge \bar{a} = a \Rightarrow$ $\Rightarrow x \vee a \in A \subseteq J \subseteq I$). As far as F is the maximal dual ideal containing E and not containing a, $T = (I \times I) \cup (F \times F)$ is a compatible tolerance relation on L. If $\bar{T}$ were an extension of T on D, then $a = a \vee a = a \vee (x \wedge \bar{a})$, $b = b \wedge b =$ $= (\bar{b} \vee x) \wedge (\bar{b} \vee b) = \bar{b} \vee (x \wedge b)$, $[a \vee (x \wedge \bar{a}), \bar{b} \vee (x \wedge b)] \in \bar{T} \Rightarrow [a, b] \in \bar{T}$ which is a contradiction. Q.E.D.

Proposition 4. *If L is a sublattice of a distributive lattice D closed under relative bicomplements, then $[D, L]$ has* τ TEP.

Proof. Let T be a compatible tolerance relation of the type τ on L formed by an ideal I and a dual ideal F. Suppose the compatible tolerance relation $\bar{T}$ on D generated by T is not an extension of T, i.e. there exist $x, y \in L$, $x < y$, $[x, y] \in \bar{T}$, $[x, y] \notin T$. It is clearly $x \in L \setminus F$, $y \in L \setminus I$.

$[x, y] \in \bar{T}$ implies the existence of an $(m + n)$-ary term t such that

$$(*) \qquad x = \mathrm{t}(a_1, \ldots, a_m, x_1, \ldots, x_n), \quad y = \mathrm{t}(b_1, \ldots, b_m, x_1, \ldots, x_n)$$

for some $a_i, b_i \in L$, $[a_i, b_i] \in T$, $x_j \in D$.

It is known that in distributive lattices every p-ary term $t \equiv t(\xi_1, \ldots, \xi_p)$ is equivalent to a term of the form

$$\bigvee \bigwedge \xi_{i_k}, \quad i_k \in \{1, \ldots, p\}.$$

It is easy to see that $(*)$ implies

$$x = \bigvee_{i=1}^{q} (c_i' \wedge y_i') \quad \text{and} \quad y = \bigvee_{i=1}^{q} (d_i' \wedge y_i')$$

for some $c_i', d_i' \in L$, $[c_i', d_i'] \in T$, $y_i' \in D$, $q \in \mathbf{N}$.

Denote

$$c_i = y \wedge (x \vee (c_i' \wedge d_i')), \quad d_i = y \wedge (x \vee c_i' \vee d_i'), \quad y_i = y \wedge (x \vee y_i').$$

Evidently, it holds

$$(**) \qquad x = \bigvee_{i=1}^{q} (c_i \wedge y_i), \quad y = \bigvee_{i=1}^{q} (d_i \wedge y_i),$$

$$c_i, d_i \in L, \quad x \leqq c_i \leqq d_i \leqq y, \quad [c_i, d_i] \in T, \quad x \leqq y_i \leqq y.$$

It can be supposed that the binomials in $(**)$ are indexed so that

$$c_i, d_i \in I \quad \text{for} \quad i = 1, \ldots, r < q \quad \text{and} \quad c_i, d_i \in F \quad \text{for} \quad i = r + 1, \ldots, q.$$

Denote $\bar{c} = \bigwedge_{i=1}^{r} c_i$, $\bar{d} = \bigvee_{i=1}^{r} d_i$, $\bar{\bar{c}} = \bigwedge_{i=r+1}^{q} c_i$, $\bar{\bar{d}} = \bigvee_{i=r+1}^{q} d_i$, $\bar{z} = \bigvee_{i=1}^{r} y_i$, $\bar{\bar{z}} = \bigvee_{i=r+1}^{q} y_i$.

Clearly $x \leqq \bar{c} \leqq \bar{d} \leqq y$, $x \leqq \bar{\bar{c}} \leqq \bar{\bar{d}} \leqq y$, $x \leqq \bar{z} \leqq y$, $x \leqq \bar{\bar{z}} \leqq y$.

$$x \leqq (\bar{c} \wedge \bar{z}) \vee (\bar{\bar{c}} \wedge \bar{\bar{z}}) = (\bigwedge_{j=1}^{r} c_j \wedge \bigvee_{i=1}^{r} y_i) \vee (\bigwedge_{j=r+1}^{q} c_j \wedge \bigvee_{i=r+1}^{q} y_i) =$$

$$= \bigvee_{i=1}^{r} (\bigwedge_{j=1}^{r} c_j \wedge y_i) \vee \bigvee_{i=r+1}^{q} (\bigwedge_{j=r+1}^{q} c_j \wedge y_i) \leqq$$

$$\leqq \bigvee_{i=1}^{r} (c_i \wedge y_i) \vee \bigvee_{i=r+1}^{q} (c_i \wedge y_i) = x, \quad \text{hence} \quad x = (\bar{c} \wedge \bar{z}) \vee (\bar{\bar{c}} \wedge \bar{\bar{z}}).$$

$$y \geqq (\bar{d} \wedge \bar{z}) \vee (\bar{\bar{d}} \wedge \bar{\bar{z}}) = (\bigvee_{j=1}^{r} d_j \wedge \bigvee_{i=1}^{r} y_i) \vee (\bigvee_{j=r+1}^{q} d_j \wedge \bigvee_{i=r+1}^{q} y_i) =$$

$$= \bigvee_{i=1}^{r} (\bigvee_{j=1}^{r} d_j \wedge y_i) \vee \bigvee_{i=r+1}^{q} (\bigvee_{j=r+1}^{q} d_j \wedge y_i) \geqq$$

$$\geqq \bigvee_{i=1}^{r} (d_i \wedge y_i) \vee \bigvee_{i=r+1}^{q} (d_i \wedge y_i) = y, \quad \text{hence} \quad y = (\bar{d} \wedge \bar{z}) \vee (\bar{\bar{d}} \wedge \bar{\bar{z}}).$$