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CUBIC FORMS ON RIEMANNIAN SURFACES

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The purpose of this paper is to produce an invariant J associated with a cubic form Φ on a Riemannian positively curved 2-manifold M with the property that its positiveness on M closed ensures the vanishing of Φ .

Let M be a two-dimensional manifold endowed with a Riemannian metric ds^2 . In a suitable domain $D \subset M$, we may choose a coframe of 1-forms (ω^1, ω^2) on D such that

$$(1) \quad ds^2 = (\omega^1)^2 + (\omega^2)^2.$$

Then there is exactly one 1-form ω_1^2 on D satisfying

$$(2) \quad d\omega^1 = -\omega^2 \wedge \omega_1^2, \quad d\omega^2 = \omega^1 \wedge \omega_1^2;$$

the Gauss curvature K of ds^2 is then given by

$$(3) \quad d\omega_1^2 = -K\omega^1 \wedge \omega^2.$$

Choosing another coframe (τ^1, τ^2) on D with the property

$$(4) \quad ds^2 = (\tau^1)^2 + (\tau^2)^2,$$

we have

$$(5) \quad \omega^1 = \cos \alpha \cdot \tau^1 - \sin \alpha \cdot \tau^2, \quad \omega^2 = \varepsilon \sin \alpha \cdot \tau^1 + \varepsilon \cos \alpha \cdot \tau^2; \quad \varepsilon = \pm 1;$$

$$(6) \quad \omega_1^2 = \varepsilon(\tau_1^2 - d\alpha).$$

Let Φ be a cubic differential form on M ; in D , it may be written as

$$(7) \quad \Phi = P(\omega^1)^3 + 3Q(\omega^1)^2\omega^2 + 3R\omega^1(\omega^2)^2 + S(\omega^2)^3$$

with respect to the coframe (ω^1, ω^2) . Let us write, with respect to the coframe (τ^1, τ^2) ,

$$(8) \quad \Phi = P^*(\tau^1)^3 + 3Q^*(\tau^1)^2\tau^2 + 3R^*\tau^1(\tau^2)^2 + S^*(\tau^2)^3.$$

Then

$$\begin{aligned}
 (9) \quad P^* &= \cos^3 \alpha \cdot P + 3\varepsilon \sin \alpha \cos^2 \alpha \cdot Q + 3 \sin^2 \alpha \cos \alpha \cdot R + \varepsilon \sin^3 \alpha \cdot S, \\
 Q^* &= -\sin \alpha \cos^2 \alpha \cdot P + \varepsilon \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot Q + \\
 &\quad + \sin \alpha (2 \cos^2 \alpha - \sin^2 \alpha) \cdot R + \varepsilon \sin^2 \alpha \cos \alpha \cdot S, \\
 R^* &= \sin^2 \alpha \cos \alpha \cdot P + \varepsilon \sin \alpha (\sin^2 \alpha - 2 \cos^2 \alpha) \cdot Q + \\
 &\quad + \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot R + \varepsilon \sin \alpha \cos^2 \alpha \cdot S, \\
 S^* &= -\sin^3 \alpha \cdot P + 3\varepsilon \sin^2 \alpha \cos \alpha \cdot Q - 3 \sin \alpha \cos^2 \alpha \cdot R + \varepsilon \cos^3 \alpha \cdot S.
 \end{aligned}$$

Let us introduce the covariant derivatives $P_1, \dots, S_2$ of $P, \dots, S$ with respect to the coframe (ω^1, ω^2) by means of

$$\begin{aligned}
 (10) \quad dP - 3Q\omega_1^2 &= P_1\omega^1 + P_2\omega^2, \\
 dQ + (P - 2R)\omega_1^2 &= Q_1\omega^1 + Q_2\omega^2, \\
 dR + (2Q - S)\omega_1^2 &= R_1\omega^1 + R_2\omega^2, \\
 dS + 3R\omega_1^2 &= S_1\omega^1 + S_2\omega^2.
 \end{aligned}$$

Then

$$\begin{aligned}
 (11) \quad \{dP_1 - (P_2 + 3Q_1)\omega_1^2\} \wedge \omega^1 + \{dP_2 + (P_1 - 3Q_2)\omega_1^2\} \wedge \omega^2 &= \\
 = 3QK\omega^1 \wedge \omega^2, \\
 \{dQ_1 + (P_1 - Q_2 - 2R_1)\omega_1^2\} \wedge \omega^1 + \\
 + \{dQ_2 + (P_2 + Q_1 - 2R_2)\omega_1^2\} \wedge \omega^2 &= (2R - P)K\omega^1 \wedge \omega^2, \\
 \{dR_1 + (2Q_1 - R_2 - S_1)\omega_1^2\} \wedge \omega^1 + \\
 + \{dR_2 + (2Q_2 + R_1 - S_2)\omega_1^2\} \wedge \omega^2 &= (S - 2Q)K\omega^1 \wedge \omega^2, \\
 \{dS_1 + (3R_1 - S_2)\omega_1^2\} \wedge \omega^1 + \\
 + \{dS_2 + (3R_2 + S_1)\omega_1^2\} \wedge \omega^2 &= -3RK\omega^1 \wedge \omega^2,
 \end{aligned}$$

and we get the existence of the second order covariant derivatives $P_{11}, \dots, S_{22}$ such that

$$\begin{aligned}
 (12) \quad dP_1 - (P_2 + 3Q_1)\omega_1^2 &= P_{11}\omega^1 + (P_{12} - \frac{3}{2}QK)\omega^2, \\
 dP_2 + (P_1 - 3Q_2)\omega_1^2 &= (P_{12} + \frac{3}{2}QK)\omega^1 + P_{22}\omega^2, \\
 dQ_1 + (P_1 - Q_2 - 2R_1)\omega_1^2 &= Q_{11}\omega^1 + (Q_{22} + PK)\omega^2, \\
 dQ_2 + (P_2 + Q_1 - 2R_2)\omega_1^2 &= (Q_{12} + 2RK)\omega^1 + Q_{22}\omega^2, \\
 dR_1 + (2Q_1 - R_2 - S_1)\omega_1^2 &= R_{11}\omega^1 + (R_{12} + 2QK)\omega^2,
 \end{aligned}$$

$$\begin{aligned}
dR_2 + (2Q_2 + R_1 - S_2)\omega_1^2 &= (R_{12} + SK)\omega^1 + R_{22}\omega^2, \\
dS_1 + (3R_1 - S_2)\omega_1^2 &= S_{11}\omega^1 + (S_{12} + \frac{3}{2}RK)\omega^2, \\
dS_2 + (3R_2 + S_1)\omega_1^2 &= (S_{12} - \frac{3}{2}RK)\omega^1 + S_{22}\omega^2.
\end{aligned}$$

Let $P_1^*, \dots, S_2^*$ be the covariant derivatives of $P^*, \dots, S^*$ with respect to the coframe (τ^1, τ^2) . Then

$$\begin{aligned}
(13) \quad P_1^* &= \cos^4 \alpha \cdot P_1 + \varepsilon \sin \alpha \cos^3 \alpha \cdot P_2 + 3\varepsilon \sin \alpha \cos^3 \alpha \cdot Q_1 + \\
&\quad + 3 \sin^2 \alpha \cos^2 \alpha \cdot Q_2 + 3 \sin^2 \alpha \cos^2 \alpha \cdot R_1 + \\
&\quad + 3\varepsilon \sin^3 \alpha \cos \alpha \cdot R_2 + \varepsilon \sin^3 \alpha \cos \alpha \cdot S_1 + \sin^4 \alpha \cdot S_2, \\
P_2^* &= -\sin \alpha \cos^3 \alpha \cdot P_1 + \varepsilon \cos^4 \alpha \cdot P_2 - 3\varepsilon \sin^2 \alpha \cos^2 \alpha \cdot Q_1 + \\
&\quad + 3 \sin \alpha \cos^3 \alpha \cdot Q_2 - 3 \sin^3 \alpha \cos \alpha \cdot R_1 + \\
&\quad + 3\varepsilon \sin^2 \alpha \cos^2 \alpha \cdot R_2 - \varepsilon \sin^4 \alpha \cdot S_1 + \sin^3 \alpha \cos \alpha \cdot S_2, \\
Q_1^* &= -\sin \alpha \cos^3 \alpha \cdot P_1 - \varepsilon \sin^2 \alpha \cos^2 \alpha \cdot P_2 + \\
&\quad + \varepsilon \cos^2 \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot Q_1 + \\
&\quad + \sin \alpha \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot Q_2 + \\
&\quad + \sin \alpha \cos \alpha (2 \cos^2 \alpha - \sin^2 \alpha) \cdot R_1 + \\
&\quad + \varepsilon \sin^2 \alpha (2 \cos^2 \alpha - \sin^2 \alpha) \cdot R_2 + \varepsilon \sin^2 \alpha \cos^2 \alpha \cdot S_1 + \\
&\quad + \sin^3 \alpha \cos \alpha \cdot S_2, \\
Q_2^* &= \sin^2 \alpha \cos^2 \alpha \cdot P_1 - \varepsilon \sin \alpha \cos^3 \alpha \cdot P_2 - \\
&\quad - \varepsilon \sin \alpha \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot Q_1 + \\
&\quad + \cos^2 \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot Q_2 - \sin^2 \alpha (2 \cos^2 \alpha - \sin^2 \alpha) \cdot R_1 + \\
&\quad + \varepsilon \sin \alpha \cos \alpha (2 \cos^2 \alpha - \sin^2 \alpha) \cdot R_2 - \\
&\quad - \varepsilon \sin^3 \alpha \cos \alpha \cdot S_1 + \sin^2 \alpha \cos^2 \alpha \cdot S_2, \\
R_1^* &= \sin^2 \alpha \cos^2 \alpha \cdot P_1 + \varepsilon \sin^3 \alpha \cos \alpha \cdot P_2 + \\
&\quad + \varepsilon \sin \alpha \cos \alpha (\sin^2 \alpha - 2 \cos^2 \alpha) \cdot Q_1 + \\
&\quad + \sin^2 \alpha (\sin^2 \alpha - 2 \cos^2 \alpha) \cdot Q_2 + \cos^2 \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot R_1 + \\
&\quad + \varepsilon \sin \alpha \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot R_2 + \\
&\quad + \varepsilon \sin \alpha \cos^3 \alpha \cdot S_1 + \sin^2 \alpha \cos^2 \alpha \cdot S_2, \\
R_2^* &= -\sin^3 \alpha \cos \alpha \cdot P_1 + \varepsilon \sin^2 \alpha \cos^2 \alpha \cdot P_2 - \\
&\quad - \varepsilon \sin^2 \alpha (\sin^2 \alpha - 2 \cos^2 \alpha) \cdot Q_1 +
\end{aligned}$$

$$\begin{aligned}
& + \sin \alpha \cos \alpha (\sin^2 \alpha - 2 \cos^2 \alpha) \cdot Q_2 - \\
& - \sin \alpha \cos \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot R_1 + \\
& + \varepsilon \cos^2 \alpha (\cos^2 \alpha - 2 \sin^2 \alpha) \cdot R_2 - \\
& - \varepsilon \sin^2 \alpha \cos^2 \alpha \cdot S_1 + \sin \alpha \cos^3 \alpha \cdot S_2, \\
S_1^* & = -\sin^3 \alpha \cos \alpha \cdot P_1 - \varepsilon \sin^4 \alpha \cdot P_2 + 3\varepsilon \sin^2 \alpha \cos^2 \alpha \cdot Q_1 + \\
& + 3 \sin^3 \alpha \cos \alpha \cdot Q_2 - 3 \sin \alpha \cos^3 \alpha \cdot R_1 - \\
& - 3\varepsilon \sin^2 \alpha \cos^2 \alpha \cdot R_2 + \varepsilon \cos^4 \alpha \cdot S_1 + \sin \alpha \cos^3 \alpha \cdot S_2, \\
S_2^* & = \sin^4 \alpha \cdot P_1 - \varepsilon \sin^3 \alpha \cos \alpha \cdot P_2 - 3\varepsilon \sin^3 \alpha \cos \alpha \cdot Q_1 + \\
& + 3 \sin^2 \alpha \cos^2 \alpha \cdot Q_2 + 3 \sin^2 \alpha \cos^2 \alpha \cdot R_1 - \\
& - 3\varepsilon \sin \alpha \cos^3 \alpha \cdot R_2 - \varepsilon \sin \alpha \cos^3 \alpha \cdot S_1 + \cos^4 \alpha \cdot S_2.
\end{aligned}$$

By means of (9) + (13) and (10) + (12), it is easy to prove

Lemma. In D , consider the 1-form

$$\begin{aligned}
(14) \quad \tau & = \{(Q + \frac{2}{3}S)P_1 - PQ_1 + SR_1 - (R + \frac{2}{3}P)S_1\} \omega^1 + \\
& + \{(Q + \frac{2}{3}S)P_2 - PQ_2 + SR_2 - (R + \frac{2}{3}P)S_2\} \omega^2
\end{aligned}$$

and the function

$$(15) \quad J = P_2Q_1 - P_1Q_2 + R_2S_1 - R_1S_2 + \frac{2}{3}(P_2S_1 - P_1S_2).$$

Then, in the obvious notation,

$$(16) \quad \tau^* = \varepsilon \tau, \quad J^* = J.$$

Further,

$$(17) \quad d\tau = \{2J + (P^2 + 3Q^2 + 3R^2 + S^2)K\} \omega^1 \wedge \omega^2.$$

Thus J is an invariant of Φ on M , and τ is globally defined on the orientable M . From the Stokes theorem, we get the following

Theorem. Let M be an orientable two-dimensional Riemannian manifold with a positive Gauss curvature endowed with a cubic differential form Φ . Let J be the above introduced invariant associated with Φ . If $\Phi \equiv 0$ on the boundary ∂M of M and $J \geq 0$ on M , then $\Phi \equiv 0$ on M .

It remains to clarify the geometric interpretation of the invariant J .

The associated Euclidean connection on M is determined by means of the formulas

$$(18) \quad \nabla m = \omega^1 v_1 + \omega^2 v_2, \quad \nabla v_1 = -\omega_1^2 v_2, \quad \nabla v_2 = \omega_1^2 v_1.$$

Let γ be a curve on M , and let

$$(19) \quad v = xv_1 + yv_2$$

be a tangent vector field along γ . Because of

$$(20) \quad \nabla v = (dx - y\omega_1^2)v_1 + (dy + x\omega_1^2)v_2,$$

v is parallel along γ if and only if

$$(21) \quad dx = y\omega_1^2, \quad dy = -x\omega_1^2 \text{ along } \gamma.$$

Now, let us choose $m_0 \in M$ and $v_0, w_0 \in T_{m_0}(M)$; let

$$(22) \quad w_0 = \xi v_1 + \eta v_2.$$

Further, choose a curve $\gamma \subset M$ going through m_0 and having w_0 for its tangent vector at m_0 and a parallel vector field v (19) along γ such that $v(m_0) = v_0$. Then

$$(23) \quad \begin{aligned} w_0 \Phi(v) = & (P_1x^3 + 3Q_1x^2y + 3R_1xy^2 + S_1y^3)\xi + \\ & + (P_2x^3 + 3Q_2x^2y + 3R_2xy^2 + S_2y^3)\eta \end{aligned}$$

does not depend on γ . Thus we are in the position to define, for each vector $v_0 \in T_{m_0}(M)$, the 1-form φ_{v_0} by means of

$$(24) \quad \varphi_{v_0}(w_0) = w_0 \Phi(v); \quad w_0 \in T_{m_0}(M).$$

At m_0 , choose an orthonormal frame; let the coframe (ω^1, ω^2) be chosen in such a way that the dual frame (v_1, v_2) at m_0 is exactly our frame. Then

$$(25) \quad \begin{aligned} \varphi_{v_1} &= P_1\omega^1 + P_2\omega^2, \quad \varphi_{v_2} = S_1\omega^1 + S_2\omega^2, \\ \varphi_{v_1+v_2} &= (P_1 + 3Q_1 + 3R_1 + S_1)\omega^1 + (P_2 + 3Q_2 + 3R_2 + S_2)\omega^2, \\ \varphi_{v_1-v_2} &= (P_1 - 3Q_1 + 3R_1 - S_1)\omega^1 + (P_2 - 3Q_2 + 3R_2 - S_2)\omega^2 \end{aligned}$$

and

$$(26) \quad (\varphi_{v_1+v_2} + \varphi_{v_1-v_2}) \wedge \varphi_{v_1} - (\varphi_{v_1+v_2} - \varphi_{v_1-v_2}) \wedge \varphi_{v_2} = 6J \, do,$$

do = $\omega^1 \wedge \omega^2$ being the area element of M .

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