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DETERMINATION OF A SURFACE BY ITS MEAN CURVATURE

ALOIS ŠVEC, Olomouc

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M. MATSUMOTO [2] and T. Y. THOMAS [3] have shown how to reconstruct a surface of the Euclidean 3-space from its metric form and its mean curvature; see also [1]. In what follows, a simpler and more complete solution of the same problem is presented.

1. Let be given a domain $D \subset \mathbb{R}^2$ and a metric

(1) $$ds^2 = A(x, y)\,dx^2 + 2\,B(x, y)\,dx\,dy + C(x, y)\,dy^2$$

on it. Let us choose the forms $\omega^1 = \Gamma_1^1\,dx + \Gamma_2^1\,dy$, $\omega^2 = \Gamma_1^2\,dx + \Gamma_2^2\,dy$ such that

(2) $$ds^2 = (\omega^1)^2 + (\omega^2)^2 .$$

Then there is exactly one form ω_1^2 such that

(3) $$d\omega^1 = -\omega^2 \wedge \omega_1^2, \quad d\omega^2 = \omega^1 \wedge \omega_1^2 .$$

If

(4) $$d\omega^1 = r\omega^1 \wedge \omega^2, \quad d\omega^2 = s\omega^1 \wedge \omega^2,$$

we have

(5) $$\omega_1^2 = r\omega^1 + s\omega^2 .$$

The Gauss curvature K of the metric (1) is defined by the formula

(6) $$d\omega_1^2 = -K\omega^1 \wedge \omega^2 .$$

Let $f : D \to \mathbb{R}$ be a function. Its covariant derivatives $f_i, f_{ij} = f_{ji}$ with respect to the chosen coframe (ω^1, ω^2) let be defined by the equations

(7) $$df = f_1\omega^1 + f_2\omega^2 ;$$

(8) $$df_1 - f_2\omega_1^2 = f_{11}\omega^1 + f_{12}\omega^2, \quad df_2 + f_1\omega_1^2 = f_{12}\omega^1 + f_{22}\omega^2 .$$

Let $f, g : D \to R$ be functions. Let us introduce the following differential operators:

(9) $$\nabla(f, g) = f_1 g_1 + f_2 g_2, \quad \nabla f = \nabla(f, f),$$

(10) $$\Delta f = f_{11} + f_{22}, \quad \Psi f = (f_{11} - f_{22})^2 + 4f_{12}^2,$$

(11) $$\Phi(f, g) = (f_{11} - f_{22})(f_1 g_1 - f_2 g_2) + 2f_{12}(f_1 g_2 + f_2 g_1), \quad \Phi f = \Phi(f, f).$$

Let

(12) $$ds^2 = (\tau^1)^2 + (\tau^2)^2$$

be another expression of the form (2). Then

(13) $$\tau^1 = \omega^1 . \cos \varphi - \omega^2 . \sin \varphi, \quad \tau^2 = \varepsilon(\omega^1 . \sin \varphi + \omega^2 . \cos \varphi);$$
$$\varepsilon = \pm 1.$$

From

(14) $$d\tau^1 = -\tau^2 \wedge \varepsilon(\omega_1^2 - d\varphi), \quad d\tau^2 = \tau^1 \wedge \varepsilon(\omega_1^2 - d\varphi),$$

we see that

(15) $$\tau_1^2 = \varepsilon(\omega_1^2 - d\varphi).$$

Denote by f_i^*, f_{ij}^* the covariant derivatives of the function f with respect to the coframe (τ^1, τ^2). Then

(16) $$f_1 = \cos \varphi . f_1^* + \varepsilon \sin \varphi . f_2^*, \quad f_2 = -\sin \varphi . f_1^* + \varepsilon \cos \varphi . f_2^*;$$

(17) $$f_{11} = \cos^2 \varphi . f_{11}^* + 2\varepsilon \sin \varphi \cos \varphi . f_{12}^* + \sin^2 \varphi . f_{22}^*,$$
$$f_{12} = -\sin \varphi \cos \varphi . f_{11}^* + \varepsilon(\cos^2 \varphi - \sin^2 \varphi) f_{12}^* + \sin \varphi \cos \varphi . f_{22}^*,$$
$$f_{22} = \sin^2 \varphi . f_{11}^* - 2\varepsilon \sin \varphi \cos \varphi . f_{12}^* + \cos^2 \varphi . f_{22}^*.$$

This implies

(18) $$\nabla^*(f, g) = \nabla(f, g), \quad \Delta^* f = \Delta f, \quad \Psi^* f = \Psi f, \quad \Phi^*(f, g) = \Phi(f, g).$$

2. Let $M : D \to E^3$ be a surface. The frame (w_1, w_2) on D being dual to (ω^1, ω^2), let the orthonormal frame (v_1, v_2, v_3) associated with M be $v_1 = (dM) w_1$, $v_2 = (dM) w_2$ and v_3 the unit normal vector. Then the fundamental equations of M are

(19) $$dM = \omega^1 v_1 + \omega^2 v_2, \quad dv_1 = \omega_1^2 v_2 + \omega_1^3 v_3,$$
$$dv_2 = -\omega_1^2 v_1 + \omega_2^3 v_3, \quad dv_3 = -\omega_1^3 v_1 - \omega_2^3 v_2$$

with the integrability conditions (3),

(20) $$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0$$

and

$$(21)\qquad d\omega_1^2 = -\omega_1^3 \wedge \omega_2^3\,, \quad d\omega_1^3 = \omega_1^2 \wedge \omega_2^3\,, \quad d\omega_2^3 = -\omega_1^2 \wedge \omega_1^3\,.$$

From (20), we get the existence of functions $x, y : D \to \mathbb{R}$ such that

$$(22)\qquad \omega_1^3 = (H + x)\,\omega^1 + y\omega^2\,, \quad \omega_2^3 = y\omega^1 + (H - x)\,\omega^2\,,$$

H being the mean curvature of M. From (21_1) and (6),

$$(23)\qquad K = (H + x)(H - x) - y^2\,.$$

Let us introduce the functions

$$(24)\qquad l = \sqrt{(H^2 - K)}\,, \quad L = l^2 = H^2 - K\,.$$

Then

$$(25)\qquad x^2 + y^2 = l^2\,,$$

and we are in the position to write

$$(26)\qquad \begin{aligned} \omega_1^3 &= (H + l\cos\alpha)\,\omega^1 + l\sin\alpha\,.\,\omega^2\,,\\ \omega_2^3 &= l\sin\alpha\,.\,\omega^1 + (H - l\cos\alpha)\,\omega^2\,. \end{aligned}$$

Our task is to produce, the forms ω^1, ω^2 and the function H being given, a function α such that the forms (26) satisfy $(21_{2,3})$.

By direct calculation, we get

$$(27)\qquad \begin{aligned} l\alpha_1 &= -H_1\sin\alpha + H_2\cos\alpha + l_2 - 2rl\,,\\ l\alpha_2 &= H_1\cos\alpha + H_2\sin\alpha - l_1 - 2sl\,, \end{aligned}$$

the indices denoting the above introduced covariant derivatives. Let us write

$$(28)\qquad d\alpha = \alpha_1\omega^1 + \alpha_2\omega^2\,,$$

$$(29)\qquad d\alpha_1 - \alpha_2\omega_1^2 = \alpha_{11}\omega^1 + \alpha_{12}\omega^2\,, \quad d\alpha_2 + \alpha_1\omega_1^2 = \alpha_{21}\omega^1 + \alpha_{22}\omega^2\,;$$

the equation

$$(30)\qquad \alpha_{12} = \alpha_{21}$$

is then the integrability condition of (28). The differentiation of (27) yields

$$(31)\qquad \begin{aligned} l\alpha_{11} &= -(l_1 + H_1\cos\alpha + H_2\sin\alpha)\,\alpha_1 - rl\alpha_2 - (H_{11} + rH_2)\sin\alpha +\\ &\quad + (H_{12} - rH_1)\cos\alpha + l_{12} - 3rl_1 - 2r_1 l\,,\\ l\alpha_{12} &= -l_2\alpha_1 - (sl + H_1\cos\alpha + H_2\sin\alpha)\,\alpha_2 - (H_{12} + sH_2)\sin\alpha + \end{aligned}$$

$$+ (H_{22} - sH_1)\cos\alpha + l_{22} - sl_1 - 2rl_2 - 2r_2 l\,,$$

$$l\alpha_{21} = (rl - H_1 \sin\alpha + H_2\cos\alpha)\,\alpha_1 - l_1\alpha_2 + (H_{12} - rH_1)\sin\alpha +$$
$$+ (H_{11} + rH_2)\cos\alpha - l_{11} - 2sl_1 - rl_2 - 2s_1 l\,,$$

$$l\alpha_{22} = sl\alpha_1 - (l_2 + H_1\sin\alpha - H_2\cos\alpha)\,\alpha_2 + (H_{22} - sH_1)\sin\alpha +$$
$$+ (H_{12} + sH_2)\cos\alpha - l_{12} - 3sl_2 + 2s_2 l\,.$$

Let us recall that (5) and (6) imply

(32) $$K = r_2 - s_1 - r^2 - s^2\,.$$

From ($31_{2,3}$) and (27), we get

(33) $$L(\alpha_{12} - \alpha_{21}) = -\,2(H_{12}l - H_2 l_1 - H_1 l_2)\sin\alpha +$$
$$+ (H_{22}l - H_{11}l + 2H_1 l_1 - 2H_2 l_2)\cos\alpha -$$
$$- \nabla H + l\,\Delta l - \nabla l - 2KL.$$

Further,

(34) $$L_1 = 2ll_1\,,\quad L_2 = 2ll_2\,,$$
$$L_{11} = 2l_1^2 + 2ll_{11}\,,\quad L_{12} = 2l_1 l_2 + 2ll_{12}\,,\quad L_{22} = 2l_2^2 + 2ll_{22}$$

and

(35) $$\nabla L = 2L\nabla l\,,\quad \Delta L = 2\,\nabla l + 2l\,\Delta l\,.$$

The equation (33) may be rewritten as

(36) $$2L^2(\alpha_{12} - \alpha_{21}) = -4L(H_{12}l - H_2 l_1 - H_1 l_2)\sin\alpha +$$
$$+ 2L(H_{22}l - H_{11}l + 2H_1 l_1 - 2H_2 l_2)\cos\alpha - 2L\nabla H + L\Delta L - \nabla L - 4KL^2\,.$$

Further,

(37) $$L_1 = 2HH_1 - K_1\,,\quad L_2 = 2HH_2 - K_2\,,$$
$$L_{11} = 2H_1^2 + 2HH_{11} - K_{11}\,,\quad L_{12} = 2H_1H_2 + 2HH_{12} - K_{12}\,,$$
$$L_{22} = 2H_2^2 + 2HH_{22} - K_{22}\,,$$

which implies

(38) $$\nabla L = 4H^2\,\nabla H - 4H\,\nabla(H, K) + \nabla K\,,\quad \Delta L = 2\,\nabla H + 2H\,\Delta H - \Delta K\,.$$

Because of this, the integrability condition (28) may be written as

(39) $$-4LP_1\sin\alpha + 2LP_2\cos\alpha + P = 0$$

with

$$(40) \qquad P_1 = H_{12}l - H_2 l_1 - H_1 l_2 \,, \quad P_2 = (H_{22} - H_{11})\, l + 2H_1 l_1 - 2H_2 l_2 \,,$$

$$(41) \qquad P = -4KH^4 + 2\,\Delta H\,.\,H^3 + (8K^2 - \Delta K - 4\,\nabla H)\,H^2 + \\ + 2\{2\,\nabla(H, K) - K\,\Delta H\}\,H + K\,\Delta K - \nabla K - 4K^3\,.$$

Further, it is easy to see that

$$(42) \qquad (4P_1^2 + P_2^2)\,L = (H^2 - K)^2\,\Psi H + 4H^2(\nabla H)^2 + \\ + \nabla H\,.\,\{\nabla K - 4H\,\nabla(H, K)\} + 2(H^2 - K)\,\{\Phi(H, K) - 2H\,\Phi H\}\,.$$

3. Let us recall that the second fundamental form of M is given by

$$(43) \qquad II = \omega^1\omega_1^3 + \omega^2\omega_2^3 = \\ = (H + l\cos\alpha)\,(\omega^1)^2 + 2l\sin\alpha\omega^1\omega^2 + (H - l\cos\alpha)\,(\omega^2)^2\,;$$

the vectors v_1, v_2 are principal at $p \in D$ if $\sin\alpha(p) = 0$.

Now, it is easy to see the validity of the following

Theorem. *In a domain $D \subset \mathbb{R}^2$, let a metric $\mathrm{d}s^2$ be given. Let K be its Gauss curvature, and let $H : D \to \mathbb{R}$ be a function satisfying $H^2 > K$. Let $p \in D$ be a fixed point, and let the vectors $w_1(p)$, $w_2(p)$ be orthonormal with respect to $\mathrm{d}s^2$.*

1° *Let $\nabla H = 0$. If there is a surface $M : D \to E^3$ with its first form equal to $\mathrm{d}s^2$ and the mean curvature H, H is a solution of the equation*

$$(44) \qquad 4KH^4 + (\Delta K - 8K^2)\,H^2 + \nabla K - K\,\Delta K + 4K^3 = 0\,.$$

Let $\mathrm{d}s^2$ be such that there exists a constant solution H of (44) satisfying $H^2 > K$. Then there is a neighborhood $U \subset D$ of p and a unique surface $M : U \to E^3$ having $\mathrm{d}s^2$ for its first form and H for its mean curvature, the vectors $\mathrm{d}M_p\, w_1(p)$, $\mathrm{d}M_p\, w_2(p)$ being principal.

2° *Let*

$$(45) \qquad (H^2 - K)^2\,\Psi H + 4H^2(\nabla H)^2 + \nabla H\,.\,\{\nabla K - 4H\,\nabla(H, K)\} + \\ + 2(H^2 - K)\,\{\Phi(H, K) - 2H\,\Phi H\} = 0\,.$$

If there is a surface $M : D \to E^3$ with its first form equal to $\mathrm{d}s^2$ and the mean curvature H, we have

$$(46) \qquad 4KH^4 - 2\,\Delta H\,.\,H^3 + (\Delta K + 4\,\nabla H - 8K^2)\,H^2 + \\ + 2\{K\,\Delta H - 2\,\nabla(H, K)\}\,H + \nabla K - K\,\Delta K + 4K^3 = 0\,.$$