Časopis pro pěstování matematiky, roč. 103 (1978), Praha

INFINITE DIRECTED PATHS IN LOCALLY FINITE DIGRAPHS

BOHDAN ZELINKA, Liberec

(Received February 18, 1976)

We shall consider infinite locally finite directed graphs (shortly *ILF*-digraphs). A locally finite digraph is a digraph in which the indegree and the outdegree of each vertex is finite. We introduce three types of infinite directed paths (or shortly dipaths), namely one-way infinite sourcing dipaths, one-way infinite sinking dipaths and two-way infinite dipaths.

A one-way infinite sourcing dipath (or shortly sourcing dipath) in a digraph G is a one-way infinite sequence

$$v_0, e_0, v_1, e_1, v_2, e_2, \dots,$$

where v_i are vertices and e_i are edges of G, $e_i = \overrightarrow{v_i v_{i+1}}$ for all non-negative integers i and all terms of the sequence are pairwise distinct.

A one-way infinite sinking dipath (or shortly sinking dipath) in a digraph G is defined similarly as a sourcing dipath, the only difference being that $e_i = \overrightarrow{v_{i+1} v_i}$ for all non-negative integers i.

A two-way infinite dipath in a digraph G is a two-way infinite sequence

$$\dots, v_{-2}, e_{-2}, v_{-1}, e_{-1}, v_0, e_0, v_1, e_1, v_2, e_2, \dots,$$

where v_i are vertices and e_i are edges of G, $e_i = \overrightarrow{v_i v_{i+1}}$ for all integers i and all terms of this sequence are pairwise distinct.

Finite dipaths are defined in a well-known way.

We shall prove some lemmas.

Lemma 1. *Let G be a strongly connected ILF-digraph. Then G contains at least one one-way infinite sourcing dipath.*

Proof. Let v_0 be a vertex of G. As G is strongly connected, to any vertex v of G there exists a finite dipath from v_0 into v. If n is a non-negative integer, let V_n be the set of all vertices v of G such that there exists a dipath of the length n from v_0 into v, but there exists no such dipath of a length smaller than n. Evidently $V_0 = \{v_0\}$

and $V = \bigcup_{n=0}^{\infty} V_n$, where V is the vertex set of G. As G is locally finite, V_n is a finite set for each n; as G is infinite, $V_n \neq \emptyset$ for each n. Let E_0 be the set of all edges $\overrightarrow{uv}$ where $u \in V_n$, $v \in V_{n+1}$ for some n. Now we shall describe a labyrinth excursion on G by the following rules:

I. The excursion starts at v_0; at the starting moment all edges of G have the black colour.

II. If we are at a vertex v which is the initial vertex of an edge $e \in E_0$, we go through e into its terminal vertex and change the colour of e to green.

III. If we are at a vertex v which is not the initial vertex of a black edge $e \in E_0$, we go through the green edge whose terminal vertex is v into its initial vertex and change its colour to red.

We shall prove that this labyrinth excursion is infinite. If we are at a vertex $v \neq v_0$, then there exists exactly one green edge incoming into v, namely the edge through which we have come into v for the first time. Thus we cannot stop at a vertex $v \neq v_0$. Suppose that we stop at v_0. This means that we have traversed all edges outgoing from v_0 in both directions (this means that they are red). Let M be the set of all vertices which we have traversed; as we have stopped after a finite number of steps, the set M is finite. This implies that there exists a non-negative integer such that $M \cap V_n = \emptyset$. Let $v \in V_n$ and consider a finite dipath P of the length n from v_0 into v. We have $v_0 \in M$, $v \notin M$, therefore there exists a vertex w of P which is in M and such that no vertex of P between w and v, except w itself, is in M. Let x be the vertex of P immediately succeeding w. Then $\overrightarrow{wx} \in E_0$, because it belongs to P and P has the length n. We traversed w but we did not go through $\overrightarrow{wx}$ which was in E_0 and black and instead of this we returned from w through a green edge, thus violating the rule II. Therefore the labyrinth excursion is infinite. We make no circuits at this excursion, because by the rule II we can go only from V_n into V_{n+1} for some n and by the rule III we can only return through an edge already traversed. Thus the result of this excursion is a sequence of edges some of which are green and some red. The subsequence of this sequence consisting of all green edges is evidently the sequence of edges of a sourcing dipath.

Lemma 1'. *Let G be a strongly connected ILF-digraph. Then G contains at least one one-way infinite sinking dipath.*

Proof is dual to the proof of Lemma 1.

These two lemmas are the digraph analoga of Theorem 2.4.2 from [1] which concerns undirected graphs.

Lemma 2. *Let G be an acyclic ILF-digraph which contains no one-way infinite sourcing dipath. Then G has at least one sink.*

Lemma 2'. *Let G be an acyclic ILF-digraph which contains no one-way infinite sinking dipath. Then G has at least one source.*

Proofs are evident.

Lemma 3. *Let G be an acyclic ILF-digraph which contains neither one-way infinite sourcing dipaths nor sinking ones. Then G has infinitely many sources and infinitely many sinks.*

Proof. According to Lemma 2′ the set S of sources of G is non-emty. Let v be a vertex of G. Consider a sequence $v = u_0, u_1, u_2, \ldots$ such that $\overrightarrow{u_{n+1}u_n}$ is an edge of G for $n = 0, 1, \ldots$ and suppose that this sequence continues as long as possible. In this sequence no vertex is repeated because G is acyclic. The sequence must end at a certain vertex because otherwise it would be the sequence of vertices of a sinking dipath. Thus this sequence has its last vertex which is in S. We have proved that to each vertex v of G there exists a finite dipath from a vertex of S into v. For each non-negative integer n let V_n be the set of all vertices v of G with the property that there exists a dipath of the length n from a vertex of S into v and there exists no shorter dipath with this property. Suppose that S is finite. As G is locally finite, each V_n is a finite set. We have $V = \bigcup_{n=1}^{\infty} V_n$, where V is the vertex set of G. As G is infinite, we have $V_n \neq \emptyset$ for each n. Now by means of a labyrinth excursion similarly as in the proof of Lemma 1 we can prove that there exists a sourcing dipath in G, which is a contradiction. Thus G has infinitely many sources. Dually we prove that G has infinitely many sinks.

A leaf (or a quasi-component) of a digraph G is a subgraph of G induced by a class of the equivalence defined of the vertex set of G so that two vertices u, v are in this equivalence if and only if there exists a dipath from u into v and a dipath from v into u. The leaf composition graph $L(G)$ of G is the image of G in the homomorphism τ which maps two vertices onto the same vertex if and only if they belong to the same leaf of G. This concept was defined in [1].

Theorem 1. *Let G be an ILF-digraph which contains neither one-way infinite sourcing dipaths nor sinking ones. Then*

(α) *each leaf of G is a finite digraph;*

(β) *the leaf composition graph L(G) of G has infinitely many sources and infinitely many sinks.*

Proof. Each leaf of G is strongly connected, therefore if (α) is not fulfilled, there exists an infinite leaf of G and it contains a sourcing dipath and a sinking dipath by Lemmas 1 and 1′, which is a contradiction. If (β) is not fulfilled, then $L(G)$ has a sourcing dipath or a sinking one by Lemma 3. Let P be a one-way infinite dipath in $L(G)$. Let v be a vertex of P which is neither the first nor the last in P. Let e_1 (or e_2) be the edge of P incoming into v (or outgoing from v, respectively). Let e'_1, e'_2 be edges of G such that $\tau(e'_1) = e_1$, $\tau(e'_2) = e_2$, where τ is the homomorphism from the definition of the leaf composition graph. Let v' be the terminal vertex of e'_1, let v''

be the initial vertex of e'_2. We have $\tau(v') = \tau(v'') = v$, therefore v' and v'' are in the same leaf of G. As any leaf is strongly connected, there exists a dipath $P(v)$ from v' into v'' in this leaf. For each edge e of P we choose an edge e' such that $\tau(e') = e$ and for the vertices of P we find dipaths $P(v)$ as described; thus we obtain an infinite dipath in G.

Now we prove a theorem concerning two-way infinite dipaths.

Theorem 2. *For every positive integer n there exists a strongly connected ILF-digraph in which there exist n vertex-disjoint sourcing dipaths and n vertex-disjoint sinking dipaths, but no two-way infinite dipath.*

Proof. Let the vertex set V of the required digraph G consist of all ordered pairs $[p, q]$, where p is a positive integer and q is an integer such that $1 \leqq q \leqq n$. An edge goes from $[p, q]$ into $[p + 1, q]$ and from $[p + 1, q]$ into $[p, q + 1]$ for each p and q, the sum $q + 1$ being taken modulo n. Let P_i be the sourcing dipath whose sequence of vertices is $[1, i], [2, i], [3, i], \ldots$ for $i = 1, \ldots, n$. Let Q_j be the sinking dipath whose sequence of vertices is $[1, j], [2, j - 1], [3, j - 2], \ldots$ for $j = 1, \ldots, n$, where the differences $j - 1, j - 2, \ldots$ are taken modulo n. The paths $P_1, \ldots, P_n$ (or $Q_1, \ldots, Q_n$) form a system of n pairwise vertex-disjoint sourcing (or sinking, respectively) dipaths. Now let $[p_1, q_1]$ and $[p_2, q_2]$ be two vertices of G. We go along P_{q_1} from $[p_1, q_1]$ into $[p', q_1]$, where p' is the least integer such that $p' \geqq p_1$ and $p' + q_1 - 1 \equiv q_2 \pmod n$. The vertex $[p', q_1]$ lies on Q_{q_2}. We go along Q_{q_2} from $[p', q_1]$ into $[p'', q_2]$, where p'' is the greatest integer such that $p'' \leqq p_2$ and $p'' \equiv 1 \pmod n$; the vertex $[p'', q_2]$ lies on Q_{q_2}. Then we go along P_{q_2} from $[p'', q_2]$ into $[p_2, q_2]$. We have proved that G is strongly connected. Now let R be a sinking dipath in G. Suppose that there exists q_0 such that $1 \leqq q_0 \leqq n$ and R has no common vertex with P_{q_0}. The dipath R must have a common vertex with some P_i because each vertex of G belongs to some P_i. Thus we may choose q_0 so that R has a common vertex with $P_{q_0 - 1}$ (subscript taken modulo n). Let $[p_0, q_0 - 1]$ be such a common vertex with the property that p_0 is minimal. Let e be the edge of R whose terminal vertex is $[p_0, q_0 - 1]$.Its initial vertex cannot be $[p_0 - 1, q_0 + 1]$, because of the minimality of p_0, therefore it is $[p_0 + 1, q_0]$. But this vertex belongs to P_{q_0}, which is a contradiction. Thus we have proved that each sinking path in G has common vertices with all paths $P_1, \ldots, P_n$. As this must hold also for all infinite sinking subpaths of such a dipath, each sinking dipath in G has infinitely many common vertices with each P_i for $i = 1, \ldots, n$. Now let R_1 be a sourcing dipath in G, let its initial vertex be $[p^*, q^*]$. Let M be the set of all vertices of G of the form $[p, q]$, where $p \leqq p^*$. This set is finite; it has np^* elements. Let R_2 be a sinking dipath in G. Only a finite number of vertices of R_2 are in M and thus there exists a sinking dipath R_3 which is a subpath of R_2 and such that none of its vertices is in M. Now R_3 has infinitely many common vertices with P_{q^*}. Consider the sequence $\mathscr{S}$ of the common vertices of P_{q^*} and R_3 in the ordering in which they occur when going along R_3 in the direction opposite to the orientation of its edges. From this sequence we con-