Proof. Recall that the family of Borel sets in a topology is a smallest σ-algebra containing all open sets, and the sets with the Baire property form the smallest σ-algebra containing all open sets as well the sets of the first category. Since any nowhere dense set is closed (in the density topology), the Borel sets coincide with sets with the Baire property. Thus, it is sufficient to prove (iv) $\Rightarrow$ (i) only. But any Lebesgue measurable set is of the form $G \setminus N$, where G is of type G_δ (in the Euclidean topology) and N has Lebesgue measure zero. It follows (use Proposition A, (i) $\Rightarrow$ (iv)) that any such set is G_δ in the density topology. ∎

A real function f on a topological space X has the *Baire property* if $f^{-1}(U)$ has the Baire property for any open set $U \subset \mathbb{R}$. The following assertion substitutes the Lusin theorem in the proof of the Denjoy theorem and forms, in fact, its categorial counterpart.

Proposition C. *A real function f on X has the Baire property if and only if there exists a set M of the first category such that the restriction of f to $X \setminus M$ is continuous.*

Proof. See e.g. [3], Theorem 8.1. ∎

Continuous functions in the density topology are exactly *approximately continuous* functions. Now, we are able to pass to the proof of the main theorem.

Theorem (Denjoy-Stepanoff). *A real function is Lebesgue measurable if and only if it is approximately continuous almost everywhere.*

Proof. A real function f is Lebesgue measurable iff it has the Baire property in the density topology (Proposition B). In view of Propositions C and A, f is Lebesgue measurable iff there is a null set $N \subset \mathbb{R}$ such that the restriction of f to $\mathbb{R} \setminus N$ is approximately continuous on $\mathbb{R} \setminus N$. But this is exactly the case when f is approximately continuous at all points of density-open set $\mathbb{R} \setminus N$. ∎

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