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A TOPOLOGICAL PROOF OF DENJOY-STEPANOFF THEOREM

JAROSLAV LUKEŠ, Praha

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In 1915, A. DENJOY proved that any Lebesgue measurable function is approximately continuous almost everywhere. (A simple proof using the Lusin theorem on characterization of measurable functions is due to W. SIERPIŃSKI [4], 1922.) The converse of this theorem is also true (see W. STEPANOFF [5] 1924, and for a simplified proof E. KAMKE [2] 1927), so that measurable functions are completely characterized as those approximately continuous at almost all points.

The purpose of this short note is to give a slightly different proof of the mentioned theorems. In what follows, $\mathbb{R}$ will denote the set of reals, λ stands for the Lebesgue measure in $\mathbb{R}$.

The family of all measurable sets on $\mathbb{R}$ having density one at each of its points forms a certain topology which will be called the *density topology*. Thus, a set M is open in the density topology iff M is measurable and $\lim_{h \to 0} (2h)^{-1} \lambda([x - h, x + h] \cap \cap M) = 1$ for any $x \in M$. (For a collection of its properties see e.g. [6].)

The following two propositions are crucial in our investigation.

Proposition A. *The following families of sets coincide*:

(i) *Lebesgue null sets,*
(ii) *nowhere dense sets in the density topology,*
(iii) *sets of the first category in the density topology,*
(iv) (*closed*) *discrete sets in the density topology.*

Proof. The proof is easy and is left to the reader (cf. [3], Theorem 22.6, [6], Theorem 2.7). ■

Proposition B ([3], Theorem 22.7, [6], Theorem 2.6). *The following families of sets coincide*:

(i) *sets of the type G_δ in the density topology,*
(ii) *Borel sets in the density topology,*
(iii) *sets with the Baire property in the density topology,*
(iv) *Lebesgue measurable sets.*