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NONLINEAR POTENTIAL EQUATIONS WITH LINEAR PARTS AT RESONANCE

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1. INTRODUCTION

1.1. This paper deals with the solvability of the equation

$$Lu = Su\,, \tag{1.1}$$

where L, S are operators acting from a Hilbert space H into H, L is a linear non-invertible selfadjoint and Fredholm operator, S is nonlinear completely continuous.

1.2. Denote by Ker $[L]$ and Im $[L]$ the null-space and the range of the operator L, respectively. Let $P : H \to \text{Ker}\,[L]$ be the orthogonal projection from H onto Ker $[L]$. Put

$$P^c u = u - Pu\,, \quad u \in H\,.$$

The solvability of the equation (1.1) is usually established by solving the so-called bifurcation system

$$PS(w + v) = 0\,, \quad v = KP^cS(w + v)\,, \tag{1.2}$$

where $w \in \text{Ker}\,[L]$, $v \in \text{Im}\,[L]$ and $K : \text{Im}\,[L] \to H$ is the right inverse of the operator L. The Schauder fixed point theorem was originally used to obtain the solvability of (1.2) in the case of boundary value problems for second order partial differential equations by E. M. LANDESMAN and A. C. LAZER [14]. The abstract setting of this method is given in [6], [7], [10], [16], ..., where also the applications to existence theorems for various boundary value problems are given.

1.3. In the papers of J. MAWHIN (for the references see [11]) the coincidence degree theory is established which is useful for proving the existence results for equations of the type (1.1). Let us remark that the topological approach to the solvability of (1.1) also in the special cases of differential equations has been used during the last seven years in many papers — the long list may be found e.g. in [4], [11].

1.4. The type of results obtained by the above method may best be illustrated by the following example:

Let n be a positive integer. We consider the existence of a solution of nonlinear two-point boundary value problem

$$(1.3) \qquad -u''(x) - n^2 u(x) + g(u(x)) = f(x), \quad x \in (0, \pi)$$
$$u(0) = u(\pi) = 0,$$

where $g(\xi)$ is a bounded continuous real valued function defined on the real line $\mathbb{R}^1$ with a finite limit

$$g(\infty) = \lim_{\xi \to \infty} g(\xi).$$

Suppose that there exists $\xi_0 \in \mathbb{R}^1$ such that

$$g(\xi) = -g(-\xi)$$

for $|\xi| \geqq \xi_0$. Let $f \in L_1(0, \pi)$.

Then the boundary value problem (1.3) has at least one weak solution $u \in W_0^{1,2}(0, \pi)$ provided

$$(1.4) \qquad \left| \int_0^\pi f(x) \sin nx \, dx \right| < 2\, g(\infty).$$

1.5. In order that the set of functions $f \in L_1(0, \pi)$ satisfying the condition (1.4) be nonempty, we must suppose $g(\infty) > 0$. In the case $g(\infty) = 0$ the procedure from Section 1.2 does not work. The solvability of boundary value problems for ordinary and partial differential equations with such a type of nonlinearities are solved in [2], [3], [5], [9], [12], [13].

1.6. A new idea how to establish the solvability of boundary value problems for second order partial differential equations (whose abstract formulations correspond to (1.1)) is included in the paper [1], where the following elementary critical point principle is proved.

1.7. Notation. Let (x, y, z) be a point in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q = \mathbb{R}^{n+m+q}$ and let

$$F : \mathbb{R}^{n+m+q} \to \mathbb{R}^1$$

be assumed to be of class C^1. Denote by $\langle , \rangle$ and $| \, |$ the inner product and the norm in $\mathbb{R}^k$, respectively, where k may equal n, m, q or $n + m + q$. We set

$$\frac{\partial F}{\partial x} = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right),$$

$$\frac{\partial F}{\partial y} = \left(\frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial y_m} \right),$$

$$\frac{\partial F}{\partial z} = \left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_q} \right)$$

so that, identifying the gradient ∇F at a point $(\bar{x}, \bar{y}, \bar{z})$ with a point in $\mathbb{R}^{n+m+q}$ we may write

$$\nabla F(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\partial F}{\partial x}(\bar{x}, \bar{y}, \bar{z}), \frac{\partial F}{\partial y}(\bar{x}, \bar{y}, \bar{z}), \frac{\partial F}{\partial z}(\bar{x}, \bar{y}, \bar{z})\right).$$

1.8. Elementary Critical Point Principle. *Let $n > 0$, $m \geqq 0$ and $q \geqq 0$. Suppose that there exist numbers $c_0 > 0$, $r_0 > 0$ such that:*

$$\left\langle \frac{\partial F}{\partial y}(x, y, z), y \right\rangle \geqq 0 \tag{1.5}$$

for $|y| = c_0$, $|x| \leqq r_0$ and $|z| \leqq c_0$ if $m > 0$;

$$\left\langle \frac{\partial F}{\partial z}(x, y, z), z \right\rangle \leqq 0 \tag{1.6}$$

for $|x| \leqq r_0$, $|y| \leqq c_0$ and $|z| = c_0$ if $q > 0$;

$$F(x, y, z) \leqq F(0, y^*, 0) \tag{1.7}$$

for $|z| \leqq c_0$, $|y| \leqq c_0$, $|y^*| \leqq c_0$ and $|x| = r_0$.

Then there exists (x_0, y_0, z_0) with

$$|x_0| \leqq r_0\,, \quad |y_0| \leqq c_0\,, \quad |z_0| \leqq c_0 \tag{1.8}$$

and

$$\nabla F(x_0, y_0, z_0) = 0\,. \tag{1.9}$$

1.9. In this paper we shall apply Elementary Critical Point Principle to the problem of solvability of (1.1). The abstract result obtained (see Section 2) extends not only the result of S. Ahmad - A. C. Lazer - J. L. Paul (see [1]) but many various existence theorems for the weak solvability of boundary value problems for differential equations (see Sections 4 and 5). Let us note that the stated results applied to (1.3) give the existence of a solution also if $g(\infty) = 0$ (see Section 4), also in the case of sublinear nonlinearity, i.e. if

$$\lim_{\xi \to \infty} \frac{g(\xi)}{\xi^\delta} \tag{1.10}$$

is non-zero and finite for certain $\delta \in (0, 1)$ (see Section 5), and also in the case of nonlinearity which has a linear growth, i.e. if (1.10) is finite (and sufficiently small) with $\delta = 1$.

2. ABSTRACT THEOREM

2.1. The operator L. Let H be a real separable Hilbert space with the inner product $\langle u, v\rangle_H$ and with the norm

$$\|u\| = \langle u, u\rangle_H^{1/2} .$$

Suppose that $B : H \to H$ is linear completely continuous selfadjoint operator and denote by $\sigma = \sigma(B)$ the set of all eigenvalues of the operator B. Let Λ be a sequence of all eigenvalues of B considered together with their multiplicities and let $e_\lambda \in H$, $\|e_\lambda\| = 1$, be the eigenvector corresponding to $\lambda \in \Lambda$, i.e.

$$\lambda e_\lambda = Be_\lambda , \quad \lambda \in \Lambda .$$

Let $0 \notin \delta$. Choose $\lambda_0 \in \sigma$ fixed and denote

$$\mu = \lambda_0 - \inf \sigma , \tag{2.1}$$

$$d = \text{distance of } \lambda_0 \text{ to } \sigma - \{\lambda_0\} . \tag{2.2}$$

Let W be a null-space of the operator

$$L : u \mapsto \lambda_0 u - Bu , \quad u \in H \tag{2.3}$$

(i.e. $W = \text{Ker}\,[L]$).

2.2. The operator S. Let $S : H \to H$ be a strongly continuous operator (i.e. it maps weakly convergent sequences in H onto strongly convergent sequences in H) and suppose that there exist $\alpha \geqq 0$, $\beta \geqq 0$, $\delta \in [0, 1]$ such that

$$\|Su\| \leqq \alpha + \beta\|u\|^\delta , \quad u \in H . \tag{2.4}$$

Suppose that

$$\delta = 0 \quad \text{if and only if} \quad \beta = 0 , \tag{2.5}$$

$$\beta < \tfrac{1}{2}d \quad \text{if} \quad \delta = 1 . \tag{2.6}$$

Moreover, let the operator S be potential with a potential $\mathscr{S} : H \to \mathbb{R}^1$, i.e. the functional $\mathscr{S}$ possesses the Fréchet derivative $\mathscr{S}'u$ on the whole space H and

$$\mathscr{S}'u = Su , \quad u \in H : \lim_{\|h\| \to 0} \frac{\mathscr{S}(u + h) - \mathscr{S}(u) - \langle Su, h\rangle_H}{\|h\|} = 0 .$$

Define

$$\varkappa : r \mapsto \inf_{\substack{w \in W \\ \|w\| = r}} \mathscr{S}(w) . \tag{2.7}$$

The main result is the following theorem.

2.3. Theorem. *Let the above assumptions be fulfilled. Then the equation*

$$Lu = Su \tag{2.8}$$

is solvable in H provided

$$\nu(\delta) = \liminf_{r \to \infty} \frac{\varkappa(r)}{(\alpha + \beta r^{\delta})^2} > \varrho(\delta), \tag{2.9}$$

where

$$\varrho(\delta) = \begin{cases} \dfrac{\mu + 5d}{2d^2} & \text{if } \delta = 0, \\ \dfrac{\mu + 4d}{2d^2} & \text{if } 0 < \delta < 1, \\ \left(4\beta + \dfrac{\mu}{2}\right)(d - 2\beta)^{-2} + 2(d - 2\beta)^{-1} & \text{if } \delta = 1. \end{cases} \tag{2.10}$$

2.4. Proof of Theorem 2.3. Denote

$$\Lambda^{\sim} = \{\lambda \in \Lambda;\ \lambda > \lambda_0\}, \quad \Lambda_{\sim} = \{\lambda \in \Lambda;\ \lambda < \lambda_0\}$$

and let Z and V be the closures of linear hulls of all eigenvectors e_λ, $\lambda \in \Lambda$ for which $\lambda \in \Lambda^{\sim}$ and $\lambda \in \Lambda_{\sim}$, respectively. Then

$$H = W \oplus V \oplus Z$$

(the direct sum). We define a functional

$$\Phi : W \times V \times Z \to \mathbb{R}^1$$

by

$$\Phi : (w, v, z) \mapsto \tfrac{1}{2}\langle Lv, v\rangle_H + \tfrac{1}{2}\langle Lz, z\rangle_H - \mathscr{S}(w + v + z). \tag{2.11}$$

Obviously

$$\langle Lv, v\rangle_H \geqq d\|v\|^2, \quad v \in V, \tag{2.12}$$

$$\langle Lz, z\rangle_H \leqq -d\|z\|^2, \quad z \in Z. \tag{2.13}$$

Put

$$A(\delta) = \min_{\tau \in [0,\infty)} \left\{\frac{d}{2}\tau^2 - \alpha\tau - \beta\tau^{1+\delta}\right\}. \tag{2.14}$$

Let $c = c(r) > 0$ be the (unique) solution of the algebraic equation

$$dc - (\alpha + \beta r^{\delta}) - 2\beta c^{\delta} = 0. \tag{2.15}$$

If $\delta = \beta = 0$ then

$$c(r) = \alpha d^{-1}. \tag{2.16}$$

Let $\delta \in (0, 1]$. Obviously

(2.17) $$c(r) \geqq (\alpha + \beta r^\delta)\, d^{-1}$$

and thus

(2.18) $$\lim_{r \to \infty} c(r) = \infty\,.$$

The implicit function theorem implies that there exists the derivative $c'(r)$ and

(2.19) $$c'(r)\,(d - 2\beta\delta\, c^{\delta-1}(r)) - \beta\delta r^{\delta-1} = 0\,.$$

Thus

(2.20) $$\lim_{r \to \infty} \frac{c'(r)}{\beta\delta r^{\delta-1}} = \lim_{r \to \infty} \frac{1}{d - 2\beta c^{\delta-1}(r)} = \omega(\delta)\,,$$

where

(2.21) $$\omega(\delta) = \begin{cases} d^{-1} & \text{if} \quad \delta \in (0, 1) \\ (d - 2\beta)^{-1} & \text{if} \quad \delta = 1\,, \end{cases}$$

and, the l'Hospital rule implies

(2.22) $$\lim_{r \to \infty} \frac{c(r)}{\alpha + \beta r^\delta} = \omega\,(\delta)\,.$$

The above results give

(2.23) $$\nu(\delta) > \varrho(\delta) = \tfrac{1}{2}\mu\, \omega^2(\delta) + 2\,\omega(\delta) \quad \text{if} \quad \delta \in (0, 1)$$

and

(2.24) $$\nu(0) > \frac{1}{2d} + \tfrac{1}{2}\mu d^{-2} + \frac{2}{d} = -\frac{A(0)}{\alpha^2} + \tfrac{1}{2}\mu d^{-2} + \frac{2}{d}\,.$$

According to the assumptions (2.9), (2.10) there exists $r_0 > 0$ such that

$$\frac{\varkappa(r_0)}{(\alpha + \beta r_0^\delta)^2} > -\frac{A(\delta)}{(\alpha + \beta r_0^\delta)^2} + \tfrac{1}{2}\mu \frac{c^2(r_0)}{(\alpha + \beta r_0^\delta)^2} + 2\frac{c(r_0)}{\alpha + \beta r_0^\delta} + 4\beta \frac{c^{1+\delta}(r_0)}{(\alpha + \beta r_0^\delta)^2}\,,$$

i.e. if $c(r_0) = c_0$ then

(2.25) $$\varkappa(r_0) > -A(\delta) + \tfrac{1}{2}\mu c_0^2 + 2c_0(\alpha + \beta r_0^\delta) + 4\beta c_0^{1+\delta}\,.$$

Denote by Φ'_1, Φ'_2, and Φ'_3 the partial Fréchet derivatives of Φ with respect to the first, second and third variable, respectively.

Now the following inequalities hold:

If

(2.26) $$\|w\| \leqq r_0\,, \quad \|v\| = c_0\,, \quad \|z\| \leqq c_0$$

then

(2.27) $$\langle \Phi_2'(w, v, z), v \rangle_H \geqq 0$$

since

$$\langle \Phi_2'(w, v, z), v \rangle_H = \langle Lv, v \rangle_H - \langle S(w + v + z), v \rangle_H \geqq$$
$$\geqq d\|v\|^2 - \alpha\|v\| - \beta\|w\|^\delta \|v\| - \beta\|v\|^{1+\delta} - \beta\|z\|^\delta \|v\| \geqq$$
$$\geqq c_0\{dc_0 - (\alpha + \beta r_0^\delta) - 2\beta c_0^\delta\} = 0 \,.$$

If

(2.28) $$\|w\| \leqq r_0 \,, \quad \|v\| \leqq c_0 \,, \quad \|z\| = c_0$$

then

(2.29) $$\langle \Phi_3'(w, v, z), z \rangle_H \leqq 0$$

since

$$\langle \Phi_3'(w, v, z), z \rangle_H = \langle Lz, z \rangle_H - \langle S(w + v + z), z \rangle_H \leqq$$
$$\leqq -d\|z\|^2 + \alpha\|z\| + \beta\|w\|^\delta \|z\| + \beta\|z\|^{1+\delta} \leqq$$
$$\leqq c_0\{-dc_0 + (\alpha + \beta r_0^\delta) + 2\beta c_0^\delta\} = 0 \,.$$

If

(2.30) $$\|w\| = r_0 \,, \quad \|v\| \leqq c_0 \,, \quad \|v^*\| \leqq c_0 \,, \quad \|z\| \leqq c_0$$

then

(2.31) $$\Phi(w, v, z) \leqq \Phi(0, v^*, 0)$$

since

$$\Phi(w, v, z) \leqq \tfrac{1}{2}\mu\|v\|^2 - d\|z\|^2 + \langle S(w + \vartheta(v + z)), v + z \rangle_H - \mathscr{S}(w) \leqq$$
$$\tfrac{1}{2}\mu\|v\|^2 - d\|z\|^2 + \alpha\|v\| + \alpha\|z\| + \beta\|w\|^\delta \|v\| + \beta\|w\|^\delta \|z\| +$$
$$+ \beta\|v\|^{1+\delta} + \beta\|v\|^\delta \|z\| + \beta\|z\|^\delta \|v\| + \beta\|z\|^{1+\delta} - \mathscr{S}(w) \leqq$$
$$\leqq \tfrac{1}{2}\mu c_0^2 + 2c_0(\alpha + \beta r_0^\delta) + 4\beta c_0^{1+\delta} - \varkappa(r_0) \leqq A(\delta) \leqq$$
$$\leqq \tfrac{1}{2}d\|v^*\|^2 - \alpha\|v^*\| - \beta\|v^*\|^{1+\delta} \leqq \tfrac{1}{2}\langle Lv^*, v^* \rangle_H - \langle S(\vartheta v^*), v^* \rangle_H =$$
$$= \tfrac{1}{2}\langle Lv^*, v^* \rangle_H - \mathscr{S}(v^*) = \Phi(0, v^*, 0)\,.$$

Let $\{V_m\}_{m=1}^\infty$ and $\{Z_m\}_{m=1}^\infty$ be sequences of finite-dimensional subspaces of V and Z, respectively, such that

(2.32) $$V_1 \subset V_2 \subset \ldots \subset V_m \subset V_{m+1} \subset \ldots\,, \quad \overline{\bigcup_{m=1}^\infty V_m} = V\,;$$

(2.33) $$Z_1 \subset Z_2 \subset \ldots \subset Z_m \subset Z_{m+1} \subset \ldots\,, \quad \overline{\bigcup_{m=1}^\infty Z_m} = Z\,.$$

Now we shall apply Elementary Critical Point Principle (see Section 1.8) to the function Φ restricted to $W \times V_m \times Z_m$. The assumptions of Section 1.8 are satisfied in virtue of the relations (2.26)–(2.31).

Thus there exists $(w_m, v_m, z_m) \in W \times V_m \times Z_m \subset W \times V \times Z$ such that

$$\|w_m\| \leqq r_0\,, \quad \|v_m\| \leqq c_0\,, \quad \|z_m\| \leqq c_0 \tag{2.34}$$

and

$$\langle \Phi_1'(w_m, v_m, z_m), w\rangle_H = 0\,, \quad w \in W;$$

$$\langle \Phi_2'(w_m, v_m, z_m), v\rangle_H = 0\,, \quad v \in V_m;$$

$$\langle \Phi_3'(w_m, v_m, z_m), z\rangle_H = 0\,, \quad z \in Z_m\,,$$

i.e.

$$\langle S(w_m + v_m + z_m), w\rangle_H = 0\,, \quad w \in W; \tag{2.35}$$

$$\langle Lv_m, v\rangle_H - \langle S(w_m + v_m + z_m), v\rangle_H = 0\,, \quad v \in V_m; \tag{2.36}$$

$$\langle Lz_m, z\rangle_H - \langle S(w_m + v_m + z_m), z\rangle_H = 0, z \in Z_m\,. \tag{2.37}$$

Choose subsequences $\{w_{m_j}\}, \{v_{m_j}\}, \{z_{m_j}\}$ with the following properties ($\to$ and $\rightharpoonup$ denote the strong and weak convergences, respectively):

$$w_{m_j} \to w_0\,, \quad v_{m_j} \rightharpoonup v_0\,, \quad z_{m_j} \rightharpoonup z_0$$

(this follows from (2.34)),

$$Bv_{m_j} \to Bv_0\,, \quad Lv_{m_j} \rightharpoonup Lv_0\,, \quad S(w_{m_j} + v_{m_j} + z_{m_j}) \to S(w_0 + v_0 + z_0)$$

(this follows from the continuity properties of B and S). Then the point

$$u_0 = w_0 + v_0 + z_0 \in H$$

satisfies the equation (2.8) as follows immediately by passing to the limit in (2.35)–(2.37) and using (2.32), (2.33).

3. NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

3.1. Sobolev spaces. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 1)$ with a lipschitzian boundary $\partial\Omega$ if $N > 1$. Let us write, as usual, $j = (j_1, \ldots, j_N)$, where j_i are nonnegative integers, $i = 1, \ldots, N$, and

$$D^j = \frac{\partial^{|j|}}{\partial x_1^{j_1} \ldots \partial x_N^{j_N}}$$

with $|j| = \sum_{i=1}^{N} j_i$. We define the Sobolev space $W^{k,2}(\Omega)$ (for $k \geqq 0$, integer) of all functions u for which $D^j u \in L_2(\Omega)$ when $|j| \leqq k$, normed by

$$\|u\|_{W^{k,2}} = \left(\sum_{|j| \leqq k} \int_\Omega |D^j u(x)|^2 \, dx \right)^{1/2}$$

($D^j u$ means the derivative in the sense of distributions).

The space $W^{k,2}(\Omega)$ is a separable Hilbert space with the inner product

$$\langle u, v \rangle_{W^{k,2}} = \sum_{|j| \leqq k} \int_\Omega D^j u(x) \, D^j v(x) \, dx \, .$$

Furthermore, denoting the set of all infinitely differentiable functions on Ω with compact supports in Ω by $\mathscr{D}(\Omega)$, we define $W_0^{k,2}(\Omega)$ as the closure of $\mathscr{D}(\Omega)$ in $W^{k,2}(\Omega)$.

Let V be a closed subspace of $W^{k,2}(\Omega)$ such that

$$W_0^{k,2}(\Omega) \subset V \subset W^{k,2}(\Omega) \, . \tag{3.1}$$

3.2. Linear differential operator. Let

$$a_{ij}(x) \in L_\infty(\Omega) \, , \quad a_{ij} = a_{ji} \quad (|i|, |j| \leqq k) \, . \tag{3.2}$$

Suppose that there exists $c > 0$ such that

$$\sum_{|i| = |j| = k} a_{ij}(x) \, \xi_i \xi_j \geqq c \sum_{|i| = k} \xi_i^2 \tag{3.3}$$

for all $\xi_i \in \mathbb{R}^1$ ($|i| = k$) and almost all $x \in \Omega$. Let

$$A_{ij} \in L_\infty(\partial\Omega) \, , \quad A_{ij} = A_{ji} \quad (|i|, |j| < k) \, . \tag{3.4}$$

Put

$$\mathscr{L}(v, u) = \sum_{|i|, |j| \leqq k} \int_\Omega a_{ij}(x) \, D^i v(x) \, D^j u(x) \, dx + \sum_{|i|, |j| < k} \int_{\partial\Omega} A_{ij} \, D^i v \, D^j u \, . \tag{3.5}$$

(In the surface integral the derivatives $D^i v$, $D^j u$ are considered in the sense of traces. Since we suppose that Ω is a domain with a lipschitzian boundary $\partial\Omega$ and, moreover, $D^i v$, $D^j u \in W^{1,2}(\Omega)$ for $|i|$, $|j| < k$, the traces are well-defined — see e.g. [15, p. 15].)

The form $\mathscr{L}(v, u)$ is symmetric, bounded and bilinear on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$. Define a mapping

$$L : V \to V$$

by

$$\langle Lu, v \rangle_{W^{k,2}} = \mathscr{L}(v, u) \tag{3.6}$$

for each $u, v \in V$.

Introduce a new inner product on V by

$$\langle u, v\rangle_V = \sum_{|i|=|j|=k} \int_\Omega a_{ij}(x)\, D^i v(x)\, D^j u(x)\,\mathrm{d}x + \int_\Omega u(x)\, v(x)\,\mathrm{d}x$$

for $u, v \in V$. The norm

$$\|u\|_V = \langle u, u\rangle_V^{1/2}, \quad u \in V$$

is equivalent with $\|u\|_{W^{k,2}}$ on the space V. Define the operator $B : V \to V$ by

$$Lu = u - Bu, \quad u \in V.$$

The mapping B is selfadjoint and completely continuous by virtue of the complete continuity of the imbedding from $W^{k,2}(\Omega)$ into $W^{k-1,2}(\Omega)$ (see e.g. [15, Chapter 2]).

3.3. Nonlinear operator. It will be very convenient to denote by $\nabla_{k-1}u$ the generalized gradient of the function u, i.e. the vector containing all derivative $D^j u$ for $|j| \leqq k - 1$ (which are lexicographically ordered). Let ϱ be the number of all multi-indices of dimension N whose length is less or equal to $k - 1$.

Let $b(x; \xi)$ be defined for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^\varrho$. Suppose that the functions $b(x; \xi)$ and

$$b_i(x; \xi) = \frac{\partial b}{\partial \xi_i}(x; \xi) \quad \text{for} \quad |i| \leqq k - 1$$

satisfy the Carathéodory condition on $\Omega \times \mathbb{R}^\varrho$ (i.e. they are measurable on Ω for fixed $\xi \in \mathbb{R}^\varrho$ and continuous in ξ for fixed almost all $x \in \Omega$). Suppose that there exist $\psi_1 \in L_1(\Omega)$, $\psi_2 \in L_2(\Omega)$, $c_1 \geqq 0$, $c_2 \geqq 0$ and $\delta \in [0, 1]$,

$$c_2 = 0 \quad \text{if and only if} \quad \delta = 0. \tag{3.7}$$

such that

$$|b(x; \xi)| \leqq \psi_1(x) + c_1 \sum_{|j| \leqq k-1} |\xi_j|^{\delta+1}, \tag{3.8}$$

$$|b_i(x; \xi)| \leqq \psi_2(x) + c_2 \sum_{|j| \leqq k-1} |\xi_j|^{\delta} \tag{3.9}$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^\varrho$. Let

$$\varphi \in W^{k,2}(\Omega) \tag{3.10}$$

and define a functional

$$\mathscr{S} : V \to \mathbb{R}^1$$

by

$$\mathscr{S} : u \mapsto \int_\Omega b(x; \nabla_{k-1}u(x) + \nabla_{k-1}\varphi(x))\,\mathrm{d}x, \quad u \in V. \tag{3.11}$$

Then the functional $\mathscr{S}$ possesses the Fréchet derivative $\mathscr{S}'u = Su$ for arbitrary $u \in V$, where $S : V \to V$ is given by

$$(3.12)\quad \langle Su, v \rangle_V = \sum_{|i| \leq k-1} \int_\Omega b_i(x; \nabla_{k-1}u(x) + \nabla_{k-1}\varphi(x))\, D^i v(x)\, \mathrm{d}x\,, \quad u, v \in V.$$

It easily follows from the complete continuity of the imbedding from $W^{k,2}(\Omega)$ into $W^{k-1,2}(\Omega)$ that the operator $S : V \to V$ is strongly continuous.

Let $c_3 > 0$ be such a constant that

$$(3.13)\qquad \sum_{|i| \leq k} \|D^i u\|_{L_2} \leqq c_3 \|u\|_V\,, \quad u \in V.$$

Then (according to the assumption (3.9)) the operator $S : V \to V$ satisfies the growth condition

$$\|Su\|_V \leqq \alpha + \beta \|u\|_V^\delta\,, \quad u \in V,$$

where

$$(3.14)\qquad \alpha = c_3 \|\psi_2\|_{L_2} + c_2 c_3 (\text{meas}\ \Omega)^{(1-\delta)/2} \sum_{|j| \leq k-1} \|D^j \varphi\|_{L_2}^\delta\,,$$

$$(3.15)\qquad \beta = \begin{cases} c_2 (\text{meas}\ \Omega)^{(1-\delta)/2}\, c_3^{1+\delta} \varrho & \text{if} \quad \delta \in [0, 1) \\ c_2 c_3^2 & \text{if} \quad \delta = 1\,. \end{cases}$$

3.4. Remark. Using the imbedding theorems the condition upon ψ_2 may be generalized, e.g. $\psi_2 \in L_1(\Omega)$ if $N = 1$, etc.

3.5. Boundary value problem. As usual, we define that $u \in V$ is a weak solution of the general boundary value problem with respect to the space V (see (3.1)) and the boundary condition $\varphi \in W^{k,2}(\Omega)$ of the nonlinear partial differential equation

$$(3.16)\qquad \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij}(x)\, D^j u) = \sum_{|j| \leq k-1} (-1)^{|j|} D^j b_j(x; \nabla_{k-1} u)$$

if u satisfies the operator equation

$$(3.17)\qquad Lu = Su\,,$$

where L and S are defined by (3.5), (3.6) and (3.12), respectively.

4. BOUNDARY VALUE PROBLEMS WITH BOUNDED NONLINEARITIES

Let the notation introduced in Section 3 be observed. We shall suppose

$$(4.1)\qquad \text{Ker}\,[L] = W \neq \{0\}\,.$$

From Sections 2 and 3 we obtain immediately:

4.1. Theorem. *Suppose* (3.1)–(3.4), (3.7)–(3.9) *with* $\delta = 0$ *and* $c_2 = 0$, (3.10), (4.1). *Let*

$$\lim_{r\to\infty} \inf_{\substack{w\in \mathrm{Ker}[L] \\ \|w\|_V = 1}} \int_\Omega b(x; r\,\nabla_{k-1}w(x) + \nabla_{k-1}\varphi(x))\,\mathrm{d}x = \infty\,. \tag{4.2.}$$

Then the equation (3.17) *has at least one solution* $u \in V$.

4.2. Remarks. (i) Instead of (4.2) it suffices to suppose

$$\liminf_{r\to\infty} \inf_{\substack{w\in \mathrm{Ker}[L] \\ \|w\|_V = 1}} \int_\Omega b(x; r\,\nabla_{k-1}w(x) + \nabla_{k-1}\varphi(x))\,\mathrm{d}x > \alpha^2\,\varrho(0) \tag{4.3}$$

(see (2.7), (2.9), (2.10), (3.11)).

(ii) Theorem 4.1 extends the result from [1] mainly by considering

a) the higher order elliptic equations;

b) the general boundary value problems;

c) no continuity of the functions $b_j(x; \xi)$ in the variable $x \in \Omega$.

(iii) In the following results we give algebraic conditions upon the functions $b_j(x; \xi)$ for the assumption (4.2) to be satisfied.

4.3. Assumptions. Let M be a nonempty subset of multiindices of dimension N the length of which is less or equal to $k - 1$. Denote

$$\xi_M = \{\xi_i\}_{i\in M},\ \xi_i \in \mathbb{R}^1\,, \quad |\xi_M| = \big(\sum_{i\in M} \xi_i^2\big)^{1/2}\,, \quad \nabla_M = \{D^j u\}_{j\in M}\,.$$

Let g be an even continuously differentiable function in the variables ξ_i, $i \in M$, $g(0) = 0$. Suppose that the derivatives

$$g_j(\xi_M) = \frac{\partial g}{\partial \xi_j}(\xi_M)\,, \quad j \in M$$

are bounded. Let

$$f \in L_2(\Omega)\,, \tag{4.4}$$

$$\int_\Omega f(x)\,w(x)\,\mathrm{d}x = 0\,, \quad w \in \mathrm{Ker}\,[L]\,, \tag{4.5}$$

$$\varphi \in C^{k-1}(\bar\Omega) \cap W^{k,2}(\Omega)\,, \tag{4.6}$$

$$\mathrm{Ker}\,[L] \subset C^{k-1}(\bar\Omega)\,. \tag{4.7}$$

Put

$$b(x; \xi) = g(\xi_M) - f(x)\,\xi_0 \tag{4.8}$$

(where 0 is the multiindex with zero length) for almost all $x \in \Omega$ and every $\xi \in \mathbb{R}^e$.

4.4. Theorem. *Suppose* (3.1)–(3.4), (4.1), (4.4)–(4.7) *and*

$$\liminf_{\tau\to\infty} \tau \inf_{|\xi_M|=1} \sum_{j\in M} \xi_j\, g_j(\tau\xi_M) = \gamma > 0\,. \tag{4.9}$$

Then (4.2) *is satisfied with* $b(x;\xi)$ *given by* (4.8) *and thus the equation* (3.17) *is solvable in* V.

Proof. Obviously

$$g(\xi_M) = \int_0^{|\xi_M|} \frac{1}{\tau} \sum_{j\in M} \tau \frac{\xi_j}{|\xi_M|}\, g_j\left(\tau \frac{\xi_M}{\xi_M}\right) \mathrm{d}\tau\,. \tag{4.10}$$

Let $\varepsilon > 0$ and choose $a > 0$ such that

$$\sum_{j\in M} \tau \frac{\xi_j}{|\xi_M|}\, g_j\left(\tau \frac{\xi_M}{|\xi_M|}\right) \geqq \gamma - \varepsilon > 0$$

for each $\xi_M \neq 0$, $\tau \geqq a$. Put

$$\omega_a(\xi_M) = \begin{cases} 0 & \text{if } |\xi_M| \leqq a\,, \\ \log \dfrac{|\xi_M|}{a} & \text{if } |\xi_M| > a\,. \end{cases}$$

Then

$$g(\xi_M) \geqq -\eta a + (\gamma - \varepsilon)\, \omega_a(\xi_M)\,,$$

where

$$\sum_{j\in M} \xi_j\, g_j(\tau\xi_M) \geqq -\eta$$

for $\tau \in [0, a]$, $|\xi_M| = 1$.

Suppose that (4.2) does not hold. Then there exist sequences $r_n \in \mathbb{R}^1$, $r_n \to \infty$, $w_n \in \operatorname{Ker}[L]$, $\|w_n\|_V = 1$ such that

$$\sup_n \int_\Omega b(x; r_n \nabla_{k-1} w_n(x) + \nabla_{k-1}\varphi(x))\, \mathrm{d}x = K < \infty\,,$$

i.e.

$$\sup_n \int_\Omega g(r_n \nabla_M w_n(x) + \nabla_M \varphi(x))\, \mathrm{d}x = K\,. \tag{4.11}$$

We can suppose that $w_n \to w$ in V since $\operatorname{Ker}[L]$ is a finite dimensional space and $w_n \to w$ in $C^{k-1}(\bar{\Omega})$ according to the assumption (4.7). Choose $v > 0$ and let $n_0 \in \mathbb{N}$ be such that

$$\sup_{x\in\bar{\Omega}} |\nabla_M w_n(x) - \nabla_M w(x)| < v$$

for $n \geqq n_0$. Denote

$$\Omega_{2v}(w) = \{x \in \Omega;\ |\nabla_M w(x)| \geqq 2v\}\,.$$

Then

$$K \geqq \int_\Omega g(r_n \nabla_M w_n(x) + \nabla_M \varphi(x))\, dx \geqq$$

$$\geqq -\eta a \text{ meas } \Omega + (\gamma - \varepsilon) \int_\Omega \omega_a(r_n \nabla_M w_n(x) + \nabla_M \varphi(x))\, dx \geqq$$

$$\geqq -\eta a \text{ meas } \Omega + (\gamma - \varepsilon) \int_{\Omega_{2\nu}(w)} \omega_a(r_n \nabla_M w_n(x) + \nabla_M \varphi(x))\, dx \geqq$$

$$\geqq -\eta a \text{ meas } \Omega + (\gamma - \varepsilon) \int_{\Omega_{2\nu}(w)} \log \frac{r_n \nu - \|\varphi\|_{C^{k-1}}}{a}\, dx =$$

$$= -\eta a \text{ meas } \Omega + (\gamma - \varepsilon) \text{ meas } \Omega_{2\nu}(w) \log \frac{r_n \nu - \|\varphi\|_{C^{k-1}}}{a}$$

if n is sufficiently large so that

$$r_n > \frac{a + \|\varphi\|_{C^{k-1}}}{\nu}\,.$$

Putting $n \to \infty$ in

$$K \geqq -\eta a \text{ meas } \Omega + (\gamma - \varepsilon) \text{ meas } \Omega_{2\nu}(w) \log \frac{r_n \nu - \|\varphi\|_{C^{k-1}}}{a}$$

we obtain contradiction proving the theorem.

4.5. Theorem. *Suppose* (3.1)–(3.4), (4.1), (4.4), (4.8). *Let* $\varphi \equiv 0$. *Let* $R(\xi_M)$ *be a lower semicontinuous function in the variables* ξ_M *such that*

$$\liminf_{\tau \to \infty} \sum_{j \in M} \xi_j\, g_j(\tau \xi_M) = R(\xi_M) \tag{4.12}$$

uniformly on bounded sets of $\xi_M = \{\xi_j\}_{j \in M}$.

Then (3.17) *is solvable in* V *provided*

$$\int_\Omega R(\nabla_M w(x))\, dx > \int_\Omega f(x)\, w(x)\, dx \tag{4.13}$$

for each $w \in \operatorname{Ker}[L]$, $w \neq 0$.

Proof. The function R is bounded on bounded sets. For $p \geqq 0$ and $\xi_M = \{\xi_j\}_{j \in M}$ it is

$$R(p\xi_M) = p\, R(\xi_M)\,.$$

With respect to (4.13) we have

$$\inf_{\substack{w \in \operatorname{Ker}[L] \\ \|w\|_V = 1}} \left\{ \int_\Omega R(\nabla_M w(x)) \, dx - \int_\Omega f(x) \, w(x) \, dx \right\} = \gamma > 0 \, .$$

Let $\varepsilon > 0$ and choose $a > 0$ such that

$$\sum_{j \in M} \frac{\xi_j}{|\xi_M|} \cdot g_j \left(\tau \frac{\xi_M}{|\xi_M|} \right) \geqq R \left(\frac{\xi_M}{|\xi_M|} \right) - \varepsilon$$

for each $\xi_M \neq 0$, $\tau \geqq a$. From (4.10) we have

$$g(\xi_M) \geqq -\eta_1 a + R(\xi_M) - \varepsilon |\xi_M| - a\eta_2 + a\varepsilon$$

for arbitrary ξ_M, where

$$\sum_{j \in M} \xi_j \, g_j(\tau \xi_M) \geqq -\eta_1 \, ,$$

$$R(\xi_M) \leqq \eta_2$$

for $\tau \in [0, a]$, $|\xi_M| = 1$.

Then

$$\int_\Omega b(x; r \, \nabla_M w(x)) \, dx = \int_\Omega g(r \, \nabla_M w(x)) \, dx - r \int_\Omega f(x) \, w(x) \, dx \geqq$$

$$\geqq -\eta_1 a \text{ meas } \Omega + r \int_\Omega R(\nabla_M w(x)) \, dx - \varepsilon r \int_\Omega |\nabla_M w(x)| \, dx - a\eta_2 \text{ meas } \Omega +$$

$$+ \varepsilon a \text{ meas } \Omega - r \int_\Omega f(x) \, w(x) \, dx \geqq r(\gamma - \varepsilon \int_\Omega |\nabla_M w(x)| \, dx) -$$

$$- a\eta_2 \text{ meas } \Omega + \varepsilon a \text{ meas } \Omega \, .$$

From the previous calculation the validity of the condition (4.2) follows provided $\varepsilon > 0$ is sufficiently small.

4.6. Remarks. (i) The condition (4.4) upon "the right hand side" may be generalized in the sense of Remark 3.4 (e.g. it is possible to assume $f \in L_1(\Omega)$ if $N = 1$).

(ii) The assumption (4.7) is the regularity assumption on the solutions of the equation

$$Lu = 0$$

the validity of which is proved (under some conditions on the coefficients a_{ij}, A_{ij}) e.g. in [15].

(iii) Theorem 4.4 extends the results from the papers [3], [9], [12] mainly in the following directions:

a) instead of $f \in L_\infty(\Omega)$ we consider $f \in L_2(\Omega)$;

b) the nonlinearity contains higher order derivatives.

(iv) Theorem 4.5 extends the result from [10] and the other papers: we consider the nonlinearity in the equation (3.16) with (4.8) which depends on many variables.

5. BOUNDARY VALUE PROBLEMS WITH SUBLINEAR NONLINEARITIES

Analogously as in the proof of Theorem 4.5 we can prove (on the basis of Theorem 2.3) the following result.

5.1. Theorem. *Suppose* (3.1)–(3.4), (3.10), (4.1) *and* (4.4). *Let g be an even continuously differentiable function in the variables ξ_i $i \in M$, $g(0) = 0$. Suppose that the derivatives*

$$g_j(\xi_M) = \frac{\partial g}{\partial \xi_j}(\xi_M), \quad j \in M$$

satisfy the growth condition

$$|g_j(\xi_M)| \leqq c_4 + c_5 |\xi_M|^\delta , \tag{5.1}$$

where $c_4 \geqq 0$, $c_5 > 0$ and $\delta \in (0, 1)$. Let

$$\liminf_{\tau \to \infty} \frac{1}{\tau^\delta} \inf_{|\xi_M| = 1} \sum_{j \in M} \xi_j \, g_j(\tau \xi_M) = \gamma > 0 . \tag{5.2}$$

Then the equation (3.17) (*with $b(x; \xi)$ given by* (4.8)) *is solvable in V.*

5.2. Remark. The above theorem extends the result [10, Theorem 3.1] mainly in the following directions:

a) no monotonocity assumptions upon the functions g_j are made;

b) the nonlinearities g_j depend on many derivatives $D^j u$, $j \in M$.

5.3. In the same way it is possible to consider the boundary value problems whose nonlinearities have a linear growth. If the assumptions of Theorem 5.1 hold with $\delta = 1$ and if $c_3^2 c_5 < \frac{1}{2} d$ (for c_3 see (3.15), for d see (2.2) where $\lambda_0 = 1$ and B is defined in Section 3.2) then it is possible to generalize the result from [6] as is mentioned in Remark 5.2.