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ON KÖPCKE AND POMPEIU FUNCTIONS

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It is well known that any continuous function without derivative serves an example of nowhere monotone function. It seems that in the original Köpcke's papers, the construction of a nowhere monotone differentiable function appeared for the first time. Later, a sequence of articles followed containing a study of derivatives which change often the sign. We mention only the penetrating study of A. DENJOY 1915 [1], the papers of Z. ZALCWASSER 1927 [13], D. POMPEIU 1906 [9], S. MARCUS 1963 [7], KATZNELSON-STROMBERG 1974 [5]. Nevertheless, constructions of functions with desirable properties have been rather complicated.

The purpose of this note is to give simple constructions of such functions. A function f on an open interval I is of the Pompeiu type if f has a bounded derivative and the sets on which f' is zero or does not vanish, respectively, are both dense in I. A Köpcke function is any function of the Pompeiu type such that the sets on which f' is positive or negative, respectively, are dense in I.

In the first part of this paper we give an elementary construction of a Köpcke function. In the second part, we shall prove that the derivative of our function is even approximately continuous. Moreover, using our ideas, we shall prove a "Zahorski type" theorem in its simple version, and using Tietze's type extension procedure we shall describe an elementary method of constructing a whole scale of Köpcke functions. We mention only that similar ideas can be found in investigations of PETRUSKA-LACKOVICH [8]. Also C. GOFFMAN [3] used the complete regularity of density topology for construction of Köpcke functions. Finally, in the last part we shall propose a method of construction of Köpcke functions from functions of Pompeiu type. A completely different method using the Baire Category Theorem is due to C. E. WEIL [11].

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In this part we givę an elementary construction of a Köpcke function. We shall not use the notions of the Lebesgue measure and Lebesgue integral, we use the integral of a continuous function only. Note that Lemma 2 substitutes the assertion that every bounded approximately continuous function is a derivative (see Corollary of Lemma 3).

We believe that our construction is simpler than the elementary construction of Katznelson and Stromberg [5].

Lemma 1. *Let m be a positive integer, s and d real numbers, $d > 0$. Then the function $p : p(x) = (|x - s|/d)^{1/m}$ has the following properties:*

i) *p is continuous.*

ii) *$p(x) \geqq 0$, and $p(x) = 0$ if and only if $x = s$.*

iii) *Let $0 < \varepsilon < y \leqq p(x)$. Denote $I = \{t \in R : p(t) \leqq y - \varepsilon\}$. Let $h > 0$ and $\langle x - h, x + h\rangle \cap I \neq \emptyset$. Then I is a closed interval with a length less than*

$$2h(y - \varepsilon)^m/(y^m - (y - \varepsilon)^m).$$

Remark. The importance of the assertion (iii) consists in the convergence of the series

$$\sum_{m=1}^{\infty} \frac{(y - \varepsilon)^m}{y^m - (y - \varepsilon)^m}.$$

Proof. The properties (i) and (ii) are evident. For $z \geqq 0$, $p(t) \leqq z$ is equivalent to $|t - s| \leqq dz^m$. Choose $x_1 \in I \cap \langle x - h, x + h\rangle$. Since $p(x_1) \leqq y - \varepsilon$ and $p(x) \geqq y$ we have

$$|x_1 - s| \leqq d(y - \varepsilon)^m, \quad |x - s| \geqq dy^m, \quad |x - x_1| < h.$$

From the inequality

$$|x_1 - x| \geqq |x - s| - |x_1 - s|$$

we obtain

$$h > d(y^m - (y - \varepsilon)^m)$$

and hence

$$d(y - \varepsilon)^m \leqq h(y - \varepsilon)^m/(y^m - (y - \varepsilon)^m).$$

Since $t \in I$ if and only if $|t - s| \leqq d(y - \varepsilon)^m$, I has the required length.

Lemma 2. *Let f_n be continuous real functions such that $|f_n(x)| \leqq K$ for every $n \in N$ and $x \in R$. Let $f_n \to f$. Assume that $\lim_{n \to \infty} \int_0^x f_n = \Phi(x)$ exists for every x, and that the following condition holds for every x:*

For any $\varepsilon > 0$ there exists $\delta > 0$ and $n \in N$ such that whenever $0 < h < \delta$ and $m > n$ we can find a finite sequence of closed intervals $I_i = \langle \alpha_i, \beta_i \rangle$, $i = 1, \ldots, k$ satisfying

$$\sum_{i=1}^{k} (\beta_i - \alpha_i) < \varepsilon \, . \, h$$

and

$$\{t \in R : |f_m(t) - f(x)| \geqq \varepsilon,\ |x - t| \leqq h\} \subset \bigcup_{i=1}^{k} I_k \, .$$

Then Φ' exists everywhere and $\Phi' = f$ holds.

Proof. Consider the expression

$$\frac{1}{h}(\Phi(x + h) - \Phi(x)) = \lim_{n \to \infty} \frac{1}{h} \int_x^{x+h} f_n$$

for a fixed x. Let δ and n be found for a given $\varepsilon > 0$, let $m > n$ and $0 < |h| < \delta$. Suppose $h > 0$, the case $h < 0$, the case $h < 0$ being similar. We find points $x_1, x_2, \ldots$ $\ldots, x_{2r+1}$ such that

$$x = x_1 \leqq x_2 \leqq \ldots \leqq x_{2r+1} = x + h$$

and

$$\bigcup_{j=1}^{r} \langle x_{2j-1}, x_{2j} \rangle \cap (x, x + h) = \bigcup_{i=1}^{k} I_i \cap (x, x + h) \, .$$

We have

$$\sum_{j=1}^{r} |x_{2j} - x_{2j-1}| < \varepsilon h \, , \quad \sum_{j=1}^{r} |x_{2j+1} - x_{2j}| \leqq h \, ,$$

$$\bigcup_{j=1}^{r} \langle x_{2j}, x_{2j+1} \rangle \cap (x, x + h) \subset \{t \in (x, x + h) : |f_m(t) - f(x)| \leqq \varepsilon\} \, .$$

Thus

$$\int_x^{x+h} f_m = \sum_{j=1}^{r} \left(\int_{x_{2j-1}}^{x_{2j}} f_m + \int_{x_{2j}}^{x_{2j+1}} f_m \right) \leqq \varepsilon h K + (\varepsilon + f(x)) \, h \, .$$

Similarly

$$\int_x^{x+h} f_m \geqq -\varepsilon h K + (f(x) - \varepsilon) \, h \, .$$

Hence

$$f(x) - \varepsilon(K + 1) \leqq \frac{1}{h} \lim_{n \to \infty} \int_x^{x+h} f_n \leqq f(x) + \varepsilon(K + 1) \, .$$

This proves that

$$\Phi'(x) = \lim_{h \to 0} \frac{\Phi(x + h) - \Phi(x)}{h} = f(x) \, .$$

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, let $\emptyset \neq F_1 \subset F_2 \subset \ldots$ be a sequence of closed subsets of R. Denote*

$$A = \{a_n : n \in N\}, \quad F = \bigcup_{n=1}^{\infty} F_n.$$

If $A \cap F = \emptyset$, then there exist functions f, Φ satisfying

i) $0 \leqq f \leqq 1$; $f(t) = 0$ *for* $t \in A$ *and* $f(t) > 0$ *for* $t \in F$,
ii) $\Phi' = f$ *holds everywhere on* R.

Proof. Define $d_m = \operatorname{dist}(a_m, F_m)$, $p_0(t) = 1$ on R, $p_m(t) = (|t - a_m|/d_m)^{1/m}$ for $m \geqq 1$, $f_m(t) = \min\limits_{0 \leqq i \leqq m} p_i(t)$. Put $f = \lim\limits_{m \to \infty} f_m$, $\Phi(x) = \lim\limits_{m \to \infty} \int_0^x f_m$.

Since

$$\{t \in R : f_m(t) = 0\} = \{a_1, \ldots, a_m\}$$

and

$$p_m(t) \geqq 1 = p_0(t)$$

for any $t \in F_m$, (i) holds.

We verify the assumptions of Lemma 2: We can put $K = 1$. For each x such that $f(x) > 0$ and for any ε, $f(x) > \varepsilon > 0$, we can find $n \in N$ and $\delta > 0$ such that

$$\sum_{i=n+1}^{\infty} \frac{2(f(x) - \varepsilon)^i}{f(x)^i - (f(x) - \varepsilon)^i} < \varepsilon$$

and $|f_n(t) - f(x)| < \varepsilon$ for $t \in (x - \delta, x + \delta)$. Put

$$I_i = \{x \in R : p_i(t) \leqq f(x) - \varepsilon\}.$$

Then

$$\bigcup_{i=n+1}^{m} I_i \supset \{t \in \langle x - h, x + h\rangle : |f_m(t) - f(x)| \geqq \varepsilon\} \tag{1}$$

for any $m > n$ and h, $0 < h < \delta$. By (1) and Lemma 1, the system of intervals $\{I_j : n < j \leqq m,\ I_j \cap \langle x - h, x + h\rangle \neq \emptyset\}$ has the required properties. Indeed, the sum of their lengths is less than

$$\sum_{j=n+1}^{m} \frac{2h(y - \varepsilon)^j}{y^j - (y - \varepsilon)^j} \leqq 2h \sum_{j=n+1}^{\infty} \frac{(y - \varepsilon)^j}{y^j - (y - \varepsilon)^j} < \varepsilon h$$

where y denotes $f(x)$. For each x such that $f(x) = 0$ and any $\varepsilon > 0$ it is sufficient to find $n \in N$ and $\delta > 0$ such that $f_n(t) < \varepsilon$ for $t \in (x - \delta, x + \delta)$.

Theorem 2. *Given any two disjoint denumerable subsets A, B of R, there exists a function Ψ with a bounded derivative g such that $g > 0$ on A, $g < 0$ on B.*

(If both A and B are dense, g is a Köpcke function.)

Proof. According to Theorem 1 we find functions Ψ_1 and Ψ_2 such that Ψ_1', Ψ_2' exist everywhere and

$$0 < \Psi_1' \leqq 1 \quad \text{on} \quad A\,, \qquad \Psi_2' = 0 \quad \text{on} \quad B\,;$$

$$0 < \Psi_2' \leqq 1 \quad \text{on} \quad B\,, \qquad \Psi_1' = 0 \quad \text{on} \quad A\,.$$

Put $\Psi = \Psi_1 - \Psi_2$.

TIETZE'S TYPE EXTENSION THEOREM

In this section we shall use some theorems on approximately continuous functions and the elementary construction from the preceding section to obtain a variety of "wild" differentiable functions. The main idea using a "Tietze's type construction" is established in the paper of Petruska and Lackovich [8], where a more general theorem is proved. In our proof, in contradistinction to theirs, we shall not use a non-elementary topological lemma of Zahorski (see [12], Lemma 12).

Definition. A real function f on R is said to be *approximately continuous at* $x \in R$ if $f(t) \to f(x)$ as t tends to x on a measurable set E for which x is a point of density.

Let $\mathscr{A}$ denote the system of all measurable sets with density one at each of its points. It is not so difficult to prove that a function f is approximately continuous on R iff for any $c \in R$, the sets $\{t \in R : f(t) < c\}$, $\{t \in R : f(t) > c\}$ belong to $\mathscr{A}$.

We shall use the following well known facts on approximately continuous functions:

Theorem A. *Any approximately continuous function on R is of the Baire class* 1.

Theorem B. *If f, g, h are approximately continuous functions on R and $h(x) \neq 0$ for any $x \in R$, then the functions $f \,.\, g$, $f + g$, f/h are approximately continuous functions.*

Theorem C. *If $\sum_{n=1}^{\infty} f_n$ is a uniformly convergent series of approximately continuous functions, then $f = \sum_{n=1}^{\infty} f_n$ is approximately continuous.*

Theorem D. (Saks [10], p. 132.) *Any bounded approximately continuous function is a derivative.*

Note. For a simple proof of Theorem A see the paper of J. Lukeš and L. Zajíček [6]. Theorems B, C immediately follow from the fact that $\mathscr{A}$ is the system of open sets in a certain topology (the so called density topology, cf. [4]).

Lemma 3. *Let $f_n \searrow f$, let f_n satisfy the assumptions of Lemma 2. Then f is approximately continuous.*

Proof. Let $c \in R$. Obviously f is upper semicontinuous and thus $\{t \in R : f(t) < c\}$ belongs to $\mathscr{A}$. Denote $M = \{t \in R : f(t) > c\}$, let $x \in M$. Choose ε, $0 < \varepsilon < f(x) - c$ and find the corresponding δ and n from Lemma 2. For $m > n$ denote

$$P_m = \{t \in R : f_m(t) \leqq f(x) - \varepsilon\}\,, \quad P = \bigcup_{m=n+1}^{\infty} P_m\,.$$

Choose h, $0 < h < \delta$. By the assumptions of Lemma 2 we have

$$\lambda(P_m \cap \langle x - h, x + h\rangle) \leqq \sum_{i=1}^{k} \lambda I_i < \varepsilon h \quad (\lambda \text{ is the Lebesgue measure})$$

for any $m > n$, further $P_{n+1} \subset P_{n+2} \subset \ldots$ and therefore

$$\frac{1}{2h}\,\lambda(P \cap \langle x - h, x + h\rangle) < \varepsilon \quad \text{and} \quad \frac{1}{2h}\,\lambda(\langle x - h, x + h\rangle \setminus P) > 1 - \varepsilon\,.$$

Since $\langle x - h, x + h\rangle \setminus P \subset M$, x is a point of density for M.

Corollary. *The function f constructed in Theorem 1 is approximately continuous.*

Lemma 4. *Let A be a denumerable subset of R. Let B_0, B_1 be two disjoint G_δ-sets. Then there exists an approximately continuous function f such that $0 \leqq f \leqq 1$ and $f = 0$ on $B_0 \cap A$, $f = 1$ on $B_1 \cap A$.*

Proof. $A \cap B_0$ is denumerable, $R \setminus B_0$ is of type F_σ. By Theorem 1 we can find an approximately continuous, nonnegative function f_0 such that $f_0 > 0$ on $R \setminus B_0$, $f_0 = 0$ on $A \cap B_0$. Similarly we find an approximately continuous function f_1 such that $f_1 > 0$ on $R \setminus B_1$ and $f_1 = 0$ on $A \cap B_1$. Using Theorem B we can put $f = f_0/(f_0 + f_1)$.

Theorem 4. *Let A be a denumerable subset of R. Let g be a bounded function on A which is a restriction of a function f which is of the Baire class 1 on R. Then there exists on R a bounded approximately continuous extension g^* of g.*

Proof. We can suppose that $-1 \leqq g \leqq 1$. Put

$$H_1^* = \{x : f(x) \leqq -\tfrac{1}{3}\}\,, \quad H_2^* = \{x : f(x) \geqq \tfrac{1}{3}\}\,.$$

Since f is the function of the Baire class 1, H_1^*, H_2^* are disjoint G_δ-sets. By Lemma 4 we can find an approximately continuous function φ_1 such that $\varphi_1(x) = -\frac{1}{3}(x \in H_1^* \cap A)$, $\varphi_1(x) = \frac{1}{3}\,(x \in H_2^* \cap A)$ and $-\frac{1}{3} \leqq \varphi_1(x) \leqq \frac{1}{3}$ otherwise.

Putting $f_1 = f - \varphi_1$ we have $-\frac{2}{3} \leqq f_1(x) \leqq \frac{2}{3}$ for $x \in A$. Suppose that approximately continuous functions $\varphi_1, \ldots, \varphi_n$ have already been defined such that

(2) $$|\varphi_k(x)| \leqq \tfrac{1}{3}(\tfrac{2}{3})^{k-1}$$

for $x \in R$ and $k = 1, \ldots, n$;

(3) $$-(\tfrac{2}{3})^n \leqq f_n(x) \leqq (\tfrac{2}{3})^n$$

for $x \in A$, f_n denoting $f - \sum_{k=1}^{n} \varphi_n$. Put $H_1 = \{x; f_n(x) \leqq -\frac{1}{3}(\frac{2}{3})^n\}$, $H_2 = \{x : f_n(x) \geqq \geqq \frac{1}{3}(\frac{2}{3})^n\}$, $\hat{H}_i = A \cap H_i$. By Theorem A, φ_k, $k = 1, \ldots, n$ are functions of the Baire class 1 and therefore H_1, H_2 are disjoint G_δ-sets. Thus, by Lemma 4 we can choose an approximately continuous function $\varphi_{n+1}(x)$ such that

$$\varphi_{n+1}(x) = -\tfrac{1}{3}(\tfrac{2}{3})^n \quad (x \in \hat{H}_1), \qquad \varphi_{n+1}(x) = \tfrac{1}{3}(\tfrac{2}{3})^n \quad (x \in \hat{H}_2)$$

and

$$|\varphi_{n+1}(x)| \leqq \tfrac{1}{3}(\tfrac{2}{3})^n \quad \text{otherwise}.$$

By this construction we obviously have

$$-(\tfrac{2}{3})^{n+1} \leqq f_{n+1}(x) \leqq (\tfrac{2}{3})^{n+1}$$

for $x \in A$, where $f_{n+1} = f_n - \varphi_{n+1}$. Thus we obtain the sequence $\{\varphi_k\}_{k=1}^{\infty}$ by induction and put

$$g^*(x) = \sum_{k=1}^{\infty} \varphi_k(x).$$

By (2) the series is uniformly convergent and therefore by Theorem C g^* is a bounded approximately continuous function. From (3) it follows that $g(x) = g^*(x)$ for $x \in A$.

Remark. Let $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ be sequences of real numbers, $c_n \neq c_m$, $d_n \neq d_m$ for $n \neq m$. Let $C = \{c_n : n = 1, 2, \ldots\}$, $D = \{d_n : n = 1, 2, \ldots\}$ be disjoint dense subsets of R. If $\{z_n\}$ is a sequence of nonzero numbers tending to 0, the function g on $C \cup D$ defined by $g(c_n) = 0$, $g(d_n) = z_n$ is the restriction of a function of the Baire class 1. By Theorem 4 there exists a bounded approximately continuous extension g^* of g. By Theorem D, g^* is the derivative of a Pompeiu function. We can choose d_n and z_n such that both the sets $\{d_n : z_n > 0\}$, $\{d_n : z_n < 0\}$ are dense. Then g^* will be the derivative of a Köpcke function.

FROM POMPEIU FUNCTIONS TO KÖPCKE FUNCTIONS

In this section we demonstrate how we can construct Köpcke functions from Pompeiu functions using some essentially known simple facts concerning monotone differentiable transformations on R.

Let Ψ be a Pompeiu function. Then evidently one of the sets $\{x : \Psi'(x) > 0\}$, $\{x : \Psi'(x) < 0\}$ is dense in an open interval. Therefore it is easy from an arbitrary Pompeiu function on R to construct a Pompeiu function ω on R such that the set $\{x : \omega'(x) > 0\}$ is dense in R.

If f, g are two Pompeiu functions on R such that the sets $\{x : f'(x) > 0\}$, $\{x : g'(x) > 0\}$ are dense in R, the function $h(x) = f(x) - g(x)$ need not be a Köpcke function. But if we change g by a suitable differentiable transformation φ to $g^*(x) = g(\varphi(x))$, the function $k(x) = f(x) - g^*(x)$ will be a Köpcke function on $(0, 1)$. We shall use the following elementary lemma based on the main idea of Franklin [2].

Lemma 5. *Let $A \subset (0, 1)$, $B \subset (0, 1)$ be two disjoint denumerable sets, let C, D be two disjoint sets dense in R. Then there exists a real function φ on $(0, 1)$ such that $0 < \varphi'(x) < +\infty$ for $x \in (0, 1)$ and $\varphi(A) \subset D$, $\varphi(B) \subset C$.*

Proof. Let $A = \{a_i\}_{i=1}^{\infty}$, $B = \{b_i\}_{i=1}^{\infty}$. We may suppose that $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. Let $\varepsilon_n > 0$, $\sum_{n=1}^{\infty} \varepsilon_n < 1$. Put $\Psi_1(x) = x$, $\Psi_2(x) = x - a_1$ and

$$\Psi_{2k+1} = \prod_{i=1}^{k} (x - a_i)(x - b_i),$$
$$\Psi_{2k+2} = (x - a_{k+1}) \prod_{i=1}^{k} (x - a_i)(x - b_i) \quad \text{for} \quad k \geqq 1 .$$

We shall define a sequence $\{\omega_i\}_{i=1}^{\infty}$ for which

(4) $$|\omega_i| \sup_{x \in (0,1)} (|\Psi_i(x)| + |\Psi_i'(x)|) < \varepsilon_i \quad \text{for} \quad i = 1, 2, \ldots$$

and

(5) $$\varphi_{2n-1}(a_n) \in D, \quad \varphi_{2n}(b_n) \in C \quad \text{for} \quad n = 1, 2, \ldots,$$

where

(6) $$\varphi_0(x) = x \quad \text{and} \quad \varphi_n(x) = x + \sum_{i=1}^{n} \omega_i \Psi_i(x) \quad \text{for} \quad n = 1, 2, \ldots .$$

We proceed as follows:

Let $n \geqq 1$ and let all ω_j for $1 \leqq j < n$ be defined.

We put

$$g_n(\omega) = \varphi_{n-1}(a_m) + \omega \Psi_n(a_m) \quad \text{if} \quad n = 2m - 1$$

and

$$g_n(\omega) = \varphi_{n-1}(b_m) + \omega \Psi_n(b_m) \quad \text{if} \quad n = 2m .$$

Since g_n is a linear non-constant function and C, D are dense in R, we can find ω_n such that $g_n(\omega_n) \in D$ if n is even and $g_n(\omega_n) \in C$ if n is odd and (4) holds for $i = n$.