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A DETERMINISTIC SUBCLASS OF CONTEXT-FREE LANGUAGES

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INTRODUCTION

G. WECHSUNG in [1] has introduced a new complexity measure and has proved that the class of all context-free languages turns out to be a complexity class with respect to this measure for nondeterministic Turing machines.

We investigate the complexity class C given by the same bound and complexity measure for deterministic Turing machines in this paper. Namely, the relation of this complexity class to the class of all deterministic context-free languages is studied. It is proved that these two classes of languages are incomparable. Moreover, similar incomparability result is proved for the class C and the class of all linear languages.

WECHSUNG'S COMPLEXITY MEASURE

By a Turing machine (or simply TM) $M = (Q, X, d, q_0, F)$ we shall mean a deterministic one-tape, one-head model of Turing machine with the state space Q, the alphabet X, the next-state function d, the initial state q_0 and the accepting state space F. The alphabet of every TM will contain the blank symbol b. X_b will denote the set $X - \{b\}$.

By a computation of a TM $M = (Q, X, d, q_0, F)$ on a word $w \in X^*$ we shall mean the computation starting in the initial state q_0 on the leftmost symbol of w.

A TM $M = (Q, X, d, q_0, F)$ accepts a word $w \in X_b^*$ iff the computation of M on w halts in an accepting state.

A TM $M = (Q, X, d, q_0, F)$ recognizes a language $L \subseteq X_b^*$ iff for every word $w \in X_b^*$ the following condition holds: $w \in L \Leftrightarrow M$ accepts w.

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In case that during a computation the content of a tape square is changed, every visit of the head payed to this square after its first altering shall be called an active visit. For every word w accepted by a TM M the maximal number of all active visits on one tape square during the computation of M on w shall be denoted as $g_M(w)$.

Let k be a nonnegative integer.

A TM $M = (Q, X, d, q_0, F)$ recognizes a language $L \subseteq X_b^*$ with Wechsung's complexity k iff 1. M recognizes L and 2. for every word $w \in L$ it is $g_M(w) \leqq k$.

A language L is recognizable with Wechsung's complexity k iff there is a TM recognizing L with Wechsung's complexity k.

NOTATION AND DEFINITIONS

For every nonnegative integer k denote by $W(k)$ the class of all languages recognizable with Wechsung's complexity k. Then

$$\mathrm{C} =_{\mathrm{df}} \bigcup_{k=0}^{\infty} W(k),$$

CFL $=_{\mathrm{df}}$ the class of all context-free languages,

DCFL $=_{\mathrm{df}}$ the class of all deterministic context-free languages,

LIN $=_{\mathrm{df}}$ the class of all linear context-free languages,

$\overleftarrow{w} =_{\mathrm{df}}$ the "mirror image" of the word w,

$\Lambda =_{\mathrm{df}}$ the empty word.

By a numbering of the tape of a TM we shall understand a 1-1 mapping of the set of tape squares into the set of integers. So every tape square has a number, "square p" will denote "the square numbered by p".

Let $M = (Q, X, d, q_0, F)$ be a TM and let k be a nonnegative integer.

If $w \in X^+$ then the symbol $P(w)$ stands for "the part of the tape which was initially occupied by the characters of the input word w". If the tape of M is numbered in such a way that the square p_1 stands to the left from the square p_2, then the symbol $P(p_1, p_2)$ denotes the word formed by the sequence of characters in the squares between p_1 and p_2 ("between squares p_1 and p_2" will always implicitly include "excluding the squares p_1 and p_2").

Definition 1. Two words $u, v \in X^+$ are said to be E_1-equivalent (notation $u \sim_{E_1} v$) iff for arbitrary states $q, q' \in Q$ the following conditions hold:

1. [If M starts in the state q on the leftmost (rightmost) symbol of the word u, then M changes the content of $P(u)$ without leaving it before]

$$\Leftrightarrow$$

[If M starts in the state q on the leftmost (rightmost) symbol of the word v, then M changes the content of $P(v)$ without leaving it before].

2. [If M starts in the state q on the leftmost (rightmost) symbol of the word u, then the first exit from $P(u)$ is made leftwards in the state q']

$$\Leftrightarrow$$

[If M starts in the state q on the leftmost (rightmost) symbol of the word v, then the first exist from $P(v)$ is made leftwards in the state q'].

3. [If M starts in the state q on the leftmost (rightmost) symbol of the word u, then the first exit from $P(u)$ is made rightwards in the state q']

$$\Leftrightarrow$$

[If M starts in the state q on the leftmost (rightmost) symbol of the word v, then the first exit from $P(v)$ is made rightwards in the state q'].

4. [If M starts in the state q on the leftmost (rightmost) symbol of the word u, then M enters an accepting state without leaving $P(u)$ before]

$$\Leftrightarrow$$

[If M starts in the state q on the leftmost (rightmost) symbol of the word v, then M enters an accepting state without leaving $P(v)$ before].

For any $u \in X^+$ and $q \in Q$, the symbols $(u)_{qL}$ and $(u)_{qR}$ will denote, respectively, the content of the tape segment $P(u)$ after the first exit from $P(u)$, provided the TM M has started on the leftmost or rightmost symbol of the word u in the state q. If M does not leave the segment, the meaning of the symbols is not defined.

Definition 2. Two words $u, v \in X^+$ are said to be E_2-equivalent iff for an arbitrary sequence $q_1, A_1, q_2, A_2, \ldots, q_j, A_j$
where $j \in N$, $j \leqq 2k$,

$$A_i = \text{either } L \text{ or } R \text{ for } i = 1, 2, \ldots, j,$$

$$q_i \in Q \text{ for } i = 1, 2, \ldots, j,$$

the following condition holds:

If at least one of the symbols $(\ldots((u)_{q_1A_1})_{q_2A_2}\cdots)_{q_jA_j}$ and $(\ldots((v)_{q_1A_1})_{q_2A_2}\cdots)_{q_jA_j}$ is meaningful, then both of them are meaningful and at the same time

$$(\ldots((u)_{q_1A_1})_{q_2A_2}\cdots)_{q_jA_j} \sim_{E_1} (\ldots((v)_{q_1A_1})_{q_2A_2}\cdots)_{q_jA_j}.$$

Remark. For $j = 0$ the last relation has the form $u \sim_{E_1} v$.

Both above defined equivalences have a finite number of classes.

Lemma 1. *Let a TM $M = (Q, X, d, q_0, F)$ have s states. Let a tape segment contain the word z^s, where $z \in X$. If M enters this tape segment from the left or right and passes through it rightwards or leftwards, respectively, without any rewriting, then the first rewriting of a tape square cannot be performed before scanning a symbol different from z.*

The proof is obvious and follows from the fact that M must reach (at least) twice the same state when scanning the word z^s.

Lemma 2. *Let a* TM *M recognize a language L with Wechsung's complexity k, where $k \in N$. Then there exists such a positive integer l that during the computation of M on any word $w \in L - \{\Lambda\}$ the head reaches maximally l-1 squares out of $P(w)$.*

For the proof cf. [1].

Theorem 1. DCFL *and* C *are incomparable, i.e.* DCFL $\nsubseteq$ C & C $\nsubseteq$ DCFL.

Proof. (1.1) Let us consider the language $L = \{w\overleftarrow{w};\ w \in \{a, c\}^+\}$. It follows from [2] that $L \notin$ DCFL. We can construct a TM $M = (Q, X, d, q_0, F)$ where $X = \{a, c, b\}$ so that the computing process of M on an arbitrary word $w \in X_b^+$ will proceed as follows:

1. M will check if the leftmost of the squares of $P(w)$ which have not been rewritten contains the same character as the rightmost of the squares of $P(w)$ which have not been rewritten and if moreover these two squares are not identical. If it is so the both squares will be rewritten by the character b and then
   - either the activity No. 1 will proceed, in case some squares of $P(w)$ have not been rewritten
   - or the activity No. 2 will proceed, in case all squares contain the character b.

   If it is not so the activity No. 3 will proceed.
2. M will reach an accepting state.
3. The computation will halt in a situation for which the next-state function is not defined.

It is obvious that the next-state function of such a TM can be defined in such a way that during the computation of M on an arbitrary word $w \in \{a, c\}^+$ there will not appear more than one active visit on any square. It follows from this fact that $L \in W(1)$.

(1.2) The converse will be proved by contradiction. Consider the language $L = \{a^m c^{m+n} a^n;\ m, n = 1, 2, \ldots\}$. It is obvious that $L \in$ DCFL. Assume that $\hat{M} = (\hat{Q}, X, \hat{d}, q_0, F)$ is such a TM which recognizes L with Wechsung's complexity k, where k is a nonnegative integer. Let $q_1, q_2 \notin \hat{Q}$. Define $M = (Q, X, d, q_0, F)$, where $Q = \hat{Q} \cup \{q_1, q_2\}$ and the next-state function d is defined in the following way:

$d(q, z) = \hat{d}(q, z)$ if $(q, z) \in \hat{Q} \times X$ and $\hat{d}(q, z)$ is defined,
$d(q_1, b) = (q_1, b, R)$,
$d(q_1, a) = (q_2, a, L)$,
$d(q_2, b) = (q_0, b, R)$,
d is not defined for other arguments.

Now introduce for the TM M and k the equivalences E_1 and E_2 on X^+ according to Definitions 1 and 2. E_1 and E_2 are of finite indices, say e_1 and e_2, respectively.

Remark. If $u = b^{n_1}au_1$, $v = b^{n_2}av_1$, where $n_1, n_2 \in N$, $u_1, v_1 \in X^*$ and $u \sim_{E_1} v$, then for any state $q' \in Q$ the points 1, 2, 3, and 4 of Definition 1 hold even if we replace the words "If M starts in the state q on the leftmost (rightmost) symbol of the word u" by the words "If M starts in the state q_0 on the leftmost of the nonblank symbols of u" and if we replace the words "If M starts in the state q on the leftmost (rightmost) symbol of the word v" by the words "If M starts in the state q_0 on the leftmost of the nonblank symbols of v". This fact has been used in the proof and for this reason the TM $\hat{M}$ was extended to the TM M.

Now let us enumerate the tape of M as indicated by Fig. 1.

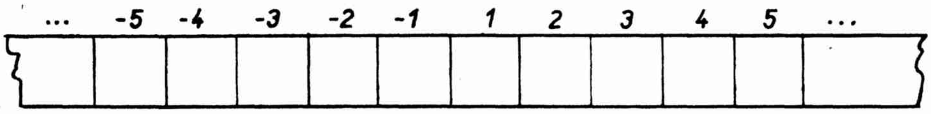


Figure 1.

Let positive integers m_1 and m_2, where $m_1 < m_2 \leqq e_1 + 1$, satisfy $a^{m_1} \sim_{E_1} a^{m_2}$ (such numbers can be found). Let l be a positive integer satisfying the assertion of Lemma 2 for the given TM M and for the language L. Define $s = \text{card } Q$.

Consider the word $w = a^n c^{2n} a^n$, where $n \in N$, $n \geqq s((s + 1)(e_2 + l - 1) + l - 2) + \max\{s + 1, m_1\}$. It holds that $w \in L$, so during the computation of M on w, not more than k active visits on any square will appear and M will reach an accepting state.

Now place w on the tape in such a way that the leftmost character of the word w will be written in the square $-2n$.

Lemma 3. *Neither between the squares $-2n - 1$ and $-n$ nor between the squares n and $2n + 1$ there exist m_1 adjacent squares the contents of which would not be changed during the computation of M on w.*

Proof. By contradiction. Let there be m_1 squares of the above described property. Let us form a word u by replacing the word situated in the assumed m_1 squares by the word a^{m_2} in the word w. M accepts u, but $u \in \{a, c\}^+ - L$.

We shall choose tape squares p_i^L and p_i^R (for $i = 1, 2, \ldots, (s+1)(e_2 + l - 1) + l$) inductively as follows:

Define $p_1^L = -2n - l$ and $p_1^R = 2n + l$.

Let p_i^L and p_i^R be defined for a positive integer i, $i < (s+1)(e_2 + l - 1) + l$. Then by Lemmas 1 and 3, there is a tape square p_i such that during the computation of M on w, p_i is rewritten as the first of squares between p_i^L and p_i^R. Then define

$$\left.\begin{aligned} p_{i+1}^L &= p_i \\ p_{i+1}^R &= p_i^R \end{aligned}\right\} \text{ if } p_i \leqq \max\{p_i^L, -2n-1\} + s \text{ or } n - s < p_i \leqq n + s\,,$$

and

$$\left.\begin{aligned} p_{i+1}^L &= p_i^L \\ p_{i+1}^R &= p_i \end{aligned}\right\} \text{ otherwise.}$$

Lemma 4. *Let i be a positive integer, $i \leqq (s+1)(e_2 + l - 1) + l$. During the computation of M on w the head can enter the part of the tape between the squares p_i^L and p_i^R at most $2k+1$-times, after rewriting these two squares.*

Proof. Let the squares p_i^L and p_i^R, where $i \in N$, $0 < i \leqq (s+1)(e_2 + l - 1) + l$, be rewritten during the computation of M on w and let then the head enter the tape segment between the squares p_i^L and p_i^R more than $2k+1$-times. At the same time at least $k+1$ active visits on the square p_i^L or p_i^R must appear.

In the following paragraphs (1.2.1) and (1.2.2), we distinguish two possible situations. (1.2.2) is again decomposed into two parts. Each of the situations leads to a contradiction as shown in the paragraph (1.2.3).

(1.2.1) Assume that for $i = 1, 2, \ldots, e_2 + 2l - 1$ the condition $p_i^L \leqq n - s$ & & $p_i^R \geqq -n + s$ holds. Then among the words $P(p_1^L, p_1^R), P(p_2^L, p_2^R), \ldots, P(p_{e_2+2l-1}^L, p_{e_2+2l-1}^R)$ there exists a pair of E_2-equivalent words such that the difference between the number of the characters c and the number of the characters a in one word is smaller than the difference between the number of the characters c and the number of the characters a in the second word. Let such a pair be formed for instance by the words $P(p_i^L, p_i^R)$ and $P(p_j^L, p_j^R)$, where $i, j \in N$, $0 < i < j < e_2 + 2l$.

The proof continues at (1.2.3).

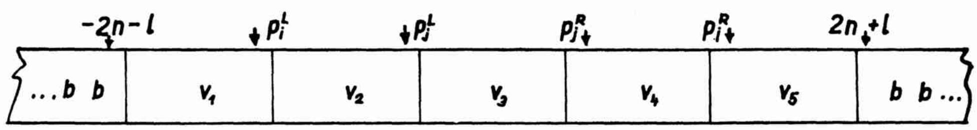
(1.2.2) Assume that $p_{e_2+2l-1}^L > n - s$. For $p_{e_2+2l-1}^R < -n + s$ the proof is quite analogous.

(1.2.2.1) Let for an integer i_0 such that $e_2 + 2l - 2 \leqq i_0 \leqq (s+1) \cdot (e_2 + l - 1) - e_2$ the condition $p_{i_0+1}^L = p_{i_0+2}^L = \ldots = p_{i_0+e_2+l}^L$ hold. Then among the words $P(p_{i_0+1}^L, p_{i_0+1}^R), P(p_{i_0+2}^L, p_{i_0+2}^R), \ldots, P(p_{i_0+e_2+l}^L, p_{i_0+e_2+l}^R)$ there exists a pair of E_2-equivalent words such that the difference between the number of the characters c and the number of the characters a in one word is smaller than the difference between the number of the characters c and the number of the characters a in the second word. Let such a pair be formed for instance by the words $P(p_i^L, p_i^R)$ and $P(p_j^L, p_j^R)$, where $i, j \in N$, $i_0 < i < j \leqq i_0 + e_2 + l$.

The proof continues at (1.2.3).

(1.2.2.2) Let the introductory assumption of the paragraph (1.2.2.1) be not fulfilled. Define $r = s(e_2 + l - 1) + l$. Then $p_r^L \geqq n$. Among the words $P(p_r^L, p_r^R)$, $P(p_{r+1}^L, p_{r+1}^R), \ldots, P(p_{r+e_2+l-1}^L, p_{r+e_2+l-1}^R)$ there exists a pair of E_2-equivalent words such that the number of the characters a in one word is greater than the number of the characters a in the second word (these words do not contain the character c). Let such a pair be formed for instance by the words $P(p_i^L, p_i^R)$ and $P(p_j^L, p_j^R)$, where $i, j \in N$, $r \leqq i < j < r + e_2 + l$.

(1.2.3) Suppose now that on the tape of the TM M the word $w_1 = b^{l-1}wb^{l-1}$ is written in such a way that the leftmost character of the word w_1 is written in the square $-2n - l + 1$. Construct a word u by replacing the tape segment between p_i^L and p_i^R by the word $P(p_j^L, p_j^R)$ in the word w_1 (cf. Fig. 2). If we remove all blank characters b in the word u we shall obtain a word u_1 accepted by M although it holds that $u_1 \in \{a, c\}^+ - L$: a contradiction.



$w_1 = v_1v_2v_3v_4v_5$ $\qquad$ $u = v_1v_3v_5$

Figure 2.

Corollary. C *is a proper subclass of* CFL.

INCOMPARABILITY OF LIN AND C

Theorem 2. LIN *and* C *are incomparable, i.e.* LIN $\nsubseteq$ C & C $\nsubseteq$ LIN.

Proof. (2.1) Consider the language $L = \{a^n c^n a^i;\ i, n = 1, 2, \ldots\} \cup \{a^i c^n a^n;\ i, n = 1, 2, \ldots\}$. It holds that $L \in$ LIN. Assume that $\hat{M} = (\hat{Q}, X, \hat{d}, q_0, F)$ is such a TM which recognizes L with Wechsung's complexity k, where k is a nonnegative integer. Let $q_1, q_2 \notin \hat{Q}$. Define $M = (Q\ X, d, q_0, F)$, where $Q = \hat{Q} \cup \{q_1, q_2\}$ and the next-state function d is defined in the following way:

$d(q, z) = \hat{d}(q, z)$ if $(q, z) \in \hat{Q} \times X$ and $\hat{d}(q, z)$ is defined,
$d(q_1, b) = (q_1, b, R)$,
$d(q_1, a) = (q_2, a, L)$,
$d(q_2, b) = (q_0, b, R)$,
d is not defined for other arguments.

Now introduce the equivalences E_1 and E_2 on X^+ according to Definitions 1 and 2 and denote their indices e_1 and e_2, respectively.

Now enumerate the tape of M as indicated by Fig. 3.

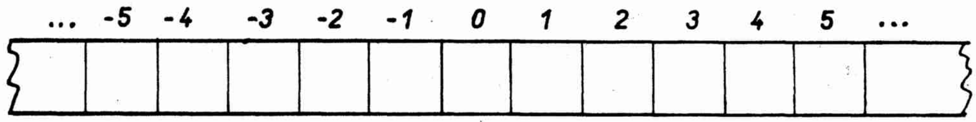


Figure 3.

Let for positive integers m_1, m_2, m_3 and m_4, where $m_1 < m_2 \leqq e_1 + 1 \;\&\; m_3 < m_4 \leqq e_1 + 1$, the condition $a^{m_1} \sim_{E_1} a^{m_2} \;\&\; c^{m_3} \sim_{E_1} c^{m_4}$ hold. Let l be a positive integer satisfying the assertion of Lemma 2 for the given TM M and the language L. Define $s = \text{card } Q$.

Take the word $w = a^n c^n a^n$, where $n \in N$, $n \geq s((e_2 + l - 1)^2 + e_2 + 2s + l - 4) + \max\{s + 1, m_1\}$. It holds that $w \in L$, hence during the computation of M on w at most k active visits on any square will occur and M will reach an accepting state.

Place w on the tape of the TM M in such a way that the leftmost character of the word w will be written in the square 1.

We shall construct inductively sequences

$$p_1^L, p_2^L, \ldots \quad \text{and} \quad p_1^R, p_2^R, \ldots .$$

Define $p_1^L = -l + 1$ and $p_1^R = 3n + l$.

Now let p_i^L and p_i^R for an i be defined. Then if there exists a square $p^{(i)}$ which is rewritten as the first of squares between p_i^L and p_i^R during the computation of M on w, define

$$\left.\begin{aligned} p_{i+1}^L &= p^{(i)} \\ p_{i+1}^R &= p_i^R \end{aligned}\right\} \quad \text{if} \quad p^{(i)} \leqq n - s \quad \text{and}$$

$$\left.\begin{aligned} p_{i+1}^L &= p_i^L \\ p_{i+1}^R &= p^{(i)} \end{aligned}\right\} \quad \text{if} \quad p^{(i)} > 2n + s\,,$$

p_{i+1}^L and p_{i+1}^R are not defined otherwise.

There are two possible cases which are studied in the paragraphs (2.1.1) and (2.1.2) in this proof. Each of this cases is decomposed into a number of subcases which are treated in the corresponding subparagraphs.

(2.1.1) Let the symbols p_i^L and p_i^R be meaningful for $i = (e_2 + l - 1)^2 + 1$.

(2.1.1.1) Let for a nonnegative integer i_0 such that $i_0 \leqq (e_2 + l - 1)(e_2 + l - 2)$, the condition $p_{i_0+1}^L = p_{i_0+2}^L = \ldots = p_{i_0+e_2+l}^L$ hold.

Consider the word $w_1 = a^{n+m_2-m_1} c^n a^n$. It holds that $w_1 \in L$. Place w_1 on the tape of M in such a way that the leftmost symbol of the word w_1 will be written in the square 1. It holds for $i = 1, 2, \ldots, (e_2 + l - 1)^2$ that during the computation of M on w_1, the first of squares between p_i^L and $m_2 - m_1 + p_i^R$ rewritten by M is

- the square p_{i+1}^L if $p_i^L \neq p_{i+1}^L$,
- the square $m_2 - m_1 + p_{i+1}^R$ otherwise.

Among the words $P(p^L_{i_0+l}, m_2 - m_1 + p^R_{i_0+l})$, $P(p^L_{i_0+l+1}, m_2 - m_1 + p^R_{i_0+l+1}), \ldots$ $\ldots, P(p^L_{i_0+e_2+l}, m_2 - m_1 + p^R_{i_0+e_2+l})$ there exists a pair of E_2-equivalent words. Let such a pair be formed by the words $P(p^L_{j_1}, m_2 - m_1 + p^R_{j_1})$ and $P(p^L_{j_2}, m_2 - m_1 + p^R_{j_2})$, where $j_1, j_2 \in N$, $i_0 + l \leqq j_1 < j_2 \leqq i_0 + e_2 + l$.

Now suppose that on the tape of M the word $w_2 = b^{l-1}w_1$ is written in such a way that the leftmost character of the word w_2 is written in the square $-l + 2$. Construct a word u by replacing the tape segment between $p^L_{j_1}$ and $m_2 - m_1 + p^R_{j_1}$ by the word $P(p^L_{j_2}, m_2 - m_1 + p^R_{j_2})$ in the word w_2. If we remove all blank characters b in the word u we shall obtain a word u_1 accepted by M although it holds that $u_1 \in \in \{a, c\}^+ - L$:

$$u_1 = a^{n+m_2-m_1}c^n a^{n_1} \quad \text{where} \quad n_1 \in N\,, \quad n_1 < n\,.$$

(2.1.1.2) Let for a nonnegative integer i_0 such that $i_0 \leqq (e_2 + l - 1)(e_2 + l - 2)$, the condition $p^R_{i_0+1} = p^R_{i_0+2} = \ldots = p^R_{i_0+e_2+l}$ hold. Contradiction can be deduced analogously as in paragraph (2.1.1.1).

(2.1.1.3) Let neither the introductory assumption of the paragraph (2.1.1.1) nor the introductory assumption of the paragraph (2.1.1.2) be fulfilled. Define $r = e_2 + + l - 1$. Among the words $P(p^L_{r(l-1)+1}, p^R_{r(l-1)+1})$, $P(p^L_{r.l+1}, p^R_{r.l+1})$, $P(p^L_{r(l+1)+1}, p^R_{r(l+1)+1}), \ldots, P(p^L_{r.r+1}, p^R_{r.r+1})$ there certainly exists a pair of E_2-equivalent words. Let such a pair be formed by the words $P(p^L_{j_1}, p^R_{j_1})$ and $P(p^L_{j_2}, p^R_{j_2})$, where $j_1 = = ri_1 + 1$, $j_2 = ri_2 + 1$, $i_1, i_2 \in N$, $l - 1 \leqq i_1 < i_2 \leqq r$.

Now construct a word u by replacing the tape segment between $p^L_{j_1}$ and $p^R_{j_1}$ by the word $P(p^L_{j_2}, p^R_{j_2})$ in the word w. M accepts u but $u \in \{a, c\}^+ - L$: $u = a^{n_1}c^n a^{n_2}$ where $n_1, n_2 \in N$, $n_1 < n$, $n_2 < n$. This contradiction completes the paragraph (2.1.1). In the paragraph (2.1.2) the following lemma is used. The proof of the lemma is evident.

Lemma 5. *There do not exist m_3 adjacent squares between n and $2n + 1$ the contents of which would not be changed during the computation of M on w.*

(2.1.2) Let the symbols p^L_i and p^R_i be meaningful for $i = j$, where j is a positive integer such that $j \leqq (e_2 + l - 1)^2$, and not meaningful for $i = j + 1$.

Let p_1 be such a square which is rewritten as the first of the squares between p^L_j and p^R_j during the computation of M on w (by Lemma 5 such a square exists).

(2.1.2.1) Let $n - s < p_1 \leqq n + s$.

Consider the word $w_1 = a^n c^n a^{n+m_2-m_1}$. It holds that $w_1 \in L$. Place w_1 on the tape of M in such a way that the leftmost symbol of the word w_1 will be written in the square 1. It holds for $i = 1, 2, \ldots, j - 1$ that during the computation of M on w_1, the first of the squares between p^L_i and $m_2 - m_1 + p^R_i$ rewritten by M is

– the square p^L_{i+1} if $p^L_i \neq p^L_{i+1}$,

– the square $m_2 - m_1 + p^R_{i+1}$ otherwise.