Really, we have

$$\operatorname{var}\left[\hat{\mathbf{Z}}_{p+N}(T) - \mathbf{Z}_{p+N+T}\right] = \operatorname{var} \boldsymbol{\varepsilon}_{p+N}(T)$$

and

$$\operatorname{var}\left[\dot{\mathbf{Z}}_{p+N}(T) - \mathbf{Z}_{p+N+T}\right] = \operatorname{var} \mathbf{L} + \operatorname{var} \boldsymbol{\varepsilon}_{p+N}(T)\,.$$

It leads to the desirable result, since var $\mathbf{L}$ is always a positive semidefinite matrix.

The influence of the random vectors $\mathbf{a}_p, \ldots, \mathbf{a}_{p-q+1}$ on the $\hat{\mathbf{Z}}_{p+N}(T)$ can also be investigated in the same way as the scalar case.

Theorem 4. *If all the roots of the equation*

$$|\boldsymbol{\Theta}(B)| = 0$$

are outside the unit circle, then $\mathbf{T}_j^N \to \mathbf{0}$ for $N \to \infty$.

Proof. Define

$$\mathbf{M} = \begin{pmatrix} \mathbf{0}, & \mathbf{I}, & \mathbf{0}, & \ldots, \mathbf{0} \\ \mathbf{0}, & \mathbf{0}, & \mathbf{I}, & \ldots, \mathbf{0} \\ \multicolumn{4}{c}{\dotfill} \\ \boldsymbol{\Theta}_q, & \boldsymbol{\Theta}_{q-1}, & \boldsymbol{\Theta}_{q-2}, & \ldots, \boldsymbol{\Theta}_1 \end{pmatrix}.$$

The roots of the matrix $\mathbf{M}$ are identical with the roots of the polynomial

$$K(x) = |\mathbf{I}x^q - \boldsymbol{\Theta}_1 x^{q-1} - \ldots - \boldsymbol{\Theta}_q| = |x^q \Theta(1/x)|$$

(see [1], p. 237). It implies that all the roots of the matrix $\mathbf{M}$ are inside the unit circle and thus

$$\mathbf{M}^n \to \mathbf{0} \quad \text{for} \quad n \to \infty\,.$$

Define

$$\boldsymbol{\Gamma}_n = (\boldsymbol{\Theta}_n^n, \boldsymbol{\Theta}_{n+1}^{n+1}, \ldots, \boldsymbol{\Theta}_{n+q-1}^{n+q-1})'\,.$$

We have

$$\boldsymbol{\Gamma}_{n+1} = \mathbf{M}^n \boldsymbol{\Gamma}_1\,.$$

Because $\mathbf{M}^n \to \mathbf{0}$, also $\boldsymbol{\Theta}_n^n \to \mathbf{0}$ for $n \to \infty$. Quite analogously as in the scalar case we can prove that $\mathbf{T}_j^N \to \mathbf{0}$.

References

[1] *Anděl, J.:* Statistical analysis of time series, (Czech), SNTL, Praha 1976.

[2] *Box G. E. P., Jenkins G. M.:* Time series analysis. Forecasting and control, Holden Day, San Francisco 1970.

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