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CLASS OF UNIMODAL DISTRIBUTIONS AND ITS TRANSFORMATIONS

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1. INTRODUCTION

The class of unimodal distributions plays a very important role in statistics and in the theory of probability. Transformations of this class given in this paper (see also KEMPERMAN [7]) enable us to describe some important properties of the class of unimodal distributions (e.g. a description of the set of all extreme points of this class) and to compactify the class in the weak topology. This approach by means of the measure theory has proved to be more efficient than the former ones (MULHOLLAND, ROGERS [9]). The class of multivariate unimodal distributions is defined so that similar transformations are possible. Therefore this definition of multivariate unimodality is rather different from the former ones (e.g. DONATH, ELSTER [3]). A relation to the class of logarithmic concave measures (see PRÉKOPA [11], [12]) is shown. Theorems on representation of the normal distribution and on the solution of the moment problem through the class of unimodal distributions demonstrate an application of the theory.

The arrangement of the work is following:

Section 2 deals with the one-dimensional case. The main results of this section are represented by Theorems 2.1, 2.2. Remark 2.2 describes the set of all extreme points of the class of unimodal distributions $U[x_0]$ with a fixed inflexion point x_0. Further, the compactification of the given class is carried out (see the class $U^*[x_0]$ in Definition 2.2 and Theorem 2.3).

Section 3 that is devoted to the class of multivariate unimodal distributions (Definition 3.1) contains the main results of the paper. There exists an advantageous relation to marginal distributions (Lemma 3.1) and analogous transformations

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$\mu = T\nu$ can be defined (Theorem 3.1 for the non-compact case and Theorem 3.2 after the compactification according to Definition 3.2). The relation to the class of logarithmic – concave measures is shown in Theorem 3.3. As some important multivariate distributions (such as the normal, beta, Wishart's distributions) are examples of the logarithmic concave measures (see PRÉKOPA [11]), these distributions belong after certain orthonormal transformations to our class of unimodal distributions or to a modified class (in the sense of Remark 3.2).

The assertions of Section 2 are given without proofs because they are special cases of more general assertions in Section 3. The proofs in Section 3 are outlined very briefly for their extensions. The detailed proofs of Section 2 may be found in CIPRA [1] and of Section 3 in CIPRA [2].

We use the notation $\mu(\boldsymbol{g}) = \boldsymbol{y}$ to denote that $\mu(g_i) = y_i$, $i = 1, \ldots, n$. Symbols $\int_x^\infty$, $\int_{x+0}^\infty$, $\int_{-x}^\infty$, $\int_{-\infty}^{x+0}$ will mean an integration over intervals $[x, +\infty)$, $(x, +\infty)$, $(-\infty, x)$, $(-\infty, x]$ respectively. A probability density is always considered relative to the Lebesgue measure.

2. CLASS OF ONE-DIMENSIONAL UNIMODAL DISTRIBUTIONS

Let us begin with a definition:

Definition 2.1. A random variable X has the unimodal distribution with an inflexion point $x_0 \in R$ if its distribution function $F(x) = P(X < x)$ is continuous and is convex in $(-\infty, x_0]$ and concave in $[x_0, +\infty)$. $U[x_0]$ will denote the class of the corresponding probability measures.

Remark 2.1. $U[x_0]$ is obviously a convex class of probability measures. Let $\mu \in U[x_0]$ have a distribution function F. Then F is even absolutely continuous according to the ε, δ – definition of the absolute continuity (see e.g. NATANSON [10]). Let us denote

$$f(x) = F'(x - 0), \quad x \neq x_0 . \tag{2.1}$$

Then f is a density of F such that

(2.2) f is continuous from the left for $x \neq x_0$, non-decreasing in $(-\infty, x_0)$ and non-increasing in $(x_0, +\infty)$.

We may conclude: a probability measure μ with a distribution function F belongs to $U[x_0]$ iff a density f of F exists such that (2.2) holds.

Let us denote by $P[x_0]$ the class of all probability measures ν on $(R, \mathscr{B})$ ($\mathscr{B}$ is the σ-field of all Borel subsets of R) such that $\nu(\{x_0\}) = 0$. We define the transformation $\mu = T\nu$, $\nu \in P[x_0]$ where the density f of μ has the form

$$f(x) = \int_{-\infty}^{x} \frac{1}{x_0 - u} \nu(\mathrm{d}u), \quad x \in (-\infty, x_0), \tag{2.3}$$

$$f(x) = \int_{x}^{\infty} \frac{1}{u - x_0} \nu(\mathrm{d}u), \quad x \in (x_0, +\infty). \tag{2.4}$$

It is easy to verify that f is really a probability density.

Theorem 2.1. *The transformation T described in* (2.3), (2.4) *is a* 1-1 *correspondence between the classes $P[x_0]$ and $U[x_0]$.*

Theorem 2.2. *This transformation T is a homomorphism in the sense of convex mixtures, i.e.: Let $\nu_u \in P[x_0]$ for each $u \in R$, let λ be an arbitrary probability measure on $(R, \mathscr{B})$. Let us define a probability measure ν as follows:*

$$\nu(A) = \int_{-\infty}^{\infty} \nu_u(A)\, \lambda(\mathrm{d}u), \quad A \in \mathscr{B}. \tag{2.5}$$

Then $\nu \in P[x_0]$ and

$$T\nu(A) = \int_{-\infty}^{\infty} T\nu_u(A)\, \lambda(\mathrm{d}u), \quad A \in \mathscr{B}. \tag{2.6}$$

Remark 2.2. Theorem 2.2 enables us to describe the set of all extreme points of the convex class $U[x_0]$ because we can give at once the set of all extreme points of $P[x_0]$:

$$\{\varepsilon_u : u \in R,\ u \neq x_0\}$$

where ε_u is a degenerate probability measure concentrated in u, i.e. $\varepsilon_u(\{u\}) = 1$. Therefore according to Theorem 2.2 the set of all extreme points of $U[x_0]$ is

$$\{\mu_u = T\varepsilon_u : u \in R,\ u \neq x_0\}. \tag{2.7}$$

A measure μ_u, $u \in (-\infty, x_0)$ or $u \in (x_0, +\infty)$, can be described explicitly as a uniform distribution concentrated in (u, x_0) or (x_0, u), respectively.

Neither $U[x_0]$ nor $P[x_0]$ is weakly compact, i.e. compact in the weak topology in the space of probability measures that induces the convergence in distribution. As a simple example we can consider this sequence of distribution functions $\{F_n\}$:

$$\begin{aligned}
F_n(x) &= 0, & x &\in \left(-\infty, x_0 - \frac{1}{n}\right), \\
&= 1, & x &\in \left[x_0 + \frac{1}{n}, +\infty\right), \\
&= \frac{n}{2}\left(x - x_0 + \frac{1}{n}\right), & x &\in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right).
\end{aligned}$$

Some applications (e.g. the theorems on minimax in Fan Ky [5]) show that the compactification of the class $U[x_0]$ is desirable.

Definition 2.2. Let x_0 be a fixed real number. $U^*[x_0]$ will denote a class of all probability measures on $(R, \mathscr{B})$ such that their distribution function F is convex in $(-\infty, x_0)$ and concave in $(x_0, +\infty)$.

Remark 2.3. It is obvious that $U[x_0] \subset U^*[x_0]$ and that $U^*[x_0]$ is a convex set. The following conclusion holds again: a probability measure μ belongs to $U^*[x_0]$ iff a function f with the property (2.2) exists such that for each Borel subset $A \subset \subset (-\infty, x_0)$ or $A \subset (x_0, +\infty)$,

$$\mu(A) = \int_A f(x)\, \mathrm{d}x\,.$$

We can give an analogous transformation $\mu = T\nu$ to the one defined in (2.3), (2.4):

$$F(x) = \int_{-\infty}^{x} f(t)\, \mathrm{d}t\,, \qquad x \in (-\infty, x_0]\,, \tag{2.8}$$

$$F(x) = \nu(\{x_0\}) + \int_{-\infty}^{x} f(t)\, \mathrm{d}t\,, \quad x \in (x_0, +\infty)\,, \tag{2.9}$$

where ν is an arbitrary probability measure on $(R, \mathscr{B})$, F is the distribution function of μ and f is defined according to (2.3), (2.4).

Theorem 2.3. *The transformation T described in* (2.8), (2.9) *is a* 1-1 *correspondence between the class of all probability measures on* $(R, \mathscr{B})$ *and* $U^*[x_0]$ *that is a homomorphism in the sense of convex mixtures* (see Theorem 2.2) *and a homeomorphism in the weak topology.*

Remark 2.4. The class of all probability measures is weakly compact, therefore the class $U^*[x_0]$ is weakly compact, too. As to the set of all extreme points of $U^*[x_0]$ it is the set described in (2.7) completed by the measure ε_{x_0}.

3. CLASS OF MULTIVARIATE UNIMODAL DISTRIBUTIONS

We formulate the results only for the two-dimensional case but the generalization for higher dimensions will be obvious.

Definition 3.1 takes advantage of Remark 2.1. Let us denote $S = \{(x, y) \in R^2 : : x = x_0 \text{ or } y = y_0\}$. This pair of axes divides the plane into four parts that will be called the open quadrants relative to (x_0, y_0).

Definition 3.1. Let $(x_0, y_0) \in R^2$ be a given point. $U[x_0, y_0]$ will denote a class of all probability measures on $(R^2, \mathscr{B}^2)$ with a density f satisfying the following properties:

(i) For each $x \neq x_0$ the function $f_x = f(x, \cdot)$ is continuous from the left for $y \neq y_0$, non-decreasing in $(-\infty, y_0)$ and non-increasing in $(y_0, +\infty)$, and analogously for each $y \neq y_0$ the function $f_y = f(\cdot, y)$ is continuous from the left for $x \neq x_0$, non-decreasing in $(-\infty, x_0)$ and non-increasing in $(x_0, +\infty)$.

(ii) $f_x(-\infty) = f_x(+\infty) = 0$ for each $x \neq x_0$ and

$f_y(-\infty) = f_y(+\infty) = 0$ for each $y \neq y_0$.

(iii) Let $(x_1, y_1), (x_2, y_2) \in R^2$ be two arbitrary points in the same open quadrant relative to (x_0, y_0) such that $|x_0| < |x_1| < |x_2|$, $|y_0| < |y_1| < |y_2|$. Then $f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2) \geqq 0$.

The following lemma shows an advantageous relation of the multivariate unimodal distributions to the marginal ones that is similar to the normal distributions.

Lemma 3.1. (i) *Let a probability measure μ on $(R^2, \mathscr{B}^2)$ be a joint distribution of independent distributions $\mu_1 \in U[x_0]$ and $\mu_2 \in U[y_0]$. Then $\mu \in U[x_0, y_0]$.*

(ii) *Let $\mu \in U[x_0, y_0]$ have a density $f(x, y)$ and let $y \neq y_0$ be such that the marginal density $f_2(y) \neq 0$. Then the distribution corresponding to the conditional density $f(x \mid y)$ belongs to $U[x_0]$.*

Proof. Both assertions can be verified very easily.

Let $P[x_0, y_0]$ be the class of all probability measures ν on $(R^2, \mathscr{B}^2)$ such that $\nu(S) = 0$. Now analogously to the one-dimensional case we are able to define a transformation $\mu = T\nu$, $\nu \in P[x_0, y_0]$ such that the density f of μ is

$$(3.1) \qquad f(x, y) = \int_x^\infty \int_y^\infty \frac{1}{(u - x_0)(v - y_0)} \nu(du, dv), \quad x > x_0, \quad y > y_0,$$

$$= \int_{-\infty}^x \int_y^\infty \frac{1}{(x_0 - u)(v - y_0)} \nu(du, dv), \quad x < x_0, \quad y > y_0$$

and analogously for the other two quadrants.

Theorem 3.1. *The transformation T described in (3.1) is a 1-1 correspondence between the classes $P[x_0, y_0]$ and $U[x_0, y_0]$ and it is a homomorphism in the sense of convex mixtures (see Theorem 2.2).*

Proof. Let $\mu \in U[x_0, y_0]$ have a density f for which the conditions from Definition 3.1 are fulfilled. If we denote by $\mathscr{B}_i^2$ the system of all Borel subsets in the i-th open quadrant relative to (x_0, y_0), we can define measures ϱ_i on $\mathscr{B}_i^2$, $i = 1, \ldots, 4$, in this way:

$$(3.2) \qquad \varrho_i(I) = f(a_1, b_1) - f(a_1, b_2) - f(a_2, b_1) + f(a_2, b_2)$$

for each semiclosed interval $I = [a_1, a_2) \times [b_1, b_2)$ lying in the i-th open quadrant relative to (x_0, y_0). Obviously, the measures ϱ_i are determined uniquely by this condition. If we define a measure ν on $(R^2, \mathscr{B}^2)$ as follows

$$(3.3) \qquad \nu(A) = \iint_A |u - x_0|\,|v - y_0|\,\varrho_i(\mathrm{d}u, \mathrm{d}v)\,, \quad A \in \mathscr{B}_i^2\,, \quad i = 1, \ldots, 4\,,$$
$$\nu(A) = 0\,, \quad A \in \mathscr{B}^2\,, \quad A \subset S\,,$$

then it is easy to verify that $\nu \in P[x_0, y_0]$ and (3.1) holds. Thorefore for each $\mu \in U[x_0, y_0]$ a measure $\nu \in P[x_0, y_0]$ exists such that $\mu = T\nu$.

Conversely, let $\mu = T\nu$, where ν is a given measure from $P[x_0, y_0]$. We will show that $\mu \in U[x_0, y_0]$. Let us define non-negative measures τ_i on $\mathscr{B}_i^2$, $i = 1, \ldots, 4$, as follows:

$$(3.4) \qquad \tau_i(\mathrm{d}u, \mathrm{d}v) = \frac{1}{|u - x_0|\,|v - y_0|}\,\nu(\mathrm{d}u, \mathrm{d}v)\,, \quad i = 1, \ldots, 4\,.$$

Then

$$(3.5) \qquad f(x, y) = \tau_1\{(u, v) : u \geqq x,\ v \geqq y\}\,, \quad x > x_0\,, \quad y > y_0\,,$$
$$f(x, y) = \tau_2\{(u, v) : u \leqq x,\ v \geqq y\}\,, \quad x < x_0\,, \quad y > y_0\,,$$

and analogously for the other quadrants.

Properties of the measures τ_i (the non-negativity and the monotonicity) imply that f fulfils the conditions from Definition 3.1.

Finally, a measure ν is determined uniquely by the measure $\mu = T\nu$, for the measures τ_i are determined uniquely by μ according to (3.5) and there is a 1-1 correspondence between τ_i and ν according to (3.4).

As to the proof of the fact that the transformation T is a homomorphism in the sense of convex mixtures it is necessary to treat $T\nu(A)$ separately for $A \in \mathscr{B}_i^2$, $i = 1, \ldots, 4$, by means of (3.1) and to use the generalized Fubini theorem (see e.g. HALMOS [6]).

The compactification of the class $U[x_0, y_0]$ is carried out in the following definition (see Definition 2.2 and Remark 2.3):

Definition 3.2. *Let $U^*[x_0, y_0]$ be the class of all probability measures on $(R^2, \mathscr{B}^2)$ such that a function f defined for $(x, y) \notin S$ exists with the following properties:*

(i) *f has the properties* (i), (ii), (iii) *from Definition* 3.1;

(ii) *for each $A \in \mathscr{B}_i^2$, $i = 1, \ldots, 4$*

$$\mu(A) = \iint_A f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

where $\mathscr{B}_i^2$ is the system of all Borel subsets in the i-th open quadrant relative to (x_0, y_0).

A transformation $\mu = T\nu$ from the class of all probability measures on $(R^2, \mathscr{B}^2)$ to the class $U^*[x_0, y_0]$ is possible again. It can be defined as follows:

$$\mu(A) = \iint_A f(x, y) \, \mathrm{d}x \, \mathrm{d}y \tag{3.6}$$

for each $A \in \mathscr{B}_i^2$, $i = 1, \ldots, 4$ where f is defined in (3.1),

$$\mu(A) = \nu(A) \tag{3.7}$$

for each $A \in \mathscr{B}^2$, $A \subset S$.

Obviously, the measure μ on $(R^2, \mathscr{B}^2)$ is determined uniquely by means of (3.6), (3.7).

Theorem 3.2. *The transformation T described in* (3.6), (3.7) *is a* 1-1 *correspondence between the class of all probability measures on $(R^2, \mathscr{B}^2)$ and the class $U^*[x_0, y_0]$ that is a homomorphism in the sense of convex mixtures* (see Theorem 2.2) *and a homeomorphism in the weak topology.*

Proof. The proof of the fact that T is a 1-1 correspondence and a homomorphism in the sense of convex mixtures is quite analogous to the proof of Theorem 3.1. We limit ourselves to prove that T is a homeomorphism in the weak topology.

As the space of all probability measures is weakly compact it is sufficient to prove that T is continuous in the weak topology. Let $\nu_j \to \nu$. We shall show that $\mu_j = T\nu_j \to \mu = T\nu$ or equivalently, $\iint\varphi \, \mathrm{d}\mu_j \to \iint\varphi \, \mathrm{d}\mu$ for each bounded continuous function φ on R^2. Let $\varphi^* = T\varphi$ be the transformation of the function φ according to (3.11), (3.12) in the next part of the paper (obviously, φ is locally integrable). It may be verified that φ^* is a bounded continuous function, too. Therefore $\iint\varphi^* \, \mathrm{d}\nu_j \to \iint\varphi^* \, \mathrm{d}\nu$. An application of the forthcoming Lemma 3.2 completes the proof because $\iint\varphi^* \, \mathrm{d}\nu_j = \iint\varphi \, \mathrm{d}\mu_j$, $\iint\varphi^* \, \mathrm{d}\nu = \iint\varphi \, \mathrm{d}\mu$.

Remark 3.1. Theorems 3.1, 3.2 enable us to find an explicit description of the set of all extreme points of the classes $U[x_0, y_0]$ and $U^*[x_0, y_0]$. It is the system of all two-dimensional uniform distributions concentrated in two-dimensional open intervals whose one extreme point is just (x_0, y_0) for the class $U[x_0, y_0]$, and it is the former one completed by the system $\{\varepsilon_{(u,v)} : (u, v) \in S\}$ for the class $U^*[x_0, y_0]$. The class $U^*[x_0, y_0]$ is really weakly compact with respect to Theorem 3.2.

Now we shall mention a certain connection of the class of multivariate unimodal distributions with the class of the so called logarithmic concave probability measures that has important applications in the stochastic programming (see PRÉKOPA [11]). A probability measure P on $(R^n, \mathscr{B}^n)$ is logarithmic concave if the inequality

$$P(\lambda A + (1 - \lambda) B) \geqq [P(A)]^\lambda [P(B)]^{1-\lambda} \tag{3.8}$$

holds for arbitrary convex sets $A, B \in R^n$ and each $0 < \lambda < 1$. According to PRÉKOPA [12] a probability measure P on $(R^n, \mathscr{B}^n)$ is logarithmic concave iff P is absolutely continuous relative to the Lebesgue measure with a density

$$f(\boldsymbol{x}) = \exp\{-Q(\boldsymbol{x})\}, \quad \boldsymbol{x} \in R^n \tag{3.9}$$

where Q is a convex function in R^n. Some important multivariate probability distributions such as the normal, beta or Wishart's distributions belong to this class (see PRÉKOPA [11]).

Theorem 3.3. *Let P be a logarithmic concave probability measure on $(R^n, \mathscr{B}^n)$ with a density (3.9) where Q is a separable convex function in R^n. Then a point $(z_1, \ldots, z_n)' \in R^n$ exists such that $P \in U[z_1, \ldots, z_n]$.*

Proof. First we prove the theorem for $n = 1$, i.e. $f(x) = \exp\{-Q(x)\}$ where Q is a convex function in R. With respect to the convexity of Q and to the fact that $\int \exp\{-Q(x)\}\,dx = 1$, a point x_0 must exist such that Q is non-increasing in $(-\infty, x_0]$ and non-decreasing in $[x_0, +\infty)$ and therefore $f(x) = \exp\{-Q(x)\}$ is non-decreasing in $(-\infty, x_0]$ and non-increasing in $[x_0, +\infty)$. Moreover, f is continuous for the convex function Q is continuous. Hence $P \in U[x_0]$ in virtue of Remark 2.1.

The case for general n is a consequence of Lemma 3.1(i) because

$$\exp\{-Q(x_1, \ldots, x_n)\} = \exp\{-\sum_{i=1}^{n} Q_i(x_i)\} = \prod_{i=1}^{n} \exp\{-Q_i(x_i)\}.$$

This theorem implies immediately the following corollary. Let P be a logarithmic concave probability measure on $(R^n, \mathscr{B}^n)$ with a density (3.9) where Q is a quadratic form $Q(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x}$ with a positive semidefinite matrix $\boldsymbol{Q}$ (hence Q is a convex function). Then there exists an orthonormal matrix $\boldsymbol{T}$ (i.e. $\boldsymbol{T}'\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}' = \boldsymbol{I}_n$) and a diagonal matrix $\boldsymbol{D}$ with non-negative eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $\boldsymbol{Q}$ in its diagonal so that $\boldsymbol{Q} = \boldsymbol{T}'\boldsymbol{D}\boldsymbol{T}$. If we consider an orthonormal transformation $\boldsymbol{y} = \boldsymbol{T}\boldsymbol{x}$ of the space R^n, the density f in (3.9) will change into

$$g(y) = \exp\{-\sum_{i=1}^{n} \lambda_i y_i^2\}. \tag{3.10}$$

This function g is a unimodal density (in the sense of Definition 3.1) according to Theorem 3.3. As a special case we obtain e.g.

Theorem 3.4. (on a representation of the normal distribution). *Let μ be an n-dimensional normal regular distribution with a vector of expectation $\boldsymbol{a}$ and a positive definite variance matrix $\boldsymbol{\Sigma}$. Then there exists a uniquely determined probability measure λ such that the normal distribution μ can be expressed as*

$$\mu(A) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mu_t(\boldsymbol{T}(A - \boldsymbol{a}))\, \lambda(d\boldsymbol{t}) \quad \text{for each} \quad A \in \mathscr{B}^n \tag{3.11}$$

where μ_t denotes an n-dimensional uniform distribution concentrated in an n-dimensional open interval such that $\boldsymbol{O}, \boldsymbol{t}$ are two of its extreme points, $\boldsymbol{T}$ is an orthonormal matrix such that $\boldsymbol{\Sigma} = \boldsymbol{T}'\boldsymbol{D}\boldsymbol{T}$ where $\boldsymbol{D}$ is a diagonal matrix with the positive eigenvalues of $\boldsymbol{\Sigma}$ in its diagonal, $\boldsymbol{T}(A - \boldsymbol{a}) = \{\boldsymbol{y} : \boldsymbol{y} = \boldsymbol{T}(\boldsymbol{x} - \boldsymbol{a}),\ \boldsymbol{x} \in A\}$.

Now it will be shown how to solve the moment problem through the class of unimodal distributions (see also KEMPERMAN [7]). An ample theory has been developed for the solution of the moment problem through the class of all probability measures on $(R^n, \mathscr{B}^n)$ (see e.g. KEMPERMAN [8], CIPRA [1]). Fortunately, the moment problem through the class of unimodal distributions can be transformed into the problem through the class of all probability distributions in virtue of the following Theorem 3.5. But first it is necessary to define the following transformation of functions.

Let $\varphi(x, y)$ be a real Borel measurable function in R^2 that is locally integrable in all open quadrants relative to (x_0, y_0), i.e. integrable in each bounded interval in these quadrants. Then we define

$$T\varphi(u, v) = \varphi^*(u, v) = \frac{1}{(u - x_0)(v - y_0)} \int_{x_0}^{u} \int_{y_0}^{v} \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y\,, \quad (u, v) \notin S\,, \tag{3.12}$$

$$\varphi^*(u, v) = \varphi(u, v)\,, \quad (u, v) \in S\,. \tag{3.13}$$

This means we use the same symbol T both for the transformation of measures and functions. The following assertion can be easily proved:

Lemma 3.2. *Let a function φ fulfil the above assumptions and let ν be an arbitrary probability measure on $(R^2, \mathscr{B}^2)$. Then*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi|\, \mathrm{d}\mu < +\infty \quad \textit{iff} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T|\varphi|\, \mathrm{d}\nu < +\infty \tag{3.14}$$

where $\mu = T\nu$.

Moreover, if at least one inequality in (3.14) holds then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi\, \mathrm{d}\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T\varphi\, \mathrm{d}\nu\,. \tag{3.15}$$

Let us confine ourselves to the moment problem consisting in finding an infimum.

Theorem 3.5. *Let $g_1, \ldots, g_n, h$ be real Borel measurable functions in R^2 that are locally integrable in each open quadrant relative to (x_0, y_0), $\boldsymbol{y} \in R^n$ a given vector. We denote*

$$L(\boldsymbol{y} \mid h) = \inf\{\mu(h) : \mu \in U^*[x_0, y_0] \cap M\,,\quad \mu(\boldsymbol{g}) = \boldsymbol{y}\} \tag{3.16}$$