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ON PERIODIC SOLUTIONS OF NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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In our previous paper ([2], Theorem 1) we established the existence of w-periodic solutions of the differential equation $x'' + Kx = F(t, x, x')$ for the case $K > 0$. In this note we prove an existence (and uniqueness; Corollary 2) theorem for this differential equation for $K \neq 0$. This theorem is stronger than Theorem 1 of [2] in the sense that there is no restriction on w (except that $[0, w] \subseteq [0, \pi/\sqrt{K}]$ for $K > 0$, and $[0, w] \subseteq [0, +\infty)$ for $K < 0$). Furthermore, its extension (which can be obtained with out difficulties) to a system of nonlinear second order differential equations provides a stronger theorem than Theorems 1 and 2 of [1].

Consider the scalar boundary value problem

$$x'' + f(t, x, x') = 0\,, \tag{1}$$

$$x(0) - x(w) = x'(0) - x'(w) = 0\,, \tag{2}$$

where f is a continuous real-valued function with domain $[0, w] \times R^2$.

Theorem 1. *Let there exist constants $K \neq 0$ and $C > 0$ such that*

$$M = \operatorname{Max}\{|Kx - f(t, x, x')| : t \in [0, w]\,,\ \ |x| \leq C\,, \\ |x'| \leq (\sqrt{|K|})\, C\} \leq |K|\, C\,. \tag{3}$$

Then in $[0, w] \subseteq [0, \pi/\sqrt{K}]$ if $K > 0$, and in $[0, w] \subseteq [0, +\infty)$ if $K < 0$, the problem (1), (2) has at least one solution $x(t)$ satisfying $|x(t)| \leq C$, $|x'(t)| \leq (\sqrt{|K|})\, C$ for $0 \leq t \leq w$.

Proof. If $K > 0$, then problem (1), (2) is equivalent to the integral equation

$$x(t) = \int_0^w G(t, s)\, F(s, x(s), x'(s))\, \mathrm{d}s\,, \tag{4}$$

where $F(t, x, x') = Kx - f(t, x, x')$ and $G(t, s)$ is Green's function

$$(5) \qquad G(t, s) = \begin{cases} \dfrac{1}{2\sqrt{K}} \cdot \dfrac{\cos(\sqrt{K})(\frac{1}{2}w + s - t)}{\sin(\sqrt{K})\, w/2} & \text{for} \quad 0 \leq s \leq t \leq w \\ \dfrac{1}{2\sqrt{K}} \cdot \dfrac{\cos(\sqrt{K})(\frac{1}{2}w + t - s)}{\sin(\sqrt{K})\, w/2} & \text{for} \quad 0 \leq t \leq s \leq w\,. \end{cases}$$

If $K < 0$, then (1), (2) is equivalent to (4) where

$$(6) \quad G(t, s) = \begin{cases} \dfrac{1}{2\sqrt{|K|}} \cdot \dfrac{\exp[-(\sqrt{|K|})(t-s)]\exp[(\sqrt{|K|})w] + \exp[(\sqrt{|K|})(t-s)]}{1 - \exp[(\sqrt{|K|})w]} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad s \leq t \\ \dfrac{1}{2\sqrt{|K|}} \cdot \dfrac{\exp[-(\sqrt{|K|})(s-t)]\exp[(\sqrt{|K|})w] + \exp[(\sqrt{|K|})(s-t)]}{1 - \exp[(\sqrt{|K|})w]} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t \leq s\,. \end{cases}$$

Let $S = \{x \in c'[0, w] : |x(t)| \leq C, |x'(t)| \leq (\sqrt{|K|})\, C\}$ and define an operator U on S by

$$U\, x(t) = \int_0^w G(t, s)\, F(s, x(s), x'(s))\, \mathrm{d}s\,.$$

From (3), it follows that

$$|U\, x(t)| \leq M \int_0^w |G(t, s)|\, \mathrm{d}s \leq \frac{M}{|K|} \leqq C\,,$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} U\, x(t)\right| \leq M \int_0^w |G_t(t, s)|\, \mathrm{d}s \leq \frac{M}{\sqrt{|K|}} \leq (\sqrt{|K|})\, C\,,$$

and hence U maps S continuously into itself. Therefore by Schauder's theorem (4) (and hence (1), (2)) has a solution with the desired properties.

Corollary 1. *If in addition to the hypotheses of Theorem* 1, *the function* $f(t, x, x')$ *is w-periodic in t and locally Lipschitzian with respect to* (x, x'), *then* (1), (2) *has a w-periodic solution.*

Corollary 2. *If in addition to the hypotheses of Theorem* 1, *the function* $f(t, x, x')$ *is w-periodic in t and if*

$$|F(t, x_1, x_1') - F(t, x_2, x_2')| \leqq C_1 \left\{|x_1 - x_2| + \frac{1}{\sqrt{|K|}} |x_1' - x_2'|\right\}, \quad 0 \leq t \leq w$$

for $(x_i, x_i') \in \Omega = \{(x, x') : |x| \leqq C, \ |x'| \leqq (\sqrt{|K|})\, C\}$, *where* $C_1 > 0$ *is a constant such that*

$$\frac{2C_1}{|K|} < 1\,,$$

then (1), (2) *has a unique w-periodic solution.*

Proof. If, for $x \in S$, we let

$$\|x\| = \operatorname{Max}\left\{|x(t)| + \frac{1}{\sqrt{|K|}}\, x'(t) : 0 \le t \le w\right\},$$

we can easily show that U is a contraction with respect to $\|\cdot\|$ on S.

Applications. Three applications of Theorem 1 for the case $K > 0$ can be found in ([2], pp. 73–75). We give below three applications for the case $K < 0$.

(A_1) *Consider the equation*

$$x'' + f(x)\, x'^n + ax = \mu\, p(t)\,, \quad a < 0\,, \quad n \ge 2 \tag{7}$$

where n is an integer, all coefficients are continuous, $f(x)$ *is locally Lipschitzian in* x, $0 \le f(x) \le b$ *for all* x, *and* $|\mu|$ *sufficiently small. If* $K < a/2$ *and if* $p(t)$ *is periodic of period* w, *then* (7) *has a w-periodic solution.*

Proof. The hypotheses of Corollary 1 are satisfied by choosing $C = |\mu|^{1/n}$ with $|\mu|$ sufficiently small.

(A_2) *Consider the equation*

$$x'' + a(t)\, x + b(t) f(x^2) = \mu\, p(t)\,, \tag{8}$$

and let

(i) $a(t)$, $b(t)$, $p(t)$ *be continuous and* $a(t)$ *non-positive,*

(ii) $f(x)$ *be locally Lipschitzian, non-negative, non-decreasing for* $x \geqq 0$ *and for some* $C > 0$

$$B\frac{f(C^2)}{C} + |\mu|\frac{D}{C} \le -E\,.$$

where

$$B = \operatorname*{Max}_{t\in[0,w]} |b(t)|\,, \quad D = \operatorname*{Max}_{t\in[0,w]} |p(t)|\,, \quad E = \operatorname*{Max}_{t\in[0,w]} a(t)\,.$$

If $K \le A = \operatorname*{Min}_{t\in[0,w]} a(t)$ *and if* $a(t)$, $b(t)$, $p(t)$ *are periodic of period* w *then* (8) *has a w-periodic solution.*