Then obviously $w(x)$ satisfies (4.1) and is non-decreasing. Further, $w(x)$ is small since $W(x) \leqq x + a_1 + a_2 + a_3 + \ldots < \infty$, and hence it satisfies (4.0), too. However, (4.4) does not hold. Moreover, for every positive k we have

$$\int_0^k \frac{x \, . \, \mathrm{d}x}{x - w(x)} = \infty$$

because the graph of $w(x)$ has a limit point $[a_n, a_n]$ for some $a_n \in (0, k)$.

Remark. There is also an increasing continuous small function $w(x)$ satisfying (4.0) and (4.1) and not satisfying (4.4). It can be constructed as an "approximation" of the function from Example 6.7. Hence we cannot replace (4.3) with a positive k by the assumption that $w(x)$ is increasing in Theorem 4.2. Analogously we cannot replace the assumption (i) in Theorem 4.7 by the assumption $w'(x) > 0$.

6.8. Example. A decreasing sequence $(a_1, a_2, a_3, \ldots)$ of positive reals such that $a_1 + a_2 + a_3 + \ldots < \infty$, $a_{n+1} - a_{n+2} \leqq a_n - a_{n+1}$ for all $n \in N$ and

$$\sum_{n=1}^{\infty} a_n \, . \, \ln \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} = \infty \, .$$

(Compare with Lemma 2.3.)

Let $c_1 = 1$, $c_{n+1} = c_n \, . \, \mathrm{e}^{c_n}$, $b_n = 1/c_n$, $a_n = b_n + b_{n+1} + b_{n+2} + \ldots$ for all $n \in N$. Then

$$\sum_{n=1}^{\infty} a_n \, . \, \ln \frac{a_n - a_{n+1}}{a_{n+1} - a_{n+2}} \geqq \sum_{n=1}^{\infty} \frac{1}{c_n} \, . \, \ln \frac{c_{n+1}}{c_n} = \sum_{n=1}^{\infty} 1 = \infty \, .$$

The other conditions can be easily verified.

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