GRADUAL PARTITION OF A GRAPH INTO COMPLETE GRAPHS

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INTRODUCTION

In this paper we investigate properties of the amalgamation operation of graphs.

Obviously every graph can be obtained by a gradual amalgamation of certain family of complete graphs. We are interested in the properties of this procedure. For every graph G we define the depth of G as a measure of *amalgamation inefficiency* of G. We prove that there are graphs of arbitrarily large depth and that for every n there exists a uniquely determined graph G_n with depth n and with a minimal number of vertices. We prove also that the depth of a planar graph is <4 which is best possible.

The paper has 2 parts:

In § 1 we introduce the notion of a gradual partition of a graph into complete graphs and state basic properties of this notion.

In § 2 we introduce and investigate the depth of a graph.

The results of this paper extend the results which were obtained at the Seminar of Applied Combinatorics at Charles University, Prague by the authors of [0].

1. GRADUAL PARTITIONS OF GRAPHS

1.1. In this paper we consider finite undirected graphs without loops and multiple edges. Explicitly, a graph G is a pair (V, E) where V is a finite set and $E \subseteq \binom{V}{2} = \{e \subseteq V; |e| = 2\}$. We shall use also the notation $G = (V(G), E(G))$.

Graph $\left(V, \binom{V}{2}\right)$ is called the complete graph on the set V and is denoted by K_V.

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Put $K_n = K_{[1,n]}$, where $[1, n] = \{1, 2, \ldots, n\}$. The graph $K_0 = K_\emptyset = (\emptyset, \emptyset)$ will be denoted sometimes shortly by $\emptyset$; this graph is called the void graph.

1.2. Definition. Let G, G' be graphs. A mapping $f : V(G) \to V(G)$ is called an *embeding* of G into G' if

1) f is a 1-1 mapping
2) $\{f(x), f(y)\} \in E(G') \Leftrightarrow \{x, y\} \in E(G)$.

1.3. Definition. Graph G is a *subgraph* of G' if $V(G) \subseteq V(G')$ and the inclusion is an embeding. This fact is denoted by $G \leqq G'$. Explicitly: $G = (V, E)$ is a subgraph of $G' = (V', E')$ if $V \subseteq V'$ and $E' \cap \binom{V}{2} = E$.

If $G \leqq G'$ and $G \neq G'$ then we write $G < G'$.

Obviously a subgraph $G = (V, E)$ of $G' = (V', E')$ is determined by the set of its vertices. In this case we also say that G is *induced* by G' on the set V. We use the notation $G'|_V = G$.

We shall find it convenient to use the following definitions:

1.4. Definition. Graph G is the *union* of G_1 and G_2 if $G_i \leqq G$, $i = 1, 2$, and $V(G) = V(G_1) \cup V(G_2)$. In this case we write $G = G_1 \vee G_2$.

Definition. Graph G is the *intersection* of graphs G_1 and G_2 if $G \leqq G_i$, $i = 1, 2$, and $V(G) = V(G_1) \cap V(G_2)$. The intersection is denoted by $G = G_1 \wedge G_2$.

Definition. Let $G_1 < G$. We say that (V', E') is the difference of graphs G and G_1 if $V' = V(G) \setminus V(G_1)$ and $(V', E') < G$. This fact is denoted by $G - G_1$.

1.5. Definition. Let $G = (V, E)$ be a graph, $v \in V$ a fixed vertex. Denote by G_v the subgraph of G induced by the set $V_v = \{v'; \{v, v'\} \in E\}$. Denote by G_v^* the subgraph of G induced on the set $V_v \cup \{v\}$.

1.6. Definition. Let G, G' be graphs. Define the graph $G + G'$ as follows:

$V(G + G') = V(G) \cup V(G')$,
$E(G + G') = E(G) \cup E(G') \cup \{\{v, v'\};\ v \in V(G),\ v' \in V(G')\}$.

Graph $G + G'$ is called the *direct sum* of graphs G and G'.

1.7. The following is the principal operation considered in this paper:

Definition. Let G, G_1, G_2, G_{12} be graphs. We say that the graph G is *partitioned* in graphs G_1 and G_2 with respect to the graph G if

1) $G_1 \leqq G$, $G_2 \leqq G$,
2) $G = G_1 \vee G_2$,
3) $G_{12} = G_1 \wedge G_2$.

In this case we write $G = (G_1, G_{12}, G_2)$. We write also $G = (G_1, G_{12}, G_2)$ if there exist isomorphisms $G \simeq G'$, $G_1 \simeq G_1'$, $G_{12} \simeq G_{12}'$, $G_2 \simeq G_2'$ such that $G' = (G_1', G_{12}', G_2')$.

If $G = (G_1, G_{12}, G_2)$ then the triple (G_1, G_{12}, G_2) is called a *partition* of G.

If $G = (G_1, G_{12}, G_2)$ and $G \neq G_1$, $G \neq G_2$, then (G_1, G_{12}, G_2) is called a *proper partition* of G.

1.8. Remarks. 1) Obviously there is not an edge $\{v_1, v_2\} \in E(G)$ for $v_1 \in V(G_1) \setminus V(G_{12})$, $v_2 \in V(G_2) \setminus V(G_{12})$.

2) If G is a disconnected graph then $G = (G_1, \emptyset, G_2)$ for convenient G_1, G_2. The graph $(G_1, \emptyset, G_2)$ is always disconnected.

3) The operation $(.,.,.)$ is the inverse operation to the amalgamation operation: if $G = (G_1, G_{12}, G_2)$ then G is an *amalgam* of G_1 and G_2 with respect to G_{12}, see [3].

4) If G is a complete graph, then there is no proper partition of G (see the above remark 1). On the other hand if G fails to be a complete graph, then there exists $v \in V(G)$ such that $G = (G_v^*, G_v, G|_{V(G)\setminus\{v\}})$. It suffices to take any vertex v for which $G|_{V(G)\setminus\{v\}} \neq G_v$ (which is equivalent to the fact that there exist $v' \in V(G)$, $v \neq v'$, $\{v, v'\} \notin E(G)$).

1.9. Remark 1.8.4 shows that every graph G may be gradually partitioned into a family of complete graphs. This gradual partition into complete graphs is introduced in the following two definitions:

1.10. Definition. A branching partition tree T is a quadruple (V, R, R', v) with the following properties:

1) $(V, R \cup R')$ is a branching from v,

2) (V, R) is a dyadic tree,

3) every vertex which fails to be a terminal vertex is incident with exactly one edge of R'.

1.11. Definition. Let G be a graph. A *gradual partition* of G into complete graphs with respect to a branching partition tree $T = (V, R, R', v)$ is a mapping $\mathcal{R} : V \to$ Gra with the following properties:

1) $\mathcal{R}(v) = G$.

2) If $(w, w_1) \in R$, $(w, w_2) \in R$, $(w, w_{12}) \in R'$ then $\mathcal{R}(w) = (\mathcal{R}(w_1), \mathcal{R}(w_{12}), \mathcal{R}(w_2))$.

3) If w is a terminal vertex of T then $\mathcal{R}(w)$ is complete graph.

If $t_1, \ldots, t_n$ are all terminal vertices of T then we say that G is *generated* by the set $\{\mathcal{R}(t_1), \ldots, \mathcal{R}(t_n)\}$. The set of all graphs generated by a set of complete graphs $\{K_{n(i)}; i \in I\}$ will be denoted by $[K_{n(i)}; i \in I]$.

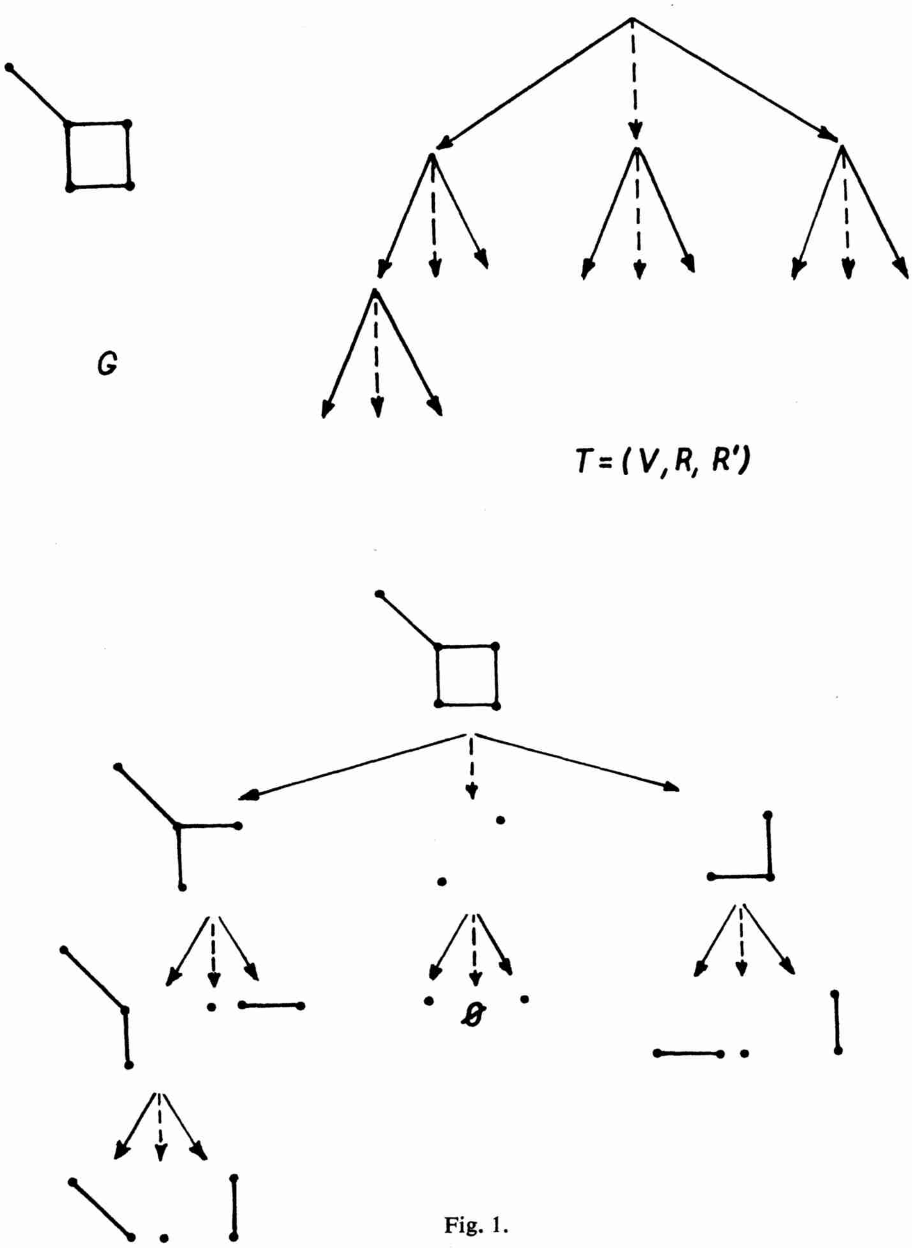

Fig. 1.

1.12. Remark. Denote by Gra the class of all finite graphs. Then

$$\text{Gra} = \bigcup_{n=1}^{\infty} [K_i;\ i = 0, \dots, n] = [K_i;\ i \in N] \quad \text{(see Remark 1.8.4)}.$$

It is easy to see that $[K_0, K_1, \dots, K_n] = \text{Gra}\,(n)$ where $\text{Gra}\,(n)$ is the class of all finite graphs which do not contain a complete subgraph with $n+1$ vertices. These classes were studied in [3]. These applications of amalgamation operation provided a motivation of this research.

1.13. Example. Fig. 1 illustrates a possible gradual partition of graph G with respect to the tree $T = (V, R, R')$. The arrows of R are depicted by straight lines, the arrows of R' by dotted lines.

1.14. Remark. Let $K^1, \ldots, K^n, L^1, \ldots, L^m$ be complete graphs. It is easy to see that

(i) $[K^1, \ldots, K^n] \cup [L^1, \ldots, L^m] \subseteq [K^1, \ldots, K^n, L^1, \ldots, L^m]$,

(ii) $[K^1, \ldots, K^n] \cap [L^1, \ldots, L^m] \supseteq [\{K^1, \ldots, K^n\} \cap \{L^1, \ldots, L^m\}]$.

However, if $[K^1, \ldots, K^n] \neq [L^1, \ldots, L^m]$ then $[K^1, \ldots, K^n] \cup [L^1, \ldots, L^m] \subsetneqq [K^1, \ldots, K^n, L^1, \ldots, L^m]$.

L. Kučera and J. Nešetřil asked when the equality in (ii) is valid. The equality in (ii) holds in most "simple cases". The smallest graph for which the equality in (ii) does not hold is in Fig. 2:

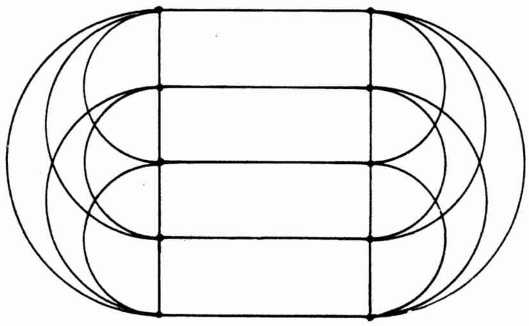

Fig. 2.

[4] contains a more detailed discussion of the equality in (ii).

2. DEPTH OF A GRAPH

In this part we introduce the notion of the depth of a gradual partition of a graph. This number characterizes the "inner" complexity of a graph.

2.1. Definition. Let $\mathscr{R}$ be a gradual partition of the graph G with respect to the tree $T = (V, R, R', v)$. Denote by $(v, x) = \{(v, x_1), \ldots, (x_n, x)\}$ the path from v to x, $x \in V$. We say that the vertex x belongs to the k-th level of $\mathscr{R}$ if $|(v, x) \cap R'| = k$.

We say that the graph $\mathscr{R}(x)$ belongs to the k-th level of $\mathscr{R}$ if $|(v, x) \cap R'| = k$.

Depth of a tree $T = (V, R, R', v)$ is the maximal level of a vertex of T. Depth of a gradual partition $\mathscr{R}$ of a graph G with respect to T is the depth of the tree T and it will be denoted by dp (R).

2.2. Definition. Depth of a graph G is the minimal depth of a gradual partition $\mathscr{R}$ of a graph G (with respect to a branching tree T).

Depth of the graph G will be denoted by $\text{dp}(G)$.

2.3. Remarks. 1) The gradual partition of the graph G in Fig. 1 has depth 2.

2) Obviously $\text{dp}(G) \leqq \max(\text{dp}(G_1), \text{dp}(G_{12}) + 1, \text{dp}(G_2))$ for every partition $G = (G_1, G_{12}, G_2)$. Moreover, for every graph G, $\text{dp}(G) > 0$ there exists a partition $G = (G_1, G_{12}, G_2)$ such that $\text{dp}(G) = \max(\text{dp}(G_1), \text{dp}(G_{12}) + 1, \text{dp}(G_2))$.

2.4. Proposition. 1) $\text{dp}(G) = 0$ *iff G is a complete graph.*

2) $\text{dp}(G) = 1$ *iff G does not contain a subgraph which is isomorphic to a cycle of length* >3.

Proof. 1) is obvious.

2) If G contains a cycle C of length >3 as a subgraph, then $\text{dp}(G) \geqq \text{dp}(C) = 2$. Now let G be a graph which does not contain a cycle of length >3 as a subgraph (graphs with this property are called triangulated graphs). We prove by induction on $|G|$ that $\text{dp}(G) \leqq 1$. It is well known that every minimal articulation set A of a triangulated graph is a complete graph. Hence there exists a partition $G = (G_1, G_{12}, G_2)$, where G_{12} is a complete graph and consequently $\text{dp}(G) = \max(\text{dp}(G_1), 1, \text{dp}(G_2))$ and we may use the induction hypothesis.

In the sequel we establish the basic properties of the depth of graphs. We prove the existence of graphs with an arbitrarily large depth.

2.5. Theorem. *Let $G \leqq H$. Then* $\text{dp}(G) \leqq \text{dp}(H)$.

Proof. If $G = H$ then $\text{dp}(G) = \text{dp}(H)$. It suffices to prove the statement of Theorem for $G = H - K_{\{a\}}$ for every $a \in V(H)$.

Let $\mathscr{R}$ be a gradual partition of H (into complete graphs) with respect to $T = (V, R, R', v)$. Define the mapping $\mathscr{R}' : V \to \text{Gra}$ by

$$\mathscr{R}'(w) = \mathscr{R}(w) - K_{\{a\}} \quad \text{if } a \text{ is a vertex of } \mathscr{R}(w),$$
$$\mathscr{R}'(w) = \mathscr{R}(w) \quad \text{otherwise}.$$

Obviously $\mathscr{R}'$ is a gradual partition of $G = H - K_{\{a\}}$ into complete graphs with respect to the same tree $T = (V, R, R', v)$. This proves $\text{dp}(G) \leqq \text{dp}(H)$.

To establish the depth of a direct sum of graphs we shall need the following

2.6. Lemma. Let H, G be non-void graphs. Let $H + G = (F_1, F_{12}, F_2)$. Then one of the following possibilities must occur (up to a permutation of symbols):

1) $F_1 = G_1 + H$, $F_{12} = G_{12} + H$, $F_2 = G_2 + H$ and $G = (G_1, G_{12}, G_2)$,

2) $F_1 = G + H_1$, $F_{12} = G + H_{12}$, $F_2 = G + H_2$ and $H = (H_1, H_{12}, H_2)$.

Proof. Let $G + H = (F_1, F_{12}, F_2)$ be a fixed partition. As $G + H \neq (H, F, G)$ for any graph F there are vertices a, b such that either $a, b \in V(H)$ or $a, b \in V(G)$ and $a \in V(F_1 - F_{12})$, $b \in V(F_2 - F_{12})$.

Assume without loss of generality that $a, b \in V(H)$. Furthermore, assume $G \nleqq F_{12}$. Then there exists a vertex $c \in V(G)$ such that either $c \in V(F_1 - F_{12})$ or $c \in V(F_2 - F_{12})$. We get a contradiction as $\{a, c\} \in E(G + H)$, $\{b, c\} \in E(G + H)$ and $F_1 \vee F_2 \neq G + H$. Thus $G \leqq F_{12}$. Put $H_1 = F_1 - G$, $H_2 = F_2 - G$, $H_{12} = F_{12} - G$. It is easy to check that $G + H = (G + H_1, G + H_{12}, G + H_2)$ and $H = (H_1, H_{12}, H_2)$.

2.7. Theorem. *Let G, H be non-void graphs. Then* $\mathrm{dp}(G + H) = \mathrm{dp}(G) + \mathrm{dp}(H)$.

Proof. Obviously $\mathrm{dp}(G) \leqq \mathrm{dp}(G + H)$, $\mathrm{dp}(H) \leqq \mathrm{dp}(G + H)$.

First, if G and H are complete graphs then Theorem is true.

Secondly, let H be a complete graph and $\mathscr{R}$ a gradual partition of G with respect to a tree $T = (V, E, E')$. Define $\mathscr{R}'$ as follows: $\mathscr{R}'(w) = \mathscr{R}(w) + H$. Obviously $\mathscr{R}'(w)$ is a complete graph iff $\mathscr{R}(w)$ is a complete graph and consequently $\mathscr{R}'$ is a gradual partition of $G + H$ into complete graphs with respect to T. Hence $\mathrm{dp}(\mathscr{R}) = \mathrm{dp}(\mathscr{R}')$ and, according to Lemma 2.6, $\mathrm{dp}(G + H) = \mathrm{dp}(G)$ if H is a complete graph. Finally, let G, H be a non-complete graphs. In this case we prove by induction on $|V(G + H)|$ that $\mathrm{dp}(G + H) = \mathrm{dp}(G) + \mathrm{dp}(H)$. The small values of $|V(G + H)|$ are obvious.

Let $|V(G + H)| = n + 1$ and let the statement of Theorem be valid for all graphs with $\leqq n$ vertices. Let $G + H = (G_1 + H, G_{12} + H, G_2 + H)$ where $G = (G_1, G_{12}, G_2)$. According to Lemma 2.6 and Remark 2.3 we may assume that

$$\mathrm{dp}(G + H) = \max\left(\mathrm{dp}(G_1 + H),\ \mathrm{dp}(G_{12} + H) + 1,\ \mathrm{dp}(G_2 + H)\right).$$

By the induction hypothesis it follows: $\mathrm{dp}(G + H) = \max(\mathrm{dp}(G_1) + \mathrm{dp}(H)$ $\mathrm{dp}(G_{12}) + \mathrm{dp}(H) + 1$, $\mathrm{dp}(G_2) + \mathrm{dp}(H)) = \mathrm{dp}(H) + \max(\mathrm{dp}(G_1),\ \mathrm{dp}(G_{12}) + 1,$ $\mathrm{dp}(G_2)) \geqq \mathrm{dp}(G) + \mathrm{dp}(H)$.

Consequently, it suffices to construct a gradual partition $\mathscr{R}$ of $G + H$ into complete graphs such that $\mathrm{dp}(\mathscr{R}) = \mathrm{dp}(G) + \mathrm{dp}(H)$. Let $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively, be gradual partitions of G and H into complete graphs with respect to branching partition trees T_1 and T_2 such that $\mathrm{dp}(\mathscr{R}_1) = \mathrm{dp}(G)$, $\mathrm{dp}(\mathscr{R}_2) = \mathrm{dp}(H)$. Let $T_i = (V_i, E_i, E'_i, v^i)$, $i = 1, 2$. Put $V_1 = \{v_1^1, \ldots, v_k^1, v_{k+1}^1, \ldots, v_n^1\}$ where $\{v_1^1, \ldots, v_k^1\}$ is the set of all end-vertices of the tree T_1. Define the branching tree $T = T_1 \circ T_2 = (V, E, E', v)$ by $V = V_1 \cup (V_2 \times [1, k])/\sim$ where $\sim$ is the equivalence generated by the set of pairs $\{(v_i^1, (v^2, i));\ i = 1, \ldots, k\}$. Let $[x]$ denote the equivalence class of $\sim$ containing the vertex x.

$$E = \{\{[x], [y]\};\ \{x, y\} \in E_1\} \cup$$
$$\cup \{\{[(x, i)], [(y, j)]\};\ i = j,\ \{x, y\} \in E_2\},$$
$$E' = \{\{[x], [y]\};\ \{x, y\} \in E'_1\} \cup$$
$$\cup \{\{[(x, i)], [(y, j)]\};\ i = j,\ \{x, y\} \in E'_2\},$$
$$v = [v^1].$$

Define the mapping $\mathscr{R}$ by

$$\mathscr{R}([x]) = \mathscr{R}_1(x) + H \quad \text{for} \quad x \in V_1,$$
$$\mathscr{R}([(x, i)]) = \mathscr{R}_1(v_i^1) + \mathscr{R}_2(x) \quad \text{for} \quad x \in V_2$$

(observe that this definition is consistent: if $[x] = [(y, i)]$ then x is an endvertex of G and hence $\mathscr{R}_1(x) + H = \mathscr{R}_1(v_i^1) + \mathscr{R}_2(y)$).

If is clear that $\mathscr{R}$ is a gradual partition of $G + H$ into complete graphs and hence $\mathrm{dp}(G + H) \leqq \mathrm{dp}(\mathscr{R}) = \mathrm{dp}(G) + \mathrm{dp}(H)$.

This proves Theorem.

2.8. Theorem. *Let* $\mathrm{dp}(G) = n > 0$. *Then there exists a subgraph* $H \leqq G$ *such that*

1) $\mathrm{dp}(H) = n$,
2) $\mathrm{dp}(H_a) = n - 1$ *for every vertex* a *of* H.

Proof. Let H be a subgraph of G such that $\mathrm{dp}(G) = \mathrm{dp}(H)$ and $\mathrm{dp}(H') < \mathrm{dp}(G)$ for every proper subgraph H' of H. Then obviously $\mathrm{dp}(H_a) \leqq n - 1$ for every $a \in V(H)$ (otherwise H would not be minimal).

Let $\mathrm{dp}(H_a) \leqq n - 2$ for a vertex $a \in V(H)$. As $H = (H_a^*, H_a, H - \{a\})$ and $\mathrm{dp}(H_a^*) = \mathrm{dp}(H_a) \leqq n - 2$ we have $\mathrm{dp}(H) = \mathrm{dp}(H - \{a\})$ which is a contradiction with the minimality of H.

2.9. Theorem. *Let* $\mathrm{dp}(G) = n$. *Then*

1) $|V(G)| \geqq 2n$,
2) $K_n \leqq G$.

Proof. We prove both statements by induction on n (the case $n = 0$ is obvious).

Let G be a graph, $\mathrm{dp}(G) = n > 0$. We may assume that $\mathrm{dp}(G') < n$ for every $G' \lneqq G$.

Let $G = (G_1, G_{12}, G_2)$ be a proper partition such that $\mathrm{dp}(G) = \max(\mathrm{dp}(G_1), \mathrm{dp}(G_{12}) + 1, \mathrm{dp}(G_2))$.

By the minimality of G, it is $\mathrm{dp}(G) = \mathrm{dp}(G_{12}) + 1$, using the induction hypothesis and the fact that (G_1, G_{12}, G_2) is a proper partition we have

$$|V(G)| \geqq |V(G_{12})| + 2 \geqq 2 . (n - 1) + 1 .$$

This proves 1.