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FUNDAMENTAL VECTOR FIELDS ON ASSOCIATED FIBER BUNDLES

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If G is a Lie groupoid and Y is a fiber bundle associated with G, then every section of the Lie algebroid LG of G determines a vector field on Y, which we call *a fundamental vector field on Y*. After deducing certain basic properties, we study the prolongations of the fundamental vector fields in connection with the general prolongation theory of projectable vector fields on arbitrary fibered manifolds, [2], [4], and with the prolongation theory of Lie algebroids, [5], [6]. We also develop a general point of view to Lie differentiation. Our consideration is in the category C^∞.

1. Given two manifolds M, N and diffeomorphisms $\varphi : M \to M$, $\psi : N \to N$, we define an induced diffeomorphism $(\varphi, \psi)^r$ on the space $J^r(M, N)$ of all r-jets of M into N by

$$j^r_x f \mapsto j^r_{\varphi(x)}(\psi \circ f \circ \varphi^{-1}) . \tag{1}$$

If ξ is a vector field on M, η is a vector field on N and ξ_t, η_t are the corresponding flows, then $(\xi_t, \eta_t)^r$ is a one-parameter family of diffeomorphisms of $J^r(M, N)$. This determines a vector field $(\xi, \eta)^r$ on $J^r(M, N)$ called the r-th prolongation of the pair (ξ, η). In coordinates, if $\xi \equiv \xi^k(u)\,(\partial/\partial u^k)$ and $\eta \equiv \eta^s(v)\,(\partial/\partial v^s)$, then

$$(\xi, \eta)^1 \equiv \xi^k \frac{\partial}{\partial u^k} + \eta^s \frac{\partial}{\partial v^s} + \left(\frac{\partial \eta^s}{\partial v^t} v^t_k - \frac{\partial \xi^l}{\partial u^k} v^s_l\right)\frac{\partial}{\partial v^s_k}, \tag{2}$$

where $v^s_k = \partial v^s/\partial u^k$ are the additional coordinates on $J^1(M, N)$, $k, l = 1, \ldots, \dim M$, $s, t = 1, \ldots, \dim N$. (In principle, the coordinate formula for $(\xi, \eta)^r$ can be deduced by iterating (2) and by the standard inclusions of the theory of non-holonomic jets.)

Consider further a fibered manifold $\pi : Y \to X$. Let ξ be a projectable vector field on Y, i.e. there is a unique vector field ξ_0 on X that is π-related with ξ. The space J^rY of all r-jets of the local sections of Y is a subset of $J^r(X, Y)$ invariant with respect to $(\xi_0, \xi)^r$. The restriction $p^r\xi$ of $(\xi_0, \xi)^r$ to J^rY is the r-th prolongation of ξ in the sense of [2], [4]. Let

$$x^i, \ y^p; \ i, j, \ldots = 1, \ldots, n = \dim X\,, \quad p, q, \ldots = 1, \ldots, \dim Y - \dim X\,,$$

be local fiber coordinates on Y and $\xi \equiv \xi^i(x)\,(\partial/\partial x^i) + \xi^p(x, y)\,(\partial/\partial y^p)$. Specializing (2), we obtain

$$p^1\xi \equiv \xi^i \frac{\partial}{\partial x^i} + \xi^p \frac{\partial}{\partial y^p} + \left(\frac{\partial \xi^p}{\partial x^i} + \frac{\partial \xi^p}{\partial y^q} y^q_i - \frac{\partial \xi^j}{\partial x^i} y^p_j\right) \frac{\partial}{\partial y^p_i}, \tag{3}$$

where $y^p_i = \partial y^p/\partial x^i$, cf. [4].

2. Let G be a Lie groupoid over X with source projection a and target projection b. Denote by LG the vector bundle (over X) of all a-vertical tangent vectors on G at the units, i.e. every element of $(LG)_x$, $x \in X$, is of the form $j^1_0\,\gamma(t)$, where $\gamma(t)$ is a curve on G satisfying $a\,\gamma(t) = x$ for all t and $\gamma(0) = e_x =$ the unit over x. Assume further that G acts on the left on a fibered manifold $\pi : Y \to X$ (in other words, Y is a fiber bundle associated with G), [9]. Every section $\varrho : X \to LG$ determines a vector field ϱ_Y on Y by

$$\varrho_Y(z) = j^1_0(\gamma(t)\,.\,z)\,, \tag{4}$$

$\pi(z) = x$, $\varrho(x) = j^1_0\,\gamma(t)$, which will be called *the fundamental field* (or G-field) *on* Y *determined by* ϱ.

Example 1. Let $E \to X$ be a vector bundle and G the groupoid of all linear isomorphisms between the fibers of E. A G-field on E will be called *a linear vector field*. In linear fiber coordinates on E, the coordinate form of a linear vector field is

$$\xi^i(x) \frac{\partial}{\partial x^i} + \xi^p_q(x)\, y^q \frac{\partial}{\partial y^p}\,. \tag{5}$$

Example 2. Similarly one introduces the affine vector fields on affine bundles. In particular, it is well-known that $J^1Y \to Y$ is an affine bundle for any fibered manifold Y.

Proposition 1. *The first prolongation $p^1\xi$ of any projectable vector field ξ on Y is an affine vector field on $J^1Y \to Y$.*

Proof is straightforward.

The target projection of G determines a fibered mahifold $G_b := (b : G \to X)$ and G acts on G_b by the left multiplication. The fundamental field on G_b defined by a section $\varrho : X \to LG$ will be denoted by ϱ_G. Such a field is characterized by the property that it is both a-vertical and right-invariant (i.e. every $g \in G$, $ag = x$, $bg = y$ determines a mapping $a^{-1}(y) \to a^{-1}(x)$, $g' \mapsto g'\,.\,g$ and ϱ_G is invariant with respect to all these mappings). If $\tau : X \to LG$ is another section, then the bracket $[\varrho_G, \tau_G]$ is also both a-vertical and right-invariant, so that there is a unique section $\{\varrho, \tau\} : X \to LG$ satisfying $\{\varrho, \tau\}_G = [\varrho_G, \tau_G]$. This endows LG with a Lie algebroid structure, [7].

Proposition 2. *If Y is a fiber bundle associated with G and ϱ, τ are two sections of LG, then the corresponding G-fields on Y satisfy*

$$(6) \qquad [\varrho_Y, \tau_Y] = \{\varrho, \tau\}_Y .$$

Proof. The source projection defines another fibered manifold $G_a := (a : G \to X)$ and the action of G on Y is a mapping $\varkappa : G_a \oplus Y \to Y$, where $\oplus$ means the fiber product over X. The zero vector field 0_Y of Y and ϱ_G determine a vector field $\varrho_G \oplus 0_Y$ on $G_a \oplus Y$. According to (4), $\varrho_G \oplus 0_Y$ is $\varkappa$-related with ϱ_Y, which proves Proposition 2.

Locally, G is isomorphic to $R^n \times H \times R^n$, where H is a Lie group and the multiplication is given by

$$(7) \qquad (x_3, h_2, x_2) \, . \, (x_2, h_1, x_1) = (x_3, h_2 h_1, x_1) ,$$

the product $h_2 h_1$ being defined in H. Further, Y is locally of the form $R^n \times F$, where F is a left H-space and the action of G on Y is given by

$$(8) \qquad (x_2, h, x_1) \, . \, (x_1, y) = (x_2, hy) ,$$

the latter product being determined by the action of H on F. Let

$$h^\alpha , \quad \alpha, \beta, \ldots = 1, \ldots, \dim H ,$$

be local coordinates on H in a neighbourhood of the unit and let e_α be the induced basis of the Lie algebra of H. Then a section ϱ of LG can be locally written as

$$(9) \qquad \varrho \equiv \varrho^i(x) \frac{\partial}{\partial x^i} + \varrho^\alpha(x) \, e_\alpha$$

and the coordinate formula for $\{\varrho, \tau\}$ is

$$(10) \qquad \{\varrho, \tau\} \equiv \left(\varrho^j \frac{\partial \tau^i}{\partial x^j} - \tau^j \frac{\partial \varrho^i}{\partial x^j} \right) \frac{\partial}{\partial x^i} + \left(\varrho^i \frac{\partial \tau^\alpha}{\partial x^i} - \tau^i \frac{\partial \varrho^\alpha}{\partial x^i} + c^\alpha_{\beta\gamma} \varrho^\beta \tau^\gamma \right) e_\alpha ,$$

provided $-c^\alpha_{\beta\gamma}$ are the structure constants of H, [8]. Further, let $A^p_\alpha(y) \, \partial/\partial y^p$ be the vector fields on F determined by e_α, [3]. Then we deduce by (8) the coordinate formula of ϱ_Y

$$(11) \qquad \varrho_Y \equiv \varrho^i(x) \frac{\partial}{\partial x^i} + A^p_\alpha(y) \, \varrho^\alpha(x) \frac{\partial}{\partial y^p} .$$

By Proposition 2 and (11), we conclude that the mapping $\varrho \mapsto \varrho_Y$ is a Lie algebroid homomorphism of LG into the Lie algebroid of all projectable vector fields on Y.

3. Denote by $\Gamma(g, t)$ the flow of the vector field ϱ_G and set $\gamma(x, t) = \Gamma(e_x, t)$. Since Γ is also right-invariant, we have

$$\Gamma(g, t) = \gamma(bg, t) . g , \tag{12}$$

i.e. Γ is determined by the values at the units of G.

The r-th prolongation G^r of G is a Lie groupoid over X defined as follows. The underlying set of G^r is the subset of all elements $A \in J^rG_a$ (= the r-th jet prolongation of fibered manifold $a : G \to X$) such that bA is an invertible r-jet of X into X, while the multiplication in G^r is given by

$$j^r_x\, g(u) . j^r_y\, h(v) = j^r_y[g(b\, h(v)) . h(v)] , \tag{13}$$

provided $b\, h(y) = x$, [1]. As ϱ_G is a-vertical, it is a-related with the zero vector field of X and we can construct its r-th prolongation $p^r\varrho_G$ on J^rG_a. Obviously, G^r is an invariant subspace of $p^r\varrho_G$.

Proposition 3. *The restriction $p^r\varrho_G \mid G^r$ is a fundamental field on G^r, i.e. there exists a unique section $\varrho^r : X \to LG^r$ such that $\varrho^r_{G^r} = p^r\varrho_G \mid G^r$.*

Proof. According to (12), the flow Γ^r induced by Γ on G^r is given by

$$\Gamma^r(j^r_x\, g(u), t) = j^r_x[\gamma(b\, g(u), t) . g(u)] . \tag{14}$$

Multiplying on the right by $j^r_y\, h(v)$, we obtain

$$j^r_y[\gamma(bg(b\, h(v)), t) . g(b\, h(v)) . h(v)] . \tag{15}$$

Using (13) we prove that Γ^r is a right-invariant flow, so that $p^r\varrho_G \mid G^r$ is a right-invariant vector field. Clearly, $p^r\varrho_G \mid G^r$ is also vertical with respect to the source projection of G^r, QED.

In the above construction, $\varrho^r(x)$ is fully determined by $j^r_x\varrho \in J^r(LG)$. This defines an identification $J^r(LG) \approx LG^r$; a detailed proof can be found in [6].

On the other hand, G^r acts on J^rY by

$$j^r_x\, g(u) . j^r_x\, \sigma(u) = j^r_y[g((bg)^{-1}(v)) . \sigma((bg)^{-1}(v))] , \tag{16}$$

where $y = b\, g(x)$ and $(bg)^{-1}$ means the inverse map of a local diffeomorphism $u \mapsto b\, g(u)$ of X into itself, [1]. Hence ϱ^r induces a G^r-field $\varrho^r_{J^rY}$ on J^rY.

Proposition 4. *The latter field coincides with the r-th prolongation of ϱ_Y, i.e.*

$$p^r(\varrho_Y) = \varrho^r_{J^rY} . \tag{17}$$

Proof consists in comparing (1), (4), (14), (16), QED.

For $r = 1$, we now deduce the coordinate expressions. Locally, we have $G^1_n = R^n \times H^1_n \times R^n$, [1], and the underlying manifold of H^1_n is the product of T^1_nH

(= the space of all n^1-velocities on H) and $L^1_n = GL(n, R)$. The induced coordinates h^α, $h^\alpha_i = \partial h^\alpha/\partial x^i$ on $T^1_n H$ and the canonical coordinates on L^1_n determine a basis e_α, e^i_α, e^i_j of the Lie algebra of H^1_n. Using (3) and Proposition 3, we find the following coordinate expression of $\varrho^1 : X \to LG^1$

$$\varrho^1 \equiv \varrho^i \frac{\partial}{\partial x^i} + \varrho^\alpha e_\alpha + \frac{\partial \varrho^\alpha}{\partial x^i} e^i_\alpha + \frac{\partial \varrho^j}{\partial x^i} e^i_j . \tag{18}$$

On the other hand, $J^1 Y$ is locally of the form $R^n \times T^1_n F$, [1]. According to [3], the vector fields on $T^1_n F$ corresponding to e_α, e^i_α, e^i_j are

$$A^p_\alpha \frac{\partial}{\partial y^p} + \frac{\partial A^p_\alpha}{\partial y^q} y^q_i \frac{\partial}{\partial y^p_i}, \quad A^p_\alpha \frac{\partial}{\partial y^p_i}, \quad -y^p_j \frac{\partial}{\partial y^p_i},$$

provided y^p_i are the additional coordinates on $T^1_n F$. Hence the coordinate form of $\varrho^1_{J^1Y}$ is

$$\varrho^1_{J^1Y} \equiv \varrho^i \frac{\partial}{\partial x^i} + A^p_\alpha \varrho^\alpha \frac{\partial}{\partial y^p} + \left(\frac{\partial A^p_\alpha}{\partial y^q} y^q_i \varrho^\alpha + A^p_\alpha \frac{\partial \varrho^\alpha}{\partial x^i} - y^p_j \frac{\partial \varrho^j}{\partial x^i} \right) \frac{\partial}{\partial y^p_i} . \tag{19}$$

On the other hand, we also obtain this formula by applying (3) to (11), which yields another proof of Proposition 4.

4. First we introduce a general concept needed in Proposition 5. Let M be a manifold and $p_M : TM \to M$ the tangent bundle of M. There are two natural projections of TTM into TM, namely the bundle projection p_{TM} and the tangent map Tp_M. Consider further the canonical involution i of TTM. Let $A, B \in TTM$ satisfy $p_{TM}(A) = Tp_M(B)$ and $Tp_M(A) = p_{TM}(B)$. Then iB lies in $T_v TM$, $v = p_{TM}(A)$, and one verifies directly that the difference $A - iB$ belongs to the tangent space of the vector space $T_x M$, $x = p_M(v)$. Hence $A - iB$ is identified with an element of $T_x M$, which will be called the strong difference of A and B and denoted by $A \dot{-} B$. In coordinates, if x^i, $X^i = \mathrm{d}x^i$ are local coordinates on TM and $A \equiv (x^i, X^i, \mathrm{d}x^i, \mathrm{d}X^i = a^i)$, $B \equiv (x^i, \mathrm{d}x^i, X^i, \mathrm{d}X^i = b^i)$, then

$$A \dot{-} B \equiv (x^i, a^i - b^i) . \tag{20}$$

Consider now a projectable vector field ξ on $Y \to X$ over ξ_0 and a section σ of Y. Taking into account the corresponding flows φ_t and φ_{0t}, we construct a curve

$$t \mapsto \varphi_t^{-1}(\sigma(\varphi_{0t}(x))) \tag{21}$$

in the fiber Y_x, whose tangent vector $(L_\xi \sigma)(x) \in T_{\sigma(x)}(Y_x)$ will be called the Lie derivative of σ with respect to ξ at x. Evaluating (21), we find

$$L_\xi \sigma = \sigma_* \xi_0 \dot{-} \xi_0 \sigma , \tag{22}$$

where $\sigma_* \xi_0$ is the image of ξ_0 by the differential of σ. In coordinates, if $\xi \equiv \xi^i(x) \,. \,(\partial/\partial x^i) + \xi^p(x, y)\,(\partial/\partial y^p)$ and $\sigma \equiv \sigma^p(x)$, then

$$L_\xi \sigma \equiv \frac{\partial \sigma^p}{\partial x^i} \xi^i(x) - \xi^p(x, \sigma(x)) \,. \tag{23}$$

In particular, if Y is a fiber bundle associated with a Lie groupoid G and ϱ is a section of LG, then we write $L_\varrho \sigma$ instead of $L_{\varrho_Y} \sigma$. Geometrically, $(L_\varrho \sigma)(x)$ is the tangent vector to the curve $\gamma^{-1}(x, t) \,. \,\sigma(b\, \gamma(x, t))$, where γ has the same meaning as in (12). In coordinates,

$$L_\varrho \sigma \equiv \frac{\partial \sigma^p}{\partial x^i} \varrho^i(x) - A^p_\alpha(\sigma(x))\, \varrho^\alpha(x) \,. \tag{24}$$

Let $T(Y/X)$ be the bundle of all vertical tangent vectors on Y. This is a vector bundle over Y, but it can be also considered as a fibered manifold over X. Similarly to § 1, every projectable vector field ξ on Y is prolonged into a projectable vector field $\bar{\xi}$ on $T(Y/X) \to X$. Taking into account the inclusion $TY \subset J^1(R, Y)$, we deduce by (2) (with zero vector field on R) that

$$\bar{\xi} \equiv \xi^i \frac{\partial}{\partial x^i} + \xi^p \frac{\partial}{\partial y^p} + \frac{\partial \xi^p}{\partial y^q} Y^q \frac{\partial}{\partial Y^p} \,, \tag{25}$$

provided $Y^p = \mathrm{d}y^p$. Consider another projectable vector field η on Y. Since $L_\xi \sigma$ is a section of $T(Y/X) \to X$, we have defined the Lie derivative $L_{\bar{\eta}}(L_\xi \sigma)$. If we construct conversely $L_{\bar{\xi}}(L_\eta \sigma)$, then the vectors $L_{\bar{\eta}}(L_\xi \sigma)(x)$, $L_{\bar{\xi}}(L_\eta \sigma)(x) \in T\,T(Y_x)$ satisfy the conditions of the definition of the strong difference. By direct evaluation, we prove

Proposition 5. *It holds*

$$L_{\bar{\xi}}(L_\eta \sigma) \dot{-} L_{\bar{\eta}}(L_\xi \sigma) = L_{[\xi, \eta]} \sigma \,. \tag{26}$$

Given a vector bundle $E \to X$, every element $A \in T(E_x)$ is identified with a vector $tA \in E_x$. In particular, $tL_\xi \sigma$ is now a section of E as well. Moreover, if ξ and η are linear vector fields on E, then (5), (25) and (26) imply

$$tL_\xi(tL_\eta \sigma) - tL_\eta(tL_\xi \sigma) = tL_{[\xi, \eta]} \sigma \,. \tag{27}$$

This formula generalizes a result by Que, [8], and includes the classical case of the first order tensor bundles. However, we underline that (27) does not hold for general projectable vector fields on E.

5. The product $X \times X$ with the trivial partial composition $(x_3, x_2) \,. \,(x_2, x_1) = (x_3, x_1)$ is a special Lie groupoid over X. The r-th prolongation of $X \times X$ is the groupoid $\Pi^r X$ of all invertible r-jets of X into X. The Lie algebroid $L(X \times X)$