SINGULAR SUPPORTS I

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The present paper, the first of a series, represents the first part of an investigation of abstract convolution equations. A preliminary communication [8] appeared already in the Soviet Doklady in 1974.

The aim of these investigations is to develop a functional-analytic theory of Hörmander's results on convolution equations. It is obvious that such a theory must contain two essential parts. The first task is to find a suitable abstract analogue of the notion of "singular support" of a distribution. This line of research started with the 1966 paper [5] and was pursued further in [11], [1] and [8], [8']. The second step consist in formulating criteria for $F' = (\varinjlim F \cap E_n)'$ or $F = \varinjlim F \cap E_n$ where E_n is a sequence of Fréchet spaces and $F \subset E = \lim E_n$ Results in this direction have been obtained in [9].

We shall use the following terminology and notation. An F_0 space will be a locally convex space the topology of which is given by a sequence of pseudonorms; it follows that a separated and complete F_0 space is a Fréchet space.

Given two topologies u_1 and u_2 on a set T we say that u_1 is coarser than u_2 or that u_2 is finer than u_1 if $u_1 \subset u_2$. In other words, a finer topology has more open sets and gives, accordingly, smaller closures. We shall denote by $u_1 \vee u_2$ the topology generated by the union $u_1 \cup u_2$, in other words, the coarsest topology which is finer than both u_1 and u_2.

(1,1) Lemma. *Let F be a linear space and w_1 and w_2 two convex topologies on F. Let $u = w_1 \vee w_2$. Then $(F, u)' = (F, w_1)' + (F, w_2)'$.*

Proof. The mapping $x \mapsto [x, x]$ is an algebraically and topologically isomorphic injection of $(F, w_1 \vee w_2)$ into $(F, w_1) \oplus (F, w_2)$. Its adjoint mapping takes the pair $[f_1, f_2] \in (F, w_1) \oplus (F, w_2)$ into its sum.

(1,2) Proposition. *Let $(E_1, u_1), (E_2, u_2), (E_3, u_3)$ be three F_0 spaces. Let*

$$T: \ (E_1, u_1) \mapsto (E_3, u_3),$$
$$A: \ (E_1, u_1) \mapsto (E_2, u_2)$$

be two continuous linear mappings. Let U be a fixed closed absolutely convex neighborhood of zero in (E_1, u_1). Denote by u the topology on E_1 generated by the set U and suppose that (E_1, u) is a normed space.

Then the following conditions are equivalent

1° $A'E_2' \subset T'E_3' + (E_1, u)'$.

2° *A is continuous from $(E_1, u \vee T^{-1}u_3)$ into (E_2, u_2); in other words: if $x_n \to 0$ in (E_1, u) and $Tx_n \to 0$ then $Ax_n \to 0$.*

3° *If x_n is sequence such that x_n is Cauchy in (E_1, u) and Tx_n is Cauchy in (E_3, u_3) then there exists a sequence x_n' such that $x_n' - x_n \to 0$ in (E_1, u), $Tx_n' - Tx_n \to 0$ in (E_3, u_3) and Ax_n' is Cauchy in (E_2, u_2); furthermore, if $z_n \to 0$ in (E_1, u), $Tz_n \to 0$ in (E_3, u_3) and Az_n is Cauchy in (E_2, u_2) then $Az_n \to 0$ in (E_2, u_2).*

4° *If x_n is sequence such that x_n is Cauchy in (E_1, u) and Tx_n is Cauchy in (E_3, u_3) then there exists a sequence x_n' such that $x_n' - x_n \to 0$ in (E_1, u) and Ax_n' is Cauchy in (E_2, u_2); at the same time, if $z_n \to 0$ in (E_1, u), $Tz_n \to 0$ in (E_3, u_3) and Az_n is Cauchy in (E_2, u_2) then $Az_n \to 0$ in (E_2, u_2).*

Proof. According to lemma (1,1) we have

$$(E_1, u \vee T^{-1}u_3)' = (E_1, u)' + T'E_3'.$$

Condition 1° may thus be restated as follows: the mapping A is continuous in the weak topologies corresponding to $u \vee T^{-1}u_3$ and u_2. All spaces in question being F_0 spaces weak and strong continuity coincides. This establishes the equivalence of 1° and 2°.

For the rest of the proof, it will be convenient to introduce some notation. Let T_0 and A_0 be the mapping from (E_1, u) respectively into (E_3, u_3) and (E_2, u_2) which coincide with T and A as mappings of linear spaces, hence $T = T_0 v$ and $A = A_0 v$ where v is the injection of (E_1, u_1) into (E_1, u).

Denote by $G(T_0)$ and $G(A_0)$ their graphs in $(E_1, u) \times (E_3, u_3)$ and $(E_1, u) \times (E_2, u_2)$. Denote by $A^\square$ the mapping of $G(T_0)$ into $G(A_0)$ defined as follows

$$A^\square[x, Tx] = [x, Ax].$$

We set

$$T^\sim = TP_1 \mid G(T_0) = P_2 T^\square, \quad A^\sim = AP_1 \mid G(T_0) = P_2 A^\square.$$

The implications 2° → 3° → 4° are immediate. Suppose now that condition 4° is satisfied.

Consider the set $M \subset (E, u)^\wedge \times E_3^\wedge \times E_2^\wedge$ defined as follows: The triple $[e_1, e_3, e_2]$ belongs to M if and only if $[e_1, e_3] \in G(T_0)^-$ and at the same time $[e_1, e_2] \in G(A_0)^-$. Here the closures are taken in the completions of the spaces in question. It follows from the definition of the set M that it is closed in $(E_1, u)^\wedge \times E_3^\wedge \times E_2^\wedge$. It follows from the second part of assumption 4° that the inclusion $[0, 0, e_2] \in M$

implies $e_2 = 0$. The set M is, therefore, the graph of a mapping from $G(T_0)^-$ into $E_2^\wedge$. Hence the mapping $A^\square$ is closable. Let us show that the domain of M is the whole of $G(T_0)^-$. Indeed, let $[e_1, e_3] \in G(T_0)^-$. It follows that there exists a sequence $x_n \in E_1$ such that $x_n \to e_1$ in $(E_1, u)^\wedge$ and $Tx_n \to e_3 \in (E_3, u_3)^\wedge$. According to 4° there exists a sequence $x'_n \in E$ such that $x'_n - x_n \to 0$ in (E_1, u) and Ax'_n is a Cauchy sequence in E_2. It follows that $x'_n \to e_1$ in (E_1, u) and $Ax'_n \to e_2$ for a suitable $e_2 \in \in (E_2, u_2)^\wedge$ so that $[e_1, e_2] \in G(A_0)^-$; hence $[e_1, e_3, e_2] \in M$. To sum up; the closure of $A^\square$ is again a mapping and is defined on the whole of $G(T_0)^\wedge$. It follows from the closed graph theorem that $A^\square$ is continuous so that $A^\sim = P_2 A^\square$ is continuous as well. We complete the proof by proving the implication 4° → 1°.

Since the mapping

$$A^\sim = P_2 A^\square : [x, Tx] \mapsto Ax$$

is continuous from $G(T_0)$ into (E_2, u_2), it follows that, for each $e'_2 \in (E_2, u_2)'$ the function

$$[x, Tx] \mapsto \langle Ax, e'_2 \rangle$$

is continuous on $G(T_0)$. Hence there exist two functionals $e'_1 \in (E_1, u)'$ and $e'_3 \in \in (E_3, u_3)'$ such that

$$\langle Ax, e'_2 \rangle = \langle x, e' \rangle + \langle Tx, e'_3 \rangle = \langle x, e' + T'e'_3 \rangle$$

whence $A'e'_2 = e' + T'e'_3 \in (E_1, u)' + T'E'_3$. This proves 1°.

Conditions 3° and 4° may be restated in the form of statements about domains of definition of certain mappings. We shall use the following notation. If G is the graph of a mapping from F_1 into F_2 we shall denote by $D(G^-)$ the projection on $F_1^\wedge$ of the closure G^- in $F_1^\wedge \times F_2^\wedge$. The set $D(G^-)$ will be called *the domain of definition* of G^-; of course, in the general case, G^- need not be the graph of a mapping from $F_1^\wedge$ into $F_2^\wedge$.

First of all, let us notice that the second part of conditions 3° and 4° asserts that the mapping $A^\sim$ is closable. Using this fact, condition 3° assumes the following form

5° The mapping A is closable and

$$G(T_0)^- \subset D(G(A^\sim)^-).$$

Since clearly $D(G(T^\sim)^-) = G(T_0)^-$, we have the following equivalent form of 3°

6° The mapping $A^\sim$ is closable and

$$D(G(T^\sim)^-) \subset D(G(A^\sim)^-).$$

Let us turn to condition 4°. Its second part may be interpreted as the closability of $A^\square$. In view of this condition 4° may be restated in each of the two following equivalent forms

7° The mapping $A^\square$ is closable and $G(T_0)^- \subset D(G(A^\square)^-)$, 8° The mapping $A^{\square}$ is closable and

$$D(G(T_0)^-) \subset D(G(A_0)^-).$$

In the sequel we shall often identify $G(T_0)$ with the space $(E_1, \tilde{u})$ where $\tilde{u} = u \vee T^{-1}u_3$. Accordingly, $T^{\sim}$ and $A^{\sim}$ will be taken as the mappings T and A considered as mappings of $(E_1, \tilde{u})$ into (E_3, u_3) and (E_2, u_2) respectively.

(1,3) Proposition. *The following conditions are equivalent:*

1° *If $x_n \in U$ and $Tx_n \to 0$ then Ax_n tends to zero in the weak topology of E_2.*

2° *For every $\varepsilon > 0$, the set $A'E'_2$ is contained in $T'E'_3 + \varepsilon U^0$.*

3° *The mapping $A^{\sim}$ is continuous and*

$$\operatorname{Ker} T^{\sim\prime\prime} \subset \operatorname{Ker} A^{\sim\prime\prime}.$$

4° *The mapping $A^{\sim}$ is continuous and* $\operatorname{Ker} T_0'' \subset \operatorname{Ker}(T_0 \oplus A_0)''$.

5° *The mapping $A^{\sim}$ is continuous and if $\xi \in (E, u)''$ annihilates $(E, u)' \cap T'E'_3$ then ξ annihilates $(E, u)' \cap (T'E'_3 + A'E'_2)$.*

6° *The mapping $A^{\sim}$ is continuous and the subspace $(E, u)' \cap (T'E'_3 + A'_2E'_2)$ is contained in the closure of $(E, u)' \cap T'E'_3$ in the strong topology of the space $(E, u)'$.*

7° *The weak topology on E_1 generated by $A'E'_2$ is coarser than that generated by $T'E'_3$ when restricted to U; in other words*

$$\sigma(E_1, A'E'_2) \mid U \subset \sigma(E_1, T'E'_3).$$

8° *The weak topology on E_1 generated by $A'E'_2$ is coarser than the topology $T^{-1}u_3$ when restricted to U; in other words if W is an arbitrary neighbourhood of zero in the topology $\sigma(E_1, A'E'_2)$ then there exists a neighbourhood of zero U_3 in (E_3, u_3) such that*

$$U \cap T^{-1}U_3 \subset W.$$

Proof. Suppose that condition 1° is satisfied and that a positive number ε is given. Let us prove that $A'(E_2, u_2)' \subset T'(E_3, u_3)' + \varepsilon U^0$. If not, then there exists a $g'_0 \in (E_2, u_2)'$ such that, for each n, the point $A'g'_0$ lies outside the set $\varepsilon U^0 + T'W_n^0$ where W_n runs over a fundamental system of neighbourhoods of zero in (E_3, u_3). The sets $\varepsilon U^0 + T'W_n^0$ being $\sigma((E_1, u_1)', E_1)$ compact, there exists, for each natural number n, an element $x_n \in E_1$ such that $\langle x_n, \varepsilon U^0 + T'W_n^0\rangle \leqq \varepsilon$ and $\langle x_n, A'g'_0\rangle > \varepsilon$.

In particular, $\langle x_n, U^0\rangle \leqq 1$ whence $x_n \in U^{00} = U$ and $\langle Tx_n, W_n^0\rangle \leqq 1$ so that $Tx_n \in W_n$. It follows from condition 1° that Ax_n tends to zero weakly in (E_2, u_2); however, $\langle Ax_n, g'_0\rangle = \langle x_n, A'g'_0\rangle > \varepsilon$ which is a contradiction. This proves condition 2°.

Now assume condition 2°. It follows that $A'E_2' \subset T'E_3' + (E_1, u)' = (E_1, \tilde{u})'$ so that A is continuous as a mapping of $(E_1, \tilde{u})$ into (E_2, u_2). Suppose now that $\xi \in (E_1, \tilde{u})'' = ((E_1, \tilde{u})', \beta((E_1, \tilde{u})', E_1))'$ is given and that $T''\xi = 0$. It follows that $\langle \xi, T'(E_3, u_3)' \rangle = 0$. Now let us denote by $P\xi$ the restriction of ξ to $(E_1, u)'$. Since ξ is bounded on the polar B^0 of some set B bounded in $(E_1, \tilde{u})$, ξ is bounded on U^0 since $B \subset \lambda U$ for some λ. It follows that $P\xi$ may be considered as an element of the second dual of the normed space (E_1, u). Let β be a number greater than $|P\xi|$, the norm of $P\xi$ in $(E_1, u)''$.

Now let $g' \in (E_2, u_2)'$ and a positive ε be given. According to our assumption, there exists an $f' \in (E_3, u_3)'$ and an $x' \in (E_1, u)'$ such that $A'g' = T'f' + x'$ and $|x'| < \varepsilon\beta^{-1}$. It follows that $\langle \xi, A'g' \rangle = \langle \xi, T'f' \rangle + \langle \xi, x' \rangle = \langle P\xi, x' \rangle$ whence $|\langle \xi, A'g' \rangle| \leqq \varepsilon$. Since ε was an arbitrary positive number, we have proved that $\langle \xi, A'g' \rangle = 0$ for every $g' \in (E_2, u_2)'$ or, in other words that $\tilde{A}''\xi = 0$.

Let us prove that condition 3° implies 1°. Let $x_n \in U$ and suppose that $Tx_n \to 0$. Denote by M the set of all elements of the sequence x_n. Since M is bounded in $(E_1, \tilde{u})$ and $\tilde{A}$ is continuous, the set AM is bounded in (E_2, u_2). Let $g' \in (E_2, u_2)'$ be given and suppose that $\langle Ax_n, g' \rangle$ does not tend to zero. The sequence $\langle Ax_n, g' \rangle$ being bounded, there exists a subsequence y_n of the sequence x_n such that $\langle Ay_n, g' \rangle$ converges to a limit different from zero. Since $y_n \in M$ there exists a cluster point η of the sequence y_n in the topology $\sigma((E_1, \tilde{u})'', (E_1, \tilde{u})')$. Let us prove that $\tilde{T}''\eta = 0$. Indeed, if $f' \in (E_3, u_3)'$ is given, the product $\langle \eta, T'f' \rangle$ is cluster point of the sequence $\langle y_n, T'f' \rangle = \langle Ty_n, f' \rangle \to 0$. It follows that $\langle \eta, T'f' \rangle = 0$. Since f' was arbitrary we have $\tilde{T}''\eta = 0$. It follows from our assumption that $\tilde{A}''\eta = 0$ so that, in particular, $\langle \eta, A'g' \rangle = 0$.

Now $\langle \eta, A'g' \rangle$ is a cluster point of the sequence $\langle y_n, A'g' \rangle$ because $\tilde{A}$ is continuous. This sequence, however, tends to a limit different from zero, a contradiction. This proves condition 1° hence the equivalence of the first three conditions.

Conditions 5° and 6° are equivalent by the Hahn-Banach theorem. Let us prove the implications $2° \to 5° \to 1°$.

Suppose 2° satisfied. It follows from Proposition (1,2) that $\tilde{A}$ is continuous. Consider a $\xi \in (E_1, u)''$ which annihilates $T'E_3' \cap (E_1, u)'$. Suppose that $e' = T'e_3' + A'e_2' \in (E_1, u)'$ and let $\varepsilon > 0$ be given. According to 2°, we have a decomposition

$$A'e_2' = T'f_3' + g$$

where $g \in (E_1, u)'$ and $|g| < \varepsilon|\xi|^{-1}$ if $\xi \neq 0$. Hence

$$e' = T'e_3' + T'f_3' + g .$$

Since e', $g \in (E_1, u)'$, we have $T'(e_3' + f_3') \in (E_1, u)'$ so that, by our assumption, $\langle \xi, T'(e_3' + f_3') \rangle = 0$. Hence $|\langle \xi, e' \rangle| = |\langle \xi, g \rangle| \leqq |\xi|\,|g| < \varepsilon$. Since ε was an arbitrary positive number, $\langle \xi, e' \rangle = 0$ and 5° is established.

Now assume 5° satisfied and let $x_n \in U$, $Tx_n \to 0$. Let $e_2' \in (E_2, u_2)'$ be given. Since $\tilde{A}$ is continuous, there exists, by Proposition (1,2), a decomposition

$$A'e_2' = T'e_3' + f$$

with $f \in (E_1, u)'$. It follows that $f \in (A'E_2' + T'E_3') \cap (E_1, u)'$. Suppose that $\langle Ax_n, e_2' \rangle$ does not tend to zero. Then $\langle x_n, f \rangle$ does not tend to zero. Otherwise we would have $\langle Ax_n, e_2' \rangle = \langle x_n, A'e_2' \rangle = \langle x_n, T'e_3' \rangle + \langle x_n, f \rangle \to 0$ which is a contradiction. Therefore there exists a cluster point $\xi \in (E_1, u)''$ such that $\langle \xi, f \rangle \neq 0$. If $h \in T'E_3' \cap (E_1, u)'$ then $h = T'e_3'$ for a suitable $e_3' \in E_3'$.

Since $h \in (E, u)'$, the number $\langle \xi, h \rangle$ is a cluster point of the sequence $\langle x_n, T'e_3' \rangle$. We have, however, $\langle x_n, T'e_3' \rangle = \langle Tx_n, e_3' \rangle \to 0$. Since h was an arbitrary element of the intersection $T'E_3' \cap (E_1, u)'$, we see that ξ annihilates $T'E_3' \cap (E_1, u)'$. It follows from our assumption that ξ annihilates $(T'E_3' + A'E_2') \cap (E_1, u)'$, in particular, ξ annihilates f. This is a contradiction.

Clearly the conditions 5° and 6° are equivalent by the Hahn-Banach theorem.

Let us prove now the equivalence of 4° and 5°. If S is linear mapping from a locally convex space P into another locally convex space Q and if $\xi \in P''$ we write $\xi \in \operatorname{Ker} S''$ if and only if ξ annihilates the range of S'. The range of S' is the set of all $x' \in P'$ such that

$$\langle Sx, y' \rangle = \langle x, x' \rangle$$

for a suitable $y' \in Q'$ and all $x \in D(S)$. The equivalence of 4° and 5° will therefore be established if we show that

$$R(T_0') = (E_1, u)' \cap T'E_3' ,$$
$$R((T_0 \oplus A_0)') = (E_1, u)' \cap (T'E_3' + A'E_2') .$$

First of all, $x' \in R(T_0')$ if and only if there exists an e_3' such that

$$\langle T_0 x, e_3' \rangle = \langle x, x' \rangle$$

for all $x \in E_1$; in other words if and only if $x' = T'e_3'$ or $x' \in (E_1, u)' \cap T'E_3'$. Similarly, $x' \in R((T_0 \oplus A_0)')$ if and only if there exist e_3' and e_2' such that

$$\langle T_0 x, e_3' \rangle + \langle A_0 x, e_2' \rangle = \langle x, x' \rangle$$

for all $x \in E_1$; in other words if and only if $x' = T'e_3' + A'e_2'$ or $x' \in (E_1, u)' \cap \cap (T'E_3' + A'E_2')$.

This completes the proof of the equivalence of 4° and 5°.

To complete the proof, we intend to prove the implications 2° → 7° → 8° → 1°. First of all, the inclusion

$$\sigma(E_1, T'E_3') \subset T^{-1}u_3$$