If $A_1A_2A_3$ is an equilateral triangle and X another point of the plane then XA_1, XA_2, XA_3 form lengths of sides of a triangle iff X does not belong to the circumscribed circle of $A_1A_2A_3$.

We shall generalize now this theorem as follows:

Theorem 8. *Let $A_1, \ldots, A_{n+1}$ be vertices of a regular n-simplex Σ in E_n. If X is a point in E_n then there exists an n-simplex with vertices $B_1, \ldots, B_{n+1}$ such that edges B_iB_k $(i \neq k,\ i, k = 1, \ldots, n+1)$ have lengths proportional to $(\varrho(A_i, X) \cdot \varrho(A_k, X))^{-1}$ iff X does not belong to the circumscribed $(n-1)$-sphere of Σ.*

Proof. Assume first that X belongs to the circumscribed $(n-1)$-sphere of Σ. If $X = A_i$ for some i, the n-simplex clearly does not exist. If $X \neq A_i$ for all $i = 1, \ldots, n+1$, the equivalence of 7° and 1° in Thm. 7 shows that the realization of the points B_i leads to a complete isodynamic system which is linearly dependent.

Assume now that X does not belong to the circumscribed $(n-1)$-sphere of Σ. Let $\mathscr{I}$ be any inversion with centre X. If B_i are points which correspond to the points A_i in $\mathscr{I}$, we have similarly as in the proof of 5° ⇒ 6° in Thm. 7,

$$\varrho(B_i, B_k) = k(\varrho(A_i, X)\, \varrho(A_k, X))^{-1}\,.$$

Moreover, the points B_i do not belong to a hyperplane since this would correspond in $\mathscr{I}$ to the circumscribed sphere of Σ and this would contain the centre of inversion X, a contradiction. The proof is complete.

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