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PERIODIC SOLUTIONS OF NONLINEAR ABSTRACT SECOND ORDER EQUATIONS WITH DISSIPATIVE TERMS

VLADIMÍR LOVICAR, Praha

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1. INTRODUCTION

Notation used in this paper is clear. In particular, R denotes the real line, $D(A)$ denotes the domain of an operator A and $x_n \rightharpoonup x$ or $x_n \overset{B}{\rightharpoonup} x$ means that a sequence x_n converges weakly to the element x in a Banach space B.

Let H_0 be a separable Hilbert space (with an inner product denoted by $(\cdot,\cdot)_0$ and the corresponding norm denoted by $|\cdot|_0$ and let A be a positive definite selfadjoint operator in H_0. For $\alpha > 0$ we shall denote by A^α the positive α-th power of A and by H_α the (separable) Hilbert space $D(A^\alpha)$ (with the inner product $(x, y)_\alpha = (A^\alpha x, A^\alpha y)_0$ $(x, y \in D(A^\alpha))$ and the corresponding norm $|\cdot|_\alpha$). Further, let F be a continuous operator on on $R \times H_1 \times H_0$ into H_0. Under the (generalized) solution of the equation

$$u''(t) + A^2 u(t) = F(t, u(t), u'(t)) \tag{1}$$

on an interval $[a, b] \subseteq R$ we understand a function $u \in C^1([a, b]; H_0) \cap C([a, b]; H_1)$ for which

$$u(t) = \cos A(t-a)\, u(a) + A^{-1} \sin A(t-a)\, u'(a) + \int_a^t A^{-1} \sin A(t-s)\, F(s, u(s), u'(s))\, ds \tag{2}$$

holds for $t \in [a, b]$.

Remark 1. It may be easily verified that if $u \in C^2([a, b]; H_0) \cap C^1([a, b]; H_1) \cap C([a, b]; H_2)$ satisfies the equation (1) in the classical sense then it is a (generalized) solution of (1). (More about (generalized) solutions of the equation (1) is found e.g. in [3] (putting $\gamma = 0$ in the notation of the paper [3]).)

Remark 2. If u is a solution of (1) on $[a, b]$ then

$$u'(t) = -A \sin A(t-a)\, u(a) + \cos A(t-a)\, u'(a) + \int_a^t \cos A(t-s)\, F(s, u(s), u'(s))\, ds \tag{3}$$

holds for $t \in [a, b]$.

Remark 3. If $u \in C([a, b]; H_1)$ and $v \in C([a, b]; H_0)$ are such that $u(t) = \cos A(t-a)\, u(a) + A^{-1} \sin A(t-a)\, v(a) + \int_a^t A^{-1} \sin A(t-s)\, F(s, u(s), v(s))\, ds$ and $v(t) = -A \sin A(t-a)\, u(a) + \cos A(t-a)\, v(a) + \int_a^t \cos A(t-s)\, F(s, u(s), v(s))\, ds$ hold for $t \in [a, b]$ then u is a solution of (1) on $[a, b]$ and $u' = v$.

The aim of this paper is to find assumptions on the operator F under which there exists at least one ω-periodic (generalized) solution of (1), i.e. such a solution u on the interval $[0, \omega]$ for which $u(0) = u(\omega)$ and $u'(0) = u'(\omega)$. The basic tool for obtaining this result is the following well known fixed point theorem which is a cosequence of the Schauder-Tichonov Fixed Point Theorem (see e.g. [1], p. 456):

Proposition. *Let B be a separable reflexive Banach space, K a nonempty closed bounded convex subset of B and T a weakly continuous operator on K into K (i.e. $x_n \in K$ and $x_n \rightharpoonup x$ implies $T(x_n) \rightharpoonup T(x)$). Then T has at least one fixed point in K, i.e. there exists $x_0 \in K$ such that $T(x_0) = x_0$.*

To prove the existence of an ω-periodic solution of (1) we shall show that there exists a nonempty closed bounded convex set $K \subseteq H_1 \times H_0$ such that to any $(x, y) \in K$ there exists a unique solution u of (1) on $[0, \omega]$ with initial values (x, y) (i.e. $u(0) = x$, $u'(0) = y$). Further, the operator T defined by $T(x, y) = (u(\omega), u'(\omega))$ is a weakly continuous operator on K and maps K into K. Thus according to Proposition, T has at least one fixed point which will prove the main result. This method was used in fact e.g. in [4].

The assumptions on F and the main theorem are stated in Section 2. In Section 3 some Lemmas are given from which the main theorem immediately follows. An example showing the applicability of the main theorem is given in Section 4. (This example deals with the equation of an extensible beam, see e.g. [5].)

2. MAIN THEOREM

We shall suppose that the right hand side of the equation (1) satisfies the following assumptions:

(4) F is continuous operator on $R \times H_1 \times H_0$ into H_0; to any $r > 0$ there exists a constant $c(r)$ such that

$$|F(t, x_1, y_1) - F(t, x_2, y_2)|_0 \leqq c(r)\,(|x_1 - x_2|_1 + |y_1 - y_2|_0)$$

holds for $t \in R$, $x_j \in H_1$, $|x_j|_1 \leqq r$, $y_j \in H_0$, $|y_j|_0 \leqq r$ $(j = 1, 2)$.

(5) There exist G, g, d, p, β_0 and r_0 such that

(5a) G is a continuous operator on H_1 into H_0;

(5b) g is a continuous convex real functional on H_1;

(5c) g is Gateaux differentiable on H_1 and $g'(x)(y) = 2\,\mathrm{Re}\,(G(x), y)_0$ holds for any $x, y \in H_1$;

(5d) $d = \min\{|x|_1^2 + g(x);\ x \in H_1\}$;

(5e) p is a real nondecreasing continuous function on $[d, \infty)$ which is locally Lipschitzian on (d, ∞);

(5f) $2\,\mathrm{Re}\,(F(t, x, y) + G(x) + 2\beta_0 y + \beta_0^2 x, y + \beta_0 x)_0 + 2\beta_0\, g(x) - 2\beta_0(G(x), x)_0 \leqq p(|x|_1^2 + |y + \beta_0 x|_0^2 + g(x))$ holds for $t \in R$, $x \in H_1$ and $y \in H_0$;

(5g) $r_0 > d$ and $-2\beta_0 r_0 + p(r_0) \leqq 0$.

(6) If $x_n \in H_1$, $x_n \overset{H_1}{\rightharpoonup} x$, $y_n \in H_0$, $y_n \overset{H_0}{\rightharpoonup} y$ then $F(t, x_n, y_n) \overset{H_0}{\rightharpoonup} F(t, x, y)$ for all $t \in R$.

Theorem. *Let $\omega > 0$. Let H_0 be a separable Hilbert space, A a positive definite selfadjoint operator in H_0, $H_1 = D(A)$ and let F be an operator on $R \times H_1 \times H_0$ into H_0 which satisfies the assumptions (4), (5) and (6). Then there exists an ω-periodic solution of the equation (1).*

The proof of Theorem follows immediately from Lemmas 5 and 6 (see next section) and from the above Proposition.

Remark 4. Obviously, it suffices to suppose that F is defined only on $[0, \omega] \times H_1 \times H_0$. We shall use this fact in Section 4.

3. AUXILIARY LEMMAS

Lemma 1. *Let F satisfy the assumption (4). Then to any $r > 0$ and $a, b \in R$ there exists $\delta > 0$ such that for $(x, y) \in H_1 \times H_0$ with $|x|_1 \leqq r$, $|y|_0 \leqq r$ and $t_0 \in [a, b]$ there exists a solution u of (1) on the interval $[t_0, t_0 + \delta]$ with $u(t_0) = x$ and $u'(t_0) = y$.*

Lemma 2. *Let F satisfy the assumption (4) and let u_1, u_2 be solutions of (1) on an interval $[a, b] \subseteq R$ with $u_1(a) = u_2(a)$ and $u_1'(a) = u_2'(a)$. Then $u_1 = u_2$.*

The above assertions may be proved in the same way as the analogous results from the theory of ordinary differential equations (the essential means being Banach Contraction Principle and Gronwall's Lemma). Therefore their proofs are omitted.

Lemma 3. *Let u be a solution of the equation*

$$u''(t) + A^2 u(t) = f(t) \tag{7}$$

on an interval $[a, b] \subseteq R$ where $f \in C([a, b]; H_0)$. Then

$$|u(t)|_1^2 + |u'(t) + \beta u(t)|_0^2 = e^{-2\beta(t-a)}(|u(a)|_1^2 + |u'(a) + \beta u(a)|_0^2) + \tag{8}$$
$$+ 2\int_a^t e^{-2\beta(t-s)} \operatorname{Re}(f(s) + 2\beta u'(s) + \beta^2 u(s), u'(s) + \beta u(s))_0 \, ds$$

holds for any $\beta \in R$ and $t \in [a, b]$.

Proof. The proof of (8) may proceed in the same way as that for the usual energy equality. If $u \in C^2([a, b]; H_0) \cap C^1([a, b]; H_1) \cap C([a, b]; H_2)$, then denoting $z(s) = |u(s)|_1^2 + |u'(s) + \beta u(s)|_0^2$ one immediately verifies that $z'(s) + 2\beta z(s) = 2 \operatorname{Re}(f(s) + 2\beta u'(s) + \beta^2 u(s), u'(s) + \beta u(s))_0$ holds for any $\beta \in R$ and $s \in [a, b]$. Multiplying this equality by $e^{-2\beta(t-s)}$ and integrating over $[a, t]$ we obtain (8). The validity of (8) for any solution u is obtained by the usual approximation process.

Lemma 4. *Let F satisfy the assumption* (4) *and let G, g satisfy the assumptions* (5a), (5b) *and* (5c). *Then for any solution u of* (1) *on an interval $[a, b] \subseteq R$ we have*

$$|u(t)|_1^2 + |u'(t) + \beta u(t)|_0^2 + g(u(t)) = \tag{9}$$
$$= e^{-2\beta(t-a)}(|u(a)|_1^2 + |u'(a) + \beta u(a)|_0^2 + g(u(a))) +$$
$$+ 2\int_a^t e^{-2\beta(t-s)}[\operatorname{Re}(F(s, u(s), u'(s)) + G(u(s)) + 2\beta u'(s) +$$
$$+ \beta^2 u(s), u'(s) + \beta u(s))_0 + \beta g(u(s)) - \beta(G(u(s)), u(s))_0] \, ds$$

for any $\beta \in R$ and $t \in [a, b]$.

Proof. It is easy to see that the function $s \to g(u(s))$ is continuously differentiable on $[a, b]$ and $g(u(\cdot))'(s) = 2 \operatorname{Re}(G(u(s)), u'(s))_0$. The relation (9) follows now immediately from (8).

Lemma 5. *Let F satisfy the assumptions* (4) *and* (5) *and let us denote*

$$K = \{(x, y) \in H_1 \times H_0;\ |x|_1^2 + |y + \beta_0 x|_0^2 + g(x) \leq r_0\}. \tag{10}$$

Then

1. *K is a nonempty closed bounded convex subset of $H_1 \times H_0$;*
2. *to any $(x, y) \in K$ there exists a solution u of* (1) *on the interval $[0, \omega]$ with initial values (x, y). Moreover, $(u(t), u'(t)) \in K$ for $t \in [0, \omega]$.*

Proof. It is easy to verify 1. To prove 2 it is sufficient to show, with respect to Lemma 1, that if u is a solution of (1) on $[0, \omega]$ with $(u(0), u'(0)) \in K$ then $(u(t), u'(t)) \in \in K$ for $t \in [0, \omega]$. For $t \in [0, \omega]$ let us denote $z(t) = |u(t)|_1^2 + |u'(t) + \beta_0 u(t)|_0^2 + g(u(t))$, $q(t) = z(0) e^{-2\beta_0(t-s)} p(z(s)) ds$. Clearly $z \in C([0, \omega])$ and $q \in C^1([0, \omega])$. With respect to (5f) and Lemma 4 $z(t) \leqq q(t)$ holds for $t \in [0, \omega]$. Since p is non-decreasing, $q'(t) = -2\beta_0 q(t) + p(z(t)) \leqq -2\beta_0 q(t) + p(q(t))$ holds for $t \in [0, \omega]$. The assumption (5g) implies that $z(t) \leqq q(t) \leqq r_0$ or, in other words, $(u(t), u'(t)) \in K$ for $t \in [0, \omega]$.

For $(x, y) \in K$ let us define

$$T(x, y) = (u(\omega), u'(\omega)) \tag{11}$$

where u is a solution of (1) on $[0, \omega]$ with initial values (x, y). Lemmas 5 and 2 imply that T is a single-valued operator which maps K into K.

Lemma 6. *Let F satisfy the assumptions (4), (5) and (6) and let K and T be defined by (10) and (11). Then T is a weakly continuous operator.*

Proof. Let $(x_n, y_n) \in K$ $(x_n, y_n) \rightharpoonup (x, y)$. First let us notice that it is sufficient to show that there exists a subsequence (x_{k_n}, y_{k_n}) such that $T(x_{k_n}, y_{k_n}) \rightharpoonup T(x, y)$. Let u_n be solutions of (1) on $[0, \omega]$ with initial values (x_n, y_n). It is easy to see from the expressions (2) and (3) that for any $z \in H_1$ $(z \in H_0)$ the functions $t \to (u_n(t), z)_1$ . . $(t \to (u_n'(t), z)_0)$ are equicontinuous on $[0, \omega]$. Hence (with respect to the separability of the spaces H_1, H_0 and using Cantor's diagonal method) we obtain that there exists a subsequence u_k such that

$$u_{k_n}(t) \overset{H_1}{\rightharpoonup} u(t)\,, \quad u'_{k_n}(t) \overset{H_0}{\rightharpoonup} v(t) \quad (t \in [0, \omega])\,. \tag{12}$$

The assumption (6) implies (with respect to theorems of Pettis and of Bochner — see e.g. [2] p. 131 and 133) that the function $t \to F(t, u(t), v(t))$ belongs to $L_1(0, \omega; H_0)$ and that the relations $u(t) = \cos At\, x + A^{-1} \sin At\, y + \int_0^t A^{-1} \sin A(t - s)$ . . $F(s, u(s), v(s))\, ds$, $v(t) = -A \sin At\, x + \cos At\, y + \int_0^t \cos A(t - s) F(s, u(s), v(s))$ . . ds hold for $t \in [0, \omega]$. Hence, by Remark 3, the function u is a solution of (1) on $[0, \omega]$ with initial values (x, y) and $u' = v$. The relation (12) for $t = \omega$ says in other words that $T(x_{k_n}, y_{k_n}) \rightharpoonup T(x, y)$.

4. EXAMPLE

On $[0, \omega] \times J$ $(J = [0, 1])$ let us consider an equation

$$u_{tt}(t. x) + a\, u_t(t, x) + u_{xxxx}(t, x) - b\left(\int_0^1 |u_x(t, \xi)|^2\, d\xi\right) u_{xx}(t, x) = f(t, x) \tag{13}$$