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CATEGORIAL APPROACH TO GLOBAL TRANSFORMATIONS OF THE n-TH ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Investigations on linear differential equations started in the middle of the last century. They were connected with the names of E. E. KUMMER [4], E. LAGUERRE [5], F. BRIOSCHI, G. H. HALPHEN, A. R. FORSYTH, P. STÄCKEL [13], S. LIE, E. J. WILCZYNSKI [15] and others. Their results, however, were of local character. The global study began with second order equations about 25 years ago by O. BORŮVKA [1], [2], and results of algebraic character form the essential part of his theory.

Here we describe algebraically the global structure of n-th order linear differential equations ($n \geqq 2$). The geometric approach was given in [6] and the importance of global transformations for studying and understanding asymptotic behavior, periodicity, boundedness, zeros, oscillatory behavior, disconjugacy and other global properties of solutions essentially connected with the whole interval of definition was demonstrated in [6], [8], [9], [10], [11].

2. GLOBAL TRANSFORMATIONS

Let $C^s(I, \mathbf{R}^k)$ denote the set of all (column) vector functions $\mathbf{u} : I \to \mathbf{R}^k$ with continuous derivatives up to and including the order s, $s \geqq 0$, let I be an open interval of $\mathbf{R}$, $k \geqq 1$, let $\mathbf{u}^T$ denote the transpose of $\mathbf{u}$. Coefficients of linear homogeneous differential equations of the n-th order are supposed to be real and continuous on the corresponding open (bounded or unbounded) intervals of definition. For $n \geqq 2$, P. STÄCKEL [13] in 1891 derived the most general pointwise transformation that converts any linear homogeneous differential equation of the n-th order into an equation of the same type. This transformation consists in changing the independent variable ($x \mapsto h(t)$) and multiplying the dependent variable by a factor $f(t)$, i.e. $y \mapsto f(t)\, y$.

With respect to this result we say that an n-th order linear homogeneous differential equation $\mathscr{L}$ on I with n linearly independent solutions $y_1, \ldots, y_n$ on I is globally transformable into an equation $\mathscr{Q}$ of the same type on J admitting n linearly independent solutions $z_1, \ldots, z_n$ if

$$\mathbf{z}(t) = A \,.\, f(t) \,.\, \mathbf{y}(h(t)) \tag{1}$$

for a real regular n by n matrix $A, f \in C^n(J, \mathbf{R})$, $h \in C^n(J, I)$, $f(t) \,.\, \mathrm{d}h(t)/\mathrm{d}t \neq 0$ on J, and $h(J) = I$, where $(y_1, \ldots, y_n)^T$ is denoted by $\mathbf{y}$ and called a fundamental solution of the corresponding equation $\mathscr{L}$. Similarly $\mathbf{z}$ is a fundamental solution of $\mathscr{Q}$.

The global transformation (1) can be expressed as $\mathscr{L} * \alpha = \mathscr{Q}$, where α is called the *transformation* of $\mathscr{L}$ into $\mathscr{Q}$. Since every fundamental solution of $\mathscr{L}$ is of the form $C\mathbf{y}$, C being an arbitrary regular n by n matrix, the transformation α essentially depends on f, called *multiplier*, and h, *parametrization*.. We shall write $\alpha = \langle f, h \rangle_{\mathscr{L}}$.

Let us note that the global character of transformations is achieved by $h(J) = I$, and linear independency of coordinates of $\mathbf{z}$ in (1) is guaranteed by the conditions on A, f, h, and $\mathbf{y}$. For more detail see [7].

Since global transformations form a reflexive, symmetric and transitive relation, the set of all n-th order linear homogeneous differential equations $(n \geqq 2)$ is decomposed into classes of globally transformable equations. Denote by $\mathbf{\Delta}$ the decomposition.

3. STATIONARY GROUPS

Proposition 1. *Let $\Delta \in \mathbf{\Delta}$ be a class of globally equivalent differential equations. The set of all global transformations, $\mathfrak{B}(\Delta)$, between every pair of equations from Δ together with the composition rule form a Brandt groupoid.*

Proof. A Brandt groupoid is a category each element of which is invertible, and such that if α and γ are its elements, there exists β for which $\alpha\beta\gamma$ is defined, see [3], p. 81–83.

Let $\mathscr{L}, \mathscr{P}, \mathscr{Q}$ be equations from Δ and let I, J, K denote the corresponding intervals of definitions. Let $\mathscr{L} * \alpha = \mathscr{P}$, $\mathscr{P} * \beta = \mathscr{Q}$, $\alpha \in \mathfrak{B}(\Delta)$, $\beta \in \mathfrak{B}(\Delta)$. Define $\alpha\beta \in \mathfrak{B}(\Delta)$ by $(\mathscr{L} * \alpha) * \beta = \mathscr{L} * (\alpha\beta)$. Evidently ${}_\alpha\varepsilon = \langle 1_I, \mathrm{id}_I \rangle_{\mathscr{L}}$ is the left unit and $\varepsilon_\alpha = \langle 1_J, \mathrm{id}_J \rangle_{\mathscr{P}}$ is the right unit of α, where $1_I : I \to \{1\} \in \mathbf{R}$, and the associativity holds. Further, if $\alpha = \langle f, h \rangle_{\mathscr{L}}$ and $\beta = \langle g, k \rangle_{\mathscr{P}}$, then $\alpha\beta = \langle (f \circ k) \,.\, g, h \circ k \rangle_{\mathscr{L}}$ which always defined provided $\varepsilon_\alpha = {}_\beta\varepsilon$; $\circ$ denotes the composition of functions. Evidently $\alpha^{-1} = \langle 1/(f \circ h^{-1}), h^{-1} \rangle_{\mathscr{P}}$, where h^{-1} is the inverse to h. For $\gamma \in \mathfrak{B}(\Delta)$ there always exists $g \in C^n(K, \mathbf{R})$ and $k \in C^n(K, J)$ such that $g \,.\, k'(t) \neq 0$ on K, $k(K) = J$, where ${}_\gamma\varepsilon = \langle 1_K, \mathrm{id}_K \rangle_{\mathscr{Q}}$. Hence for $\beta := \langle g, k \rangle_{\mathscr{P}}$, $\alpha\beta\gamma$ is defined. ■

We always consider each $\mathfrak{B}(\Delta)$ with the structure of Brandt groupoid.

For each $\mathscr{L} \in \Delta$ define $\mathfrak{A}(\mathscr{L})$ as the set of all global transformations that transform $\mathscr{L}$ into itself, $\mathfrak{A}(\mathscr{L}) := \{\alpha \in \mathfrak{B}(\Delta);\ \mathscr{L} \in \Delta \text{ and } \mathscr{L} * \alpha = \mathscr{L}\}$. Evidently $\mathfrak{A}(\mathscr{L})$ is a group called the stationary group of $\mathscr{L}$. With respect to (1), $\alpha = \langle f, h \rangle_{\mathscr{L}} \in \mathfrak{A}(\mathscr{L})$ if and only if

$$\mathbf{y}(t) = A \,.\, f(t) \,.\, \mathbf{y}(h(t))\,, \quad h(I) = I\,, \tag{2}$$

for a suitable regular n by n matrix A, where I is the interval of definition and $\mathbf{y}$ is a fundamental solution of $\mathscr{L}$.

Proposition 2. *If $\mathscr{L} \in \Delta \in \mathbf{\Delta}$, $\mathscr{L} * \alpha = \mathscr{P}$, $\alpha \in \mathfrak{B}(\Delta)$, then*

$$\mathfrak{A}(\mathscr{P}) = \alpha^{-1}\, \mathfrak{A}(\mathscr{L})\, \alpha\,. \tag{3}$$

In other words: *Each two stationary groups of any pair of globally transformable differential equations are conjugate.*

Proof. For $\beta \in \mathfrak{A}(\mathscr{P})$ we have $\mathscr{L} * \alpha\beta\alpha^{-1} = (\mathscr{P} * \beta) * \alpha^{-1} = \mathscr{P} * \alpha^{-1} = \mathscr{L}$ or $\alpha\beta\alpha^{-1} \in \mathfrak{A}(\mathscr{L})$, hence $\beta \in \alpha^{-1}\, \mathfrak{A}(\mathscr{L})\, \alpha$. For $\beta \in \alpha^{-1}\, \mathfrak{A}(\mathscr{L})\, \alpha$ we have $\alpha\beta\alpha^{-1} \in \mathfrak{A}(\mathscr{L})$ or $\mathscr{L} * \alpha\beta\alpha^{-1} = \mathscr{L}$ which gives $(\mathscr{L} * \alpha) * \beta = \mathscr{L} * \alpha$, or $\mathscr{P} * \beta = \mathscr{P}$, hence $\beta \in \mathfrak{A}(\mathscr{P})$. See also [3], [14]. ■

An interesting rôle is played by sugroups $\mathfrak{A}_G(\mathscr{L})$ of $\mathfrak{A}(\mathscr{L})$, elements of which leave invariant a certain subspace of solutions of $\mathscr{L}$, G assigning the corresponding subgroups of matrices A occuring in (2). In particular, $\mathfrak{A}_{\{E\}}(\mathscr{L})$, E being the unit matrix, is characterized by the fact that each solution of $\mathscr{L}$ is transformed into itself, or

$$\mathbf{y}(t) = f(t) \,.\, \mathbf{y}(h(t))\,, \quad h(I) = I\,. \tag{4}$$

Transformations $\alpha = \langle f, h \rangle_{\mathscr{L}}$ with increasing parametrizations $h, h' > 0$, are important for studying global properties of solutions (like periodicity, boundedness, asymptotic behavior, L^2-solutions, and others, see [6], [8], [9], [10], [11]), since they often enable us to describe the global behavior of solutions according to their local character and some information of discrete kind (e.g., conjugate points). Hence denote $\mathfrak{B}^+(\Delta) = \{\alpha = \langle f, h \rangle_{\mathscr{L}} \in \mathfrak{B}(\Delta);\ h' > 0\}$, and for $\mathscr{L} \in \Delta$ also $\mathfrak{A}^+(\mathscr{L}) = \mathfrak{A}(\mathscr{L}) \cap \mathfrak{B}^+(\Delta)$ and $\mathfrak{A}^+_G \mathscr{L} = \mathfrak{A}_G(\mathscr{L}) \cap \mathfrak{B}^+(\Delta)$. Evidently $\mathfrak{B}^+(\Delta)$ has the structure of Brandt groupoid, and $\mathfrak{A}^+(\mathscr{L})$, $\mathfrak{A}^+_G(\mathscr{L})$ are groups.

4. NONTRIVIAL STATIONARY GROUPS $\mathfrak{A}^+(\mathscr{L})$ AND $\mathfrak{A}^+_{\{E\}}(\mathscr{L})$

Functional equations (2) and (4) that correspond to $\mathfrak{A}(\mathscr{L})$ and $\mathfrak{A}_{\{E\}}(\mathscr{L})$ were studied in [12]. From the results obtained there we have

Theorem 1. *Let $\mathscr{L} \in \Delta \in \mathbf{\Delta}$, I being the interval of definition of $\mathscr{L}$. If $\mathfrak{A}^+(\mathscr{L})$ is not trivial, i.e., $\alpha = \langle f, h \rangle_{\mathscr{L}} \in \mathfrak{A}^+(\mathscr{L})$, $\alpha \neq \varepsilon_\alpha$, then $\{t \in I;\ h(t) = t\}$ has no* *accumulation point in I. On each maximal subinterval $(a, b) \subset I$ where $h(t) \neq t$, the equation $\mathscr{L}$ restricted on (a, b) is globally equivalent to a differential equation with periodic coefficients on $(-\infty, \infty)$.*

Theorem 2. *If $\mathscr{L}$ is globally equivalent to a differential equation with periodic coefficients on $(-\infty, \infty)$, then its stationary group $\mathfrak{A}^+(\mathscr{L})$ is not trivial.*

Theorem 3. *Let $\mathscr{L} \in \Delta$. $\mathfrak{A}^+_{\{E\}}(\mathscr{L})$ is not trivial if and only if there exists an equation in Δ having only periodic solutions on $(-\infty, \infty)$ with the same period.*

5. PHASES AND AMPLITUDES

Let a differential equation $\mathscr{E}(\Delta) \in \Delta$ be assigned to each $\Delta \in \boldsymbol{\Delta}$ (e.g., called canonical). For each $\mathscr{L} \in \Delta$ we have $\alpha \in \mathfrak{B}(\Delta)$ such that $\mathscr{E}(\Delta) * \alpha = \mathscr{L}$. The $\alpha = \langle f \,.\, h \rangle_{\mathscr{E}(\Delta)}$ is called a *shift* of $\mathscr{L}$ with respect to $\mathscr{E}(\Delta)$, its multiplier f is an *amplitude* and its parametrization, h, is a *phase* of $\mathscr{L}$ (with respect $\mathscr{E}(\Delta)$). The set of all shifts of all equations from Δ with respect to $\mathscr{E}(\Delta)$ will be denoted as $\mathfrak{S}_\Delta$. The stationary group $\mathfrak{A}(\mathscr{E}(\Delta))$ of $\mathscr{E}(\Delta)$ will be called the fundamental group and denoted by $\mathfrak{F}_\Delta$.

Theorem 4. *If $\mathscr{L} \in \Delta$, then*

$$\mathfrak{A}(\mathscr{L}) = \alpha^{-1} \mathfrak{F}_\Delta \alpha\,, \tag{5}$$

where α is a shift of $\mathscr{L}$.

Proof follows from Proposition 2. ■

Theorem 5. *Let $\Delta \in \boldsymbol{\Delta}$, $\mathscr{L} \in \Delta$, $\mathscr{P} \in \Delta$, let α be a shift of $\mathscr{L}$ and β a shift of $\mathscr{P}$ (with respect to $\mathscr{E}(\Delta)$). Then $\alpha^{-1}\beta$ is a transformation of $\mathscr{L}$ into $\mathscr{P}$, i.e., $\mathscr{L} * (\alpha^{-1}\beta) = \mathscr{P}$. All transformations of $\mathscr{L}$ into $\mathscr{P}$ form the set*

$$\alpha^{-1} \mathfrak{F}_\Delta \beta = \mathfrak{A}(\mathscr{L})\, \alpha^{-1}\beta = \alpha^{-1}\beta\, \mathfrak{A}(\mathscr{P})\,. \tag{6}$$

Proof. Since $\mathscr{E}(\Delta) * \alpha = \mathscr{L}$ and $\mathscr{E}(\Delta) * \beta = \mathscr{P}$, we have $\mathscr{L} * (\alpha^{-1}\beta) = \mathscr{E}(\Delta) * \beta = \mathscr{P}$. Each γ such that $\mathscr{L} * \gamma = \mathscr{P}$ satisfies $\mathscr{L} * \gamma\beta^{-1}\alpha = \mathscr{L}$, hence $\gamma\beta^{-1}\alpha \in \mathfrak{A}(\mathscr{L})$, or $\gamma \in \mathfrak{A}(\mathscr{L})\, \alpha^{-1}\beta$. Conversely, for each $\gamma \in \mathfrak{A}(\mathscr{L})\, \alpha^{-1}\beta$ we get $\mathscr{L} * \gamma = \mathscr{P}$. Finally, using (3) or (5) we obtain (6):

$$\mathfrak{A}(\mathscr{L})\, \alpha^{-1}\beta = \alpha^{-1}\mathfrak{F}_\Delta \alpha\alpha^{-1}\beta = \alpha^{-1}\mathfrak{F}_\Delta\beta = \alpha^{-1}\beta\, \mathfrak{A}(\mathscr{P})\, \beta^{-1}\beta\,. \quad ■$$

Theorem 6. *For $\Delta \in \boldsymbol{\Delta}$, $\{\mathfrak{F}_\Delta\alpha;\ \alpha \in \mathfrak{S}_\Delta\}$ is a decomposition of the set $\mathfrak{S}_\Delta$ of all shifts from Δ, called the right decomposition of $\mathfrak{S}_\Delta$ with respect to the fundamental*