Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON CONDITIONS ON RIGHT HAND SIDES OF DIFFERENTIAL RELATIONS

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(Received April 25, 1977)

0. INTRODUCTION

We are interested in the solutions of

$$\dot{x} \in F(t, x), \tag{0.1}$$

which are locally absolutely continuous and fulfil (0.1) almost everywhere. Let $\mathscr{K}_n$ be the set of nonempty convex compact subsets of R^n, $G \subset R \times R^n$ and assume that

$$F : G \to \mathscr{K}_n, \tag{0.2}$$

$$\text{for almost all } t \text{ the map } F(t, \cdot) \text{ is upper semicontinuous} \tag{0.3}$$

(of course, $F(t, \cdot)(x) = F(t, x)$ if the right-hand side is defined). In order to guarantee the existence of solutions, some conditions are to be added. For example, analogously to the ordinary differential equations it may be assumed that

$$\text{for every } x \text{ the map } F(\cdot, x) \text{ is measurable}, \tag{0.4}$$

$$F \text{ is bounded (in some sense)} \tag{0.5}$$

or (0.4) may be replaced by a more general selection property (cf. [1]).

The map F may fulfil (0.2), (0.3) and (0.4) and at the same time behave rather unreasonably as a function of the pair of variables (t, x); an example is given in Section 4. Nevertheless, if F fulfils (0.3), then (0.1) may be replaced without loss of generality by

$$\dot{x} \in \hat{F}(t, x) \tag{0.6}$$

where $\hat{F}$ is in a certain sense regular as a function of the pair (t, x).

Let $\mathscr{K}_n^0 = \mathscr{K}_n \cup \{\emptyset\}$, let Z be a metric space, $H : Z \to \mathscr{K}_n^0$. H is called upper semicontinuous, if for every open set $U \subset R^n$ the set $\{z \in Z \mid H(z) \subset U\}$ is open.

(This is equivalent to the usual definition if $H : Z \to \mathcal{K}_n$.) If $S \subset Z$, let the map $H|_S$: $S \to \mathcal{K}_n^0$ be defined by $H|_S(z) = H(z)$. Denote by $m(B)$ the Lebesgue measure of a set $B \subset R$. The above statement is made precise in the following

0.1. Theorem. *Let F fulfil (0.2), (0.3). Then there exists such an $\hat{F} : G \to \mathcal{K}_n^0$ that*

(0.7) *to every $\varepsilon > 0$ there exists such a measurable set $A_\varepsilon \subset R$ that $m(R - A_\varepsilon) <$ $< \varepsilon$ and that $\hat{F}|_{G \cap (A_\varepsilon \times R^n)}$ is upper semicontinuous,*

(0.8) $$\hat{F}(t, x) \subset F(t, x) \quad \textit{for} \quad (t, x) \in G,$$

(0.9) *every solution of (0.1) is simultaneously a solution of (0.6).*

It follows from (0.8) that every solution of (0.6) is simultaneously a solution of (0.1), so that the sets of solutions of both the equations are identical. Moreover, if the existence theorem holds for (0.1), then $m(B) = 0$ by Theorem 3.1, B being the set of such $t \in R$ that $(t, x) \in G$, $\hat{F}(t, x) = \emptyset$ for some x. Without loss of generality we may change F on $G \cap (B \times R^n)$ and obtain that $\hat{F}(t, x) \neq \emptyset$ for $(t, x) \in G$ ((0.7)–(0.9) being in force simultaneously).

Theorem 0.1 is an immediate consequence of the following more general theorem, which is not concerned with differential relations but with a selection problem. Let X be a separable metric space.

0.2. Theorem. *Assume that*

(0.10) $$G \subset R \times X, \quad F : G \to \mathcal{K}_n$$

and that (0.3) holds. Then there exists a function $\hat{F} : G \to \mathcal{K}_n^0$ satisfying (0.8),

(0.11) *to every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset R$ such that $m(R - A_\varepsilon) < \varepsilon$ and the function $\hat{F}|_{G \cap (A_\varepsilon \times X)}$ is upper semicontinuous,*

(0.12) *if $I \subset R$ is measurable, $u : I \to X$ measurable, $(t, u(t)) \in G$ for $t \in I$, $v : I \to R^n$ measurable, $v(t) \in F(t, u(t))$ for almost all $t \in I$, then $v(t) \in \hat{F}(t, u(t))$ for almost all $t \in I$.*

Theorem 0.2 is proved in Section 1 (see Theorem 1.5). As u in Theorem 0.2 is required to be measurable (and not continuous) and X is a general separable metric space, Theorem 0.1 may be extended to differential relations the right-hand sides of which depend on the values of the solution x or its derivative $\dot{x}$ with deviated arguments or on the "portion" x_t of x, $x_t : [-1, 1] \to R^n$ being defined by $x_t(\tau) = x(t + \tau)$. These extensions of Theorem 0.1 are not described in more detail, the application of Theorem 0.2 being straightforward.

The conditions (0.7), (0.11) are analogous to that of Scorza-Dragoni which had been introduced for differential equations [2]. The same condition was used in [3] where the result by Scorza-Dragoni was extended to differential relations. The necessity of admitting $\hat{F}(t, x) = \emptyset$ is clear from the following example.

Let $M_i \subset R$, $i = 1, 2$ be non-measurable sets, $m^*(M_i) = 1$ (the asterisk denotes the outer measure), $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = (0, 1)$ and put $F(t, x) = \{f(t)\}$ where $f(t) = 1$ for $t \in M_1$, $f(t) = 0$ for $t \in M_2$. The condition (0.7) implies that $\hat{F}(\cdot, x)$ is measurable. However, with regard to (0.8) this is only possible if $\hat{F}(t, x) = \emptyset$ for $(t, x) \in G \cap (A \times R)$ where $G = (0, 1) \times \{0, 1\}$, $m(R - A) = 0$.

0.3. In Section 2, assuming that the metric space X is complete separable and that the set G is of type F_σ, we prove that the function $\hat{F}$ from 0.2 is unique in the following sense: If $\hat{F}_i : G \to \mathscr{K}_n^0$, $i = 1, 2$ fulfil the conditions (0.8), (0.11), (0.12) with $\hat{F}_i$ instead of $\hat{F}$, then there is a set $A \subset R$ such that $m(R - A) = 0$ and $\hat{F}_1(t, x) = \hat{F}_2(t, x)$ for $(t, x) \in G \cap (A \times X)$. (See Note 2.7.)

In Section 3 we prove a theorem which gives sufficient conditions that $\hat{F}(t, x) = \emptyset$ occurs only for t from a set of measure zero.

The results of Sections 1–3 may be extended to the case when X is a topological space with some additional properties; e.g. in Section 1 it is sufficient to assume that X is a topological space with a countable open basis.

0.4. In Section 4 the existence of such functions $F, Q : [0, 1] \times R \to \{\{0\}, [0, 1]\}$ is proved that $Q(t, x) = F(t, x + t)$ and that

(0.13) for every $t \in [0, 1]$ the set $\{x \mid F(t, x) = [0, 1]\}$ contains at most one point,

(0.14) for every $x \in R$ the set $\{t \mid F(t, x) = [0, 1]\}$ contains at most one point,

(0.15) for every $x \in [0, 1]$, the function $Q(\cdot, x)$ is not measurable on any interval $[\alpha, \beta]$, $0 < \alpha < \beta \leqq 1$.

By (0.13) and (0.14) F fulfils (0.3) and (0.4) but Q does not fulfil (0.4) by (0.15). For this result the authors are indebted to I. Vrkoč.

1.

1.1. Definition. Let X be a metric space, R the real line, $G \subset R \times X$. The metric in X is denoted by ϱ, the metric $\hat{\varrho}$ in $R \times X$ is given by $\hat{\varrho}((t_1, x_1), (t_2, x_2)) = \max\{|t_1 - t_2|, \varrho(x_1, x_2)\}$. Let $P(G)$ be the set of such $t \in R$ that $(t, x) \in G$ for some $x \in X$. $G(t, \cdot)$ denotes the set of such $x \in X$ that $(t, x) \in G$. The set of all $f : G \to R$ such that $f(t, \cdot)$ is upper semicontinuous for almost all t is denoted by $\mathscr{U}\mathscr{S}\mathscr{C}_2(G \to R)$; the set of all $f : G \to R$ which satisfy the condition

(1.1) to every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset R$ with $m(R - A_\varepsilon) < \varepsilon$ such that $f|_{G \cap (A_\varepsilon \times X)}$ is upper semicontinuous

is denoted by $\mathscr{S}\mathscr{D}^*(G \to R)$.

1.2. Theorem. *Let X be separable, let $f \in \mathcal{U}\mathcal{S}\mathcal{C}_2(G \to R)$ have a finite range. Then there exists $\hat{f} \in \mathcal{S}\mathcal{D}^*(G \to R)$ so that*

(i) *$\hat{f}(t, x) \leqq f(t, x)$ for $(t, x) \in G \cap (A \times X)$ where $A \subset R$, $m(R - A) = 0$;*

(ii) *if $I \subset R$ is measurable, $u : I \to X$ measurable, $(t, u(t)) \in G$ for $t \in I$, $v : I \to R$ measurable and $v(t) \leqq f(t, u(t))$ almost everywhere (a.e.) in I, then $v(t) \leqq \leqq \hat{f}(t, u(t))$ a.e. in I.*

Proof. Without loss of generality we may assume that the set $P(G)$ is bounded. Let B be a subset of $P(G)$ such that $m(B) = 0$ and that $f(t, \cdot)$ is upper semicontinuous for $t \in I - B$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the values of the function f, $\alpha_1 < \alpha_2 < \ldots < \alpha_k$. Let us choose numbers β_i, $i = 1, 2, \ldots, k$ so that

$$\alpha_1 < \beta_1 < \alpha_2 \ldots < \beta_{k-1} < \alpha_k < \beta_k .$$

It follows from the upper semicontinuity of $f(t, \cdot)$ that the sets

$$U_i(t) = \{x \mid f(t, x) < \beta_i\} , \quad i = 1, 2, \ldots, k$$

are relatively open in $G(t, \cdot)$ for $t \in P(G) - B$. Let $\{V_j\}$, $j = 1, 2, \ldots$ be a countable open basis of X and

$$D_{ji} = \{t \mid V_j \cap G(t, \cdot) \subset U_i(t)\} , \quad j = 1, 2, \ldots \quad i = 1, 2, \ldots, k .$$

If D_{ji} is measurable, we denote $D_{ji} = E_{ji}$; if not, let E_{ji} be a measurable set, $D_{ji} \subset \subset E_{ji}$, $m^*(D_{ji}) = m(E_{ji})$ (the asterisk denotes the outer measure). Obviously $0 \leqq m^*(D_{ji}) < \infty$, as $P(G)$ is bounded.

Let us put

$$\vartheta_{ji} = \begin{cases} \alpha_i & \text{for } (t, x) \in E_{ji} \times V_j , \\ \infty & \text{otherwise} \end{cases}$$

and define

$$\hat{f}(t, x) = \inf_{i,j} \vartheta_{ji}(t, x) \qquad \text{for} \quad (t, x) \in G .$$

We shall prove (i). Let us assume that $(t, x) \in G \cap (C \times X)$, $C = P(G) - B$ and $f(t, x) = \alpha_i$. Then $x \in U_i(t)$, $U_i(t)$ is relatively open in $G(t, \cdot)$ and there exists a positive integer j such that $x \in V_j$, $V_j \cap G(t, \cdot) \subset U_i(t)$, i.e. $t \in D_{ji} \subset E_{ji}$. Hence $(t, x) \in \in E_{ji} \times V_j$ and $\vartheta_{ji}(t, x) = \alpha_i$,

$$\hat{f}(t, x) = \inf_{i,j} \vartheta_{ji}(t, x) \leqq \alpha_i = f(t, x)$$

and (i) holds.

Now let us prove (ii). To this aim assume that $u : I \to X$, $v : I \to R$ are measurable, $(t, u(t)) \in G$ for $t \in I$, $v(t) \leqq f(t, u(t))$ a.e. in I.

Let us denote

$$L_j = \{t \mid u(t) \in V_j\} , \quad j = 1, 2, \ldots .$$

Then L_j are measurable sets. Let the indices i, j be fixed. Choose an arbitrary $t \in D_{ji} \cap L_j$. Then $u(t) \in V_j \cap G(t, \cdot) \subset U_i(t)$, hence $f(t, u(t)) < \beta_i$ and $v(t) \leqq \alpha_i$. Consequently, $D_{ji} \cap L_j \subset H_i$, where

$$H_i = \{t \mid v(t) \leqq \alpha_i\}.$$

Since E_{ji} was chosen to satisfy $m(E_{ji}) = m^*(D_{ji})$, we have evidently

$$m(E_{ji} \cap L_j) = m^*(D_{ji} \cap L_j). \tag{1.2}$$

Moreover, we have $D_{ji} \cap L_j \subset H_i$ and thus $m(E_{ji} \cap L_j - H_i) = m(E_{ji} \cap L_j) - m(E_{ji} \cap L_j \cap H_i) = m^*(D_{ji} \cap L_j) - m(E_{ji} \cap L_j \cap H_i) = m^*(D_{ji} \cap L_j \cap H_i) - m(E_{ji} \cap L_j \cap H_i) \leqq 0$ since $D_{ji} \subset E_{ji}$. This means that $v(t) \leqq \alpha_i$ a.e. in $E_{ji} \cap L_j$.

On the other hand, if $t \in I - L_j$, then $u(t) \notin V_j$ by definition, therefore $\vartheta_{ji}(t, u(t)) = \infty$. Hence the inequality $v(t) \leqq \vartheta_{ji}(t, u(t))$ holds in $I - N_{ji}$, $m(N_{ji}) = 0$. Putting $N = \bigcup_{i,j} N_{ji}$ we have $m(N) = 0$ and $v(t) \leqq \hat{f}(t, u(t))$ for $I - N$, which proves (ii).

It remains to show that $\hat{f} \in \mathcal{SD}^*(G \to R)$. Let i, j be fixed. Let us define

$$\hat{\vartheta}_{ji}(t) = \begin{cases} \alpha_i & \text{for} \quad t \in E_{ji}, \\ \infty & \text{otherwise}. \end{cases}$$

Given $\varepsilon > 0$, find $A_\varepsilon^{ji} \subset R$, $m(R - A_\varepsilon^{ji}) < 2^{-(i+j)}\varepsilon$ so that $\hat{\vartheta}_{ji}$ is upper semicontinuous on A_ε^{ji}. (Such sets exist in virtue of Lusin's theorem.) Put $A_\varepsilon = \bigcup_{i,j} A_\varepsilon^{ji}$; then obviously $m(R - A_\varepsilon) < \varepsilon$. We have

$$\vartheta_{ji}(t, x) = \begin{cases} \hat{\vartheta}_{ji}(t) & \text{for} \quad x \in V_j, \\ \infty & \text{otherwise}. \end{cases}$$

Since the sets V_j are open, it is easily seen that ϑ_{ij} are upper semicontinuous on $G \cap (A_\varepsilon \times X)$.

By definition, $\hat{f}(t, x) = \inf_{i,j} \vartheta_{ji}(t, x)$ for $(t, x) \in G$. Consequently, $\hat{f}|_{G \cap (A_\varepsilon \times X)}$ is upper semicontinuous which completes the proof of the theorem.

1.3. Theorem. *Let X be separable, let $f \in \mathcal{USC}_2(G \to R)$. Then there exists $\hat{f} \in \mathcal{SD}^*(G \to R)$ such that* (i), (ii) *from Theorem* 1.2 *holds.*

Proof. Without loss of generality we can assume that $f(G) \subset [0, 1]$. Let us define functions $\psi_{kl} : G \to [0, 1]$ for $k = 1, 2, \ldots, l = 1, 2, \ldots, 2^k - 1$ by

$$\psi_{kl}(t, x) = \begin{cases} \dfrac{l}{2^k} & \text{if} \quad f(t, x) < \dfrac{l}{2^k}, \\ 1 & \text{if} \quad f(t, x) \geqq \dfrac{l}{2^k} \end{cases}$$

and put $f_k(t, x) = \min_{l=1,\dots,2^k-1} \psi_{kl}(t, x)$. Then evidently $f(t, x) = \inf_k f_k(t, x)$.

Since the functions $\psi_{kl}(t, \cdot)$ are upper semicontinuous, the same holds for $f_k(t, \cdot)$ (for almost all t). Moreover, the range of f_k is finite. Hence we can find $\hat{f}_k \in \in \mathscr{SD}^*(G \to R)$ by Theorem 1.2 satisfying (i), (ii) with $f_k, \hat{f}_k$ instead of $f, \hat{f}$. Put

$$\hat{f}(t, x) = \inf_k \hat{f}_k(t, x) . \tag{1.3}$$

Then evidently $\hat{f} \in \mathscr{SD}^*(G \to R)$.

It remains to prove (i), (ii). The first point is evident since $\hat{f}(t, x) \leqq \hat{f}_k(t, x) \leqq \leqq f_k(t, x)$ for $k = 1, 2, \dots$ and hence $\hat{f}(t, x) \leqq \inf_k f_k(t, x) = f(t, x)$. To prove (ii), suppose that $u : I \to X$ is measurable, $v : I \to R$ is measurable and $v(t) \leqq f(t, u(t))$ a.e. in I. Then also $v(t) \leqq f_k(t, u(t))$ and hence by Theorem 1.2 (ii) $v(t) \leqq \hat{f}_k(t, u(t))$. The assertion now follows immediately from (1.3).

1.4. Definition. Let X be a metric space, R^n the n-dimensional Euclidean space, $R = R^1$, $G \subset R \times X$. Then $\mathscr{K}_n$ denotes the family of all non-empty compact convex subsets of R^n, $\mathscr{K}_n^0 = \mathscr{K}_n \cup \{\emptyset\}$. If S is a family of subsets of R^n, then the set of all $F : G \to S$ such that $F(t, \cdot)$ is upper semicontinuous for almost all t is denoted by $\mathscr{USC}_2(G \to S)$; the set of all $F : G \to S$ which satisfy the condition

(1.4) to every $\varepsilon > 0$ there is a measurable set $A_\varepsilon \subset R$ such that $m(R - A_\varepsilon) < \varepsilon$ and the function $F|_{G \cap (A_\varepsilon \times X)}$ is upper semicontinuous

is denoted by $\mathscr{SD}^*(G \to S)$.

The main result is formulated in the following

1.5. Theorem. *Let X be separable, $F \in \mathscr{USC}_2(G \to \mathscr{K}_n)$. Then there exists $\hat{F} \in \mathscr{SD}^*(G \to \mathscr{K}_n^0)$ such that*

(i) *$\hat{F}(t, x) \subset F(t, x)$ for $(t, x) \in G$,*

(ii) *if $I \subset R$ is measurable, $u : I \to X$ measurable, $(t, u(t)) \in G$ for $t \in I$, $v : I \to R^n$ measurable, $v(t) \in F(t, u(t))$ a.e. in I, then $v(t) \in \hat{F}(t, u(t))$ a.e. in I.*

Proof. Let $\{u_j\}$, $j = 1, 2, \dots$ be a countable dense set in R^n and denote

$$\omega_j(t, x) = \sup \{(y, u_j) \mid y \in F(t, x)\} .$$

Then $\omega_j \in \mathscr{USC}_2(G \to R)$ in virtue of the assumption on F. Therefore, by Theorem 1.3 we find to every ω_j a function $\hat{\omega}_j \in \mathscr{SD}^*(G \to R)$ such that (i), (ii) from Theorem 1.3 holds with ω_j, $\hat{\omega}_j$ instead of $f, \hat{f}$. Define

$$\hat{F}(t, x) = \bigcap_{j=1}^{\infty} \{y \in R^n \mid (y, u_j) \leqq \hat{\omega}_j(t, x)\} . \tag{1.5}$$

By a standard separation theorem

$$F(t, x) = \bigcap_{j=1}^{\infty} \{y \in R^n \mid (y, u_j) \leqq \omega_j(t, x)\},$$

so that (i) holds.

If u, v are functions from Theorem 1.5 (ii), then $(v(t), u_j) \leqq \omega_j(t, u(t))$ a.e. in I, $j = 1, 2, \ldots$. By Theorem 1.3 we conclude that then

$$(v(t), u_j) \leqq \hat{\omega}_j(t, u(t)) \quad \text{a.e. in } I, \quad j = 1, 2, \ldots$$

and hence by (1.5)

$$v(t) \in \hat{F}(t, u(t)) \quad \text{a.e. in } I.$$

Theorem 1.5 is proved completely.

2.

Before we pass to the problem of uniqueness of the function $\hat{F}$ from Theorem 1.5, let us introduce several lemmas. In the sequel, if $M \subset R \times X$ then $P(M)$, $P_X(M)$ denote the projections of M onto R and X, respectively.

2.1. Lemma. *Let Y be a separable metric space, $\varphi : Y \to R$ an upper semicontinuous function. Then there exists a sequence of continuous functions $\psi_j : Y \to R$ such that*

$$\psi_{j+1}(y) \leqq \psi_j(y), \quad j = 1, 2, \ldots,$$
$$\lim_{j \to \infty} \psi_j(y) = \varphi(y)$$

for $y \in Y$.

For the proof, see [4], p. 88, Theorem 14.7.5.

2.2. Lemma. *Let X be a complete separable metric space, $Q \subset R \times X$ a compact set. Then*

(i) *both $P(Q)$ and $P_X(Q)$ are compact;*

(ii) *there is a measurable function $w : P(Q) \to X$ such that $(t, w(t)) \in Q$ for $t \in P(Q)$.*

Proof. (i) holds, as P and P_X are continuous maps and Q is compact.

Since $P_X(Q)$ is compact by (i), it can be covered by a finite number of closed balls $\bar{B}_i^{(1)}$, $i = 1, 2, \ldots, k_1$, with centers s_{i1} and radius equal to one. Denote $D_i^{(1)} = \bar{B}_i^{(1)} \cap P_X(Q)$; then $D_i^{(1)}$ are compact and cover $P_X(Q)$. Let us define a function $w_1 : P(Q) \to X$ as follows:

Put $w_1(t) = s_{11}$ for $t \in P(Q \cap (R \times D_1^{(1)})) = P_{11}$; if $P_{11}, \ldots, P_{i-1,1}$ have been defined ($i \geqq 2$), then put

$$w_1(t) = s_{i1} \quad \text{for} \quad t \in P(Q \cap ((R - \bigcup_{q=1}^{i-1} P_{q1}) \times D_i^{(1)})).$$

Thus we obtain a measurable function $w_1 : P(Q) \to X$. Evidently, it may be assumed that $B_i^{(1)} \cap P_X(Q) \neq \emptyset$; consequently, $(t, w_1(t)) \in \Omega(Q, 1)$ for $t \in P(Q)$ where $\Omega(M, \varepsilon)$ denotes the ε-neighborhood of the set M.

Given a finite covering of $P_X(Q)$ by compact sets $D_i^{(j)}$, $i = 1, 2, \ldots, k_j$ with diam $D_i^{(j)} \leqq 2^{-(j-1)}$, we find a finite number of balls $\bar{B}_i^{(j+1)}$, $i = 1, 2, \ldots, k_{j+1}$, with centers $s_{i,j+1}$ and radius 2^{-j} with the following properties: $\bar{B}_i^{(j+1)}$, $i = 1, \ldots, q_1$ cover $D_1^{(j)}$, $\bar{B}_i^{(j+1)}$, $i = q_1 + 1, \ldots, q_2$ cover $D_2^{(j)}, \ldots, \bar{B}_i^{(j+1)}$, $i = q_{k_j - 1} + 1, \ldots, q_{k_j}$ cover $D_{k_j}^{(j)}$. Further, we define $D_i^{(j+1)} = \bar{B}_i^{(j+1)} \cap D_l^{(j)}$ for $i = q_{l-1} + 1, \ldots, q_l$. Evidently $D_i^{(j+1)}$, $i = 1, \ldots, k_{j+1}$ cover $P_x(Q)$. We define a function $w_{j+1} : P(Q) \to X$ as follows:

Put $w_{j+1}(t) = s_{1,j+1}$ for $t \in P(Q \cap (R \times D_1^{(j+1)})) = P_{1,j+1}$; if $P_{1,j+1}, \ldots, P_{i-1,j+1}$ have been defined $(i \geqq 2)$, then put

$$w_{j+1}(t) = s_{i,j+1} \quad \text{for} \quad t \in P\Big(Q \cap \Big(\Big(R - \bigcup_{q=1}^{i-1} P_{q,j+1}\Big) \times D_i^{(j+1)}\Big)\Big).$$

Similarly as for w_1, we conclude that $w_{j+1} : P(Q) \to X$ is measurable and $(t, w_{j+1}(t)) \in \Omega(Q, 2^{-j})$ for $t \in P(Q)$.

The functions $w_1, w_2, \ldots$ form a Cauchy sequence. Indeed, it is easily seen that the values $w_j(t)$, $w_{j+1}(t)$ are centers of balls (in X) with radii $2^{-(j+1)}, 2^{-j}$ respectively. These balls have a non-empty intersection, hence $\varrho(w_j(t), w_{j+1}(t)) \leqq 2^{-j+1}$ and $\varrho(w_p(t), w_q(t)) \leqq 2^{-p+2}$ for $p < q$.

As X is complete, there exists a measurable function $w : P(Q) \to X$, $w(t) = \lim_{j \to \infty} w_j(t)$ for $t \in P(Q)$. Moreover $(t, w(t)) \in Q$ for $t \in P(Q)$ in virtue of $(t, w_{j+1}(t)) \in \Omega(Q, 2^{-j})$. Lemma 2.2 is proved.

2.3. Lemma. *Let X be a complete separable metric space, $\emptyset \neq Q_j \subset R \times X$, Q_j closed sets, $j = 1, 2, \ldots$. Let $Q_{j+1} \subset Q_j$, $Q = \bigcap_{j=1}^{\infty} Q_j$. Let the set Q_j have a finite 2^{-j}-net, $j = 1, 2, \ldots$.*

Then the set Q is non-empty, compact and

$$P(Q) = \bigcap_{j=1}^{\infty} P(Q_j). \tag{2.1}$$

Proof. Evidently Q is closed and has a finite ε-net for every $\varepsilon > 0$. Since X is complete, this implies that Q is compact. Since $Q_j \neq \emptyset$, we can choose a sequence $z_1, z_2, \ldots$ such that $z_j \in Q_j$, $z_j = (t_j, x_j)$, $j = 1, 2, \ldots$.

Since Q_1 has a finite 2^{-1}-net, there is a point ζ_1 of this net and a subsequence of $\{z_j\}$, say $\{z_{1j}\}$ such that

$$\hat{\varrho}(z_{1j}, \zeta_1) < 2^{-1}, \quad j = 1, 2, \ldots.$$

Given a sequence $\{z_{kj}\}$, $j = 1, 2, \ldots$ and k positive integer, we find ζ_{k+1} belonging to the finite $2^{-(k+1)}$-net of Q_{k+1} and a subsequence $\{z_{k+1,j}\}$, $j = 1, 2, \ldots$ such that

$$\hat{\varrho}(z_{k+1,j}, \zeta_{k+1}) < 2^{-(k+1)}, \quad j = 1, 2, \ldots.$$

Thus we can define by induction sequences $\{z_{kj}\}$, $j = 1, 2, \ldots$ for every positive integer k so that

$$\hat{\varrho}(z_{kp}, z_{kq}) < 2^{-k}, \quad k, p, q \text{ positive integers}.$$

Taking the diagonal sequence $\{z_{jj}\}$, $j = 1, 2, \ldots$ we conclude by a standard argument that it is a Cauchy sequence and hence in virtue of completeness of X there exists $z \in X$, $z = \lim_{j\to\infty} z_{jj}$. Since $z_{pp} \in Q_j$ for $p = j, j + 1, \ldots$ and Q_j are closed, we have $z \in Q_j$, $j = 1, 2, \ldots$ and consequently $Q \neq \emptyset$.

It remains to prove (2.1). The inclusion

$$P(Q) \subset \bigcap_{j=1}^{\infty} P(Q_j)$$

is evident. Let $t \in \bigcap_{j=1}^{\infty} P(Q_j)$. Then there exist x_j, $j = 1, 2, \ldots$ such that $(t, x_j) \in Q_j$. Similarly as above we construct subsequences of $\{x_j\}$, find that the diagonal sequence has a limit x and that $(t, x) \in Q_j$, $j = 1, 2, \ldots$. Then obviously $(t, x) \in Q$ and hence $t \in P(Q)$.

Lemma 2.3 is proved.

2.4. Theorem. *Let X be a complete separable metric space, $G \subset R \times X$ a set of class F_σ, $F_i \in \mathscr{S}\mathscr{D}^*(G \to \mathscr{K}_n^0)$, $i = 1, 2$, $\Omega = \{(t, x) \in G \mid F_1(t, x) - F_2(t, x) \neq \emptyset\}$.*

Then the set $D = P(\Omega)$ is measurable and there exist measurable functions $v : D \to R^n$, $w : D \to X$ such that

$$(t, w(t)) \in G \quad \text{and} \quad v(t) \in F_1(t, w(t)) - F_2(t, w(t)) \quad \text{a.e. in } D.$$

The theorem is an easy consequence of the following

2.5. Proposition. *Let us keep the notation of Theorem 2.4. Let $0 < m^*(D) < \infty$, $0 < \zeta < m^*(D)$. Then there exists a compact set $Q \subset G$ with $m(P(Q)) \geqq \zeta$,*

$$F_1(t, x) - F_2(t, x) \neq \emptyset \quad \textit{for} \quad (t, x) \in Q. \tag{2.2}$$

Further, there exist measurable functions $\hat{v} : P(Q) \to R^n$, $\hat{w} : P(Q) \to X$ such that

$$\hat{v}(t) \in F_1(t, \hat{w}(t)) - F_2(t, \hat{w}(t)) \quad \text{for} \quad t \in P(Q). \tag{2.3}$$

Proof. Let $\zeta + 2\eta < m^*(D)$, $\zeta > 0$, $\eta > 0$. Let $G = \bigcup_{i=1}^{\infty} H_i$, $H_i \subset H_{i+1}$ for $i = 1, 2, \ldots$, the sets H_i being closed. Without loss of generality we may assume that the set $P(Q)$ is bounded. Obviously $D = P(\Omega) = \bigcup_{i=1}^{\infty} P(\Omega \cap H_i)$, $P(\Omega \cap H_i) \subset P(\Omega \cap H_{i+1})$ for $i = 1, 2, \ldots$ so that $m^*(D) = \lim_{i\to\infty} m^*(P(\Omega \cap H_i))$ and there exists such an integer r that

$$m^*(P(\Omega \cap H_r)) > \zeta + 2\eta.$$

There exists a closed set $A_\eta \subset R$ such that $m(R - A_\eta) < \eta$ and the functions $F_i|_{G\cap(A_\eta\times X)}$, $i = 1, 2$ are upper semicontinuous. Let the set $\{u_i\}$, $i = 1, 2, \ldots$ be dense in R^n and put

$$\varphi_i(t, x) = \sup\{(u_i, y) \mid y \in F_2(t, x)\} \quad \text{for} \quad (t, x) \in G \cap (A_\eta \times X).$$

Then the function φ_i is evidently upper semicontinuous, $i = 1, 2, \ldots$ and according to Lemma 2.1 there exist continuous functions $\psi_{ij} : G \cap (A_\eta \times X) \to R$ so that

$$\psi_{i,j+1}(t, x) \leqq \psi_{i,j}(t, x), \quad \lim_{j\to\infty} \psi_{ij}(t, x) = \varphi_i(t, x).$$

For $k = 1, 2, \ldots$ put

$$\Omega_k = \{(t, x) \in H_r \cap (A_\eta \times X) \mid F_1(t, x) \cap \cap [\bigcup_{i=1}^{k} \{z \in R^n \mid (u_i, z) \geqq \psi_{ik}(t, x)\}] \neq \emptyset\}. \tag{2.4}$$

These sets are closed, since F_1 is upper semicontinuous, ψ_{ij} are continuous and the sets H_r, A_η are closed. Moreover, $\Omega_{k+1} \supset \Omega_k$, $k = 1, 2, \ldots$. Let us prove

$$\Omega \cap H_r \cap (A_\eta \times X) = \bigcup_{k=1}^{\infty} \Omega_k. \tag{2.5}$$

Let $(t, x) \in \Omega \cap H_r \cap (A_\eta \times X)$. Then there exists $h \in F_1(t, x) - F_2(t, x)$. Since $F_2(t, x)$ is closed convex, there is $u \in R^n$ such that

$$(u, h) > \sup\{(u, y) \mid y \in F_2(t, x)\}.$$

Since $F_2(t, x)$ is compact, there is an index l such that

$$(u_l, h) > \varphi_l(t, x).$$

Consequently, if k is such that $k \geqq l$ and $(u_l, h) > \psi_{lk}(t, x)$ we have $(t, x) \in \Omega_k$ and hence

$$\Omega \cap H_r \cap (A_\eta \times X) \subset \bigcup_{k=1}^{\infty} \Omega_k.$$

The converse inclusion being obvious, we conclude that (2.5) holds.

Obviously $P(\Omega \cap H_r \cap (A_\eta \times X)) = P(\Omega \cap H_r) \cap A_\eta$ and since $m(R - A_\eta) < \eta$, $m^*(P(\Omega \cap H_r)) > \zeta + 2\eta$, we have $m^*(P(\Omega \cap H_r) \cap A_\eta) > \zeta + \eta$. In virtue of (2.5) we have

$$P(\Omega \cap H_r) \cap A_\eta = \bigcup_{k=1}^{\infty} P(\Omega_k),$$

$P(\Omega_{k+1}) \supset P(\Omega_k)$ and hence there is a positive integer p such that $m^*(P(\Omega_p)) > > \zeta + \eta$.

Let the set $\{g_i\}$, $i = 1, 2, \ldots$ be dense in $R \times X$. Then

$$P(\Omega_p) = \bigcup_{l=1}^{\infty} P(\Omega_p \cap (\bigcup_{k=1}^{l} \bar{B}(g_k, 1)))$$

and there exists l_1 such that $m^*(P(Q_1)) > \zeta + \frac{1}{2}\eta$, where $Q_1 = \Omega_p \cap (\bigcup_{k=1}^{l_1} \bar{B}(g_k, 1))$.

By induction we find $l_2, l_3, \ldots$ so that $Q_{i+1} = Q_i \cap (\bigcup_{k=1}^{l_{i+1}} \bar{B}(g_k, 2^{-i}))$ and

$$m^*(P(Q_{i+1})) > \zeta + 2^{-(i+1)}\eta \,. \tag{2.6}$$

Put $Q = \bigcap_{i=1}^{\infty} Q_i$. Obviously $Q \subset Q_1 \subset \Omega_p \subset H_r \subset G$ and the sets Q_i are closed. By Lemma 2.3 the set Q is compact, non-empty and satisfies (2.1); moreover, $Q \subset G$. It follows from (2.6) that $m^*(P(Q)) \geqq \zeta$. Since $P(Q)$ is compact by Lemma 2.2 (i), we conclude

$$m(P(Q)) \geqq \zeta \,.$$

By Lemma 2.2 (ii) there exists $\hat{w} : P(Q) \to X$ measurable and such that $(t, \hat{w}(t)) \in Q$ for $t \in P(Q)$, i.e.

$$F_1(t, \hat{w}(t)) - F_2(t, \hat{w}(t)) \neq \emptyset \,.$$

Now we shall find a measurable function $\hat{v} : P(Q) \to R^n$ satisfying

$$\hat{v}(t) \in F_1(t, \hat{w}(t)) \cap [\bigcup_{i=1}^{p} \{z \in R^n \mid (u_i, z) \geqq \psi_{ip}(t, \hat{w}(t))\}] \tag{2.7}$$

for $t \in P(Q)$. In virtue of (2.4) and the inclusion $\Omega_p \supset Q$, the set on the right-hand side of (2.7) is a non-empty set for $t \in P(Q)$. Evidently (2.7) implies (2.3).

Let j be a positive integer. Find a closed set $C_j \subset P(Q)$ with

$$m(C_j) > m(P(Q))\,(1 - 2^{-j})\,,$$

such that the function $\hat{w}|_{C_j}$ is continuous and the function $F_1|_{G \cap (C_j \times R^n)}$ is upper semicontinuous. The composed function $F_1(t, \hat{w}(t))$ is upper semicontinuous on C_j. The continuity of the functions ψ_{ip}, $i = 1, 2, \ldots$ implies that the set

$$H_j = \{(t, y) \mid t \in C_j,\ y \in F_1(t, \hat{w}(t)) \cap$$
$$\cap [\bigcup_{i=1}^{p} \{z \in R^n \mid (u_i, z) \geqq \psi_{ip}(t, \hat{w}(t))\}]\}$$

is compact. Consequently, by Lemma 2.2 (ii) there exists a measurable function $v_j : C_j \to R^n$ such that $(t, v_j(t)) \in H_j$ for $t \in C_j$.

The function $\hat{v}$ defined by $\hat{v}(t) = v_1(t)$ for $t \in C_1$, $\hat{v}(t) = v_j(t)$ for $j = 2, 3, \ldots$, $t \in C_j - \bigcup_{i=1}^{j-1} C_i$ is measurable, defined a.e. in $P(Q)$ and satisfies (2.7). This completes the proof of Proposition 2.5.

2.6. Proof of Theorem 2.4. According to Proposition 2.5, to every $\varepsilon > 0$ there is a compact set $Q_\varepsilon \subset G$ such that $m^*(D) - m(P(Q_\varepsilon)) < \varepsilon$; hence D is measurable. Further, consider compact sets $Q_{2^{-j}}$ and find functions $\hat{v}_j$, $\hat{w}_j$ from Proposition 2.5 satisfying (2.3) for $t \in P(Q_{2^{-j}})$. Similarly as above, put $v(t) = \hat{v}_1(t)$, $w(t) = \hat{w}_1(t)$ for $t \in Q_{2^{-1}}$, $v(t) = \hat{v}_j(t)$, $w(t) = \hat{w}_j(t)$ for $j = 2, 3, \ldots$, $t \in Q_{2^{-j}} - \bigcup_{i=1}^{j-1} Q_{2^{-i}}$. Then the functions v, w are defined a.e. in $P(Q)$ and satisfy the assertion of Theorem 2.4.

2.7. Note. Let the metric space X be complete separable and let $G \subset R \times X$ be of type F_σ. Let $\hat{F}_i \in \mathscr{S}\mathscr{D}^*(G \to \mathscr{K}_n^0)$, $i = 1, 2$ fulfil the conditions (0.8), (0.12) with $\hat{F}_i$ instead of $\hat{F}$. Put

$$\Omega_1 = \{(t, x) \in G \mid \hat{F}_1(t, x) - \hat{F}_2(t, x) \neq \emptyset\},$$
$$\Omega_2 = \{(t, x) \in G \mid \hat{F}_2(t, x) - \hat{F}_1(t, x) \neq \emptyset\}.$$

By Theorem 2.4 the sets $D_i = P(\Omega_i)$ are measurable and there exist measurable functions $v_i : D_i \to R^n$, $w_i : D_i \to X$, $i = 1, 2$ such that

$$v_1(t) \in \hat{F}_1(t, w_1(t)) - \hat{F}_2(t, w_1(t)) \quad \text{a.e. in} \quad D_1, \tag{2.8}$$
$$v_2(t) \in \hat{F}_2(t, w_2(t)) - \hat{F}_1(t, w_2(t)) \quad \text{a.e. in} \quad D_2.$$

By (2.8) and (0.8) $v_1(t) \in F(t, w_1(t))$ a.e. in D_1 and by (0.12) $v_1(t) \in \hat{F}_2(t, w_1(t))$ a.e. in D_1. Consequently $m(D_1) = 0$ and similarly $m(D_2) = 0$. The assertion from 0.3 holds with $A = R - (D_1 \cup D_2)$.

3.

3.1. Theorem. *Let $F \in \mathscr{S}\mathscr{D}^*(G \to \mathscr{K}_n^0)$ where $G \subset R \times X$, X being a metric space. To every $(\tilde{t}, \tilde{x}) \in G$ let there exist a non-degenerate interval I and a function $w : I \to X$ such that*

$$\lim_{t \to \tilde{t}} w(t) = \tilde{x}, \tag{3.1}$$

$$F(t, w(t)) \neq \emptyset \quad \text{a.e. in} \quad I. \tag{3.2}$$

Let E be the set of all $t \in R$ such that there exists $x \in X$ with $F(t, x) = \emptyset$. Then $m(E) = 0$.

Proof. Let us assume that $m^*(E) > 0$. Let $A \subset R$ be a measurable set such that $m(R - A) < m^*(E)$ and the function $F|_{G \cap (A \times X)}$ is upper semicontinuous. Then $m^*(A \cap E) > 0$. Let B be the set of $t \in A$ which are points of metrical density of the set A. Then $m(A - B) = 0$, hence $m^*(B \cap E) > 0$.

Let us choose $\tilde{t} \in B \cap E$ and find $\tilde{x}$ so that $F(\tilde{t}, \tilde{x}) = \emptyset$. Let $w : I \to X$ be the function from the assumptions of Theorem 3.1.

Since $F|_{G\cap(A\times X)}$ is upper semicontinuous, there exists an open set $U \subset G$ so that $(\bar{t}, \tilde{x}) \in U$ and $F(t, x) = \emptyset$ for $(t, x) \in U \cap (A \times X)$. By (3.1) there exists $\delta > 0$ such that $F(t, w(t)) = \emptyset$ provided $t \in I \cap A$, $|t - \bar{t}| < \delta$. However, since $\bar{t}$ is a point of metrical density of the set A, (3.2) cannot hold.

4.

Notation. Let A be a set of real numbers and r a real number. The sets $A + r$, rA are defined by $A + r = \{a + r \mid a \in A\}$ and $rA = \{ra \mid a \in A\}$. Denote by $\varkappa(\alpha)$ the cardinal number corresponding to an ordinal type α. Let $\varkappa$ be the cardinal number of continuum and ω the first ordinal type fulfilling $\varkappa(\omega) = \varkappa$. If A is a set then $\varkappa(A)$ is the cardinal number of A.

First we shall formulate without proofs four well-known results.

4.1. Lemma. *Let $\tilde{F}$ be the system of all closed subsets of $[0, 1]$. Then $\varkappa(\tilde{F}) = \varkappa$.*

4.2. Lemma. *Let A be an uncountable closed subset of $[0, 1]$. Then $\varkappa(A) = \varkappa$.*

4.3. Lemma. *The set of ordinal types $\{\beta \mid 0 \leqq \beta < \alpha\}$ is of the ordinal type α.*

4.4. Lemma. *If σ is an infinite cardinal number, then $2\sigma = \sigma^2 = \sigma$.*

Since $\varkappa(\{\alpha \mid 0 \leqq \alpha < \omega\}) = \varkappa(\omega) = \varkappa$ (cf. Lemma 4.3), there exists a one-to-one mapping $\alpha \to r_\alpha$ of $\{\alpha \mid 0 \leqq \alpha < \omega\}$ onto the interval $[0, 1]$.

4.5. Lemma. *If Z is a set, $Z \subset [0, 1]$, $\varkappa(Z) = \varkappa$, then there exists a real function g defined on $\{\alpha \mid 0 \leqq \alpha < \omega\}$ such that $g(\alpha) \in Z$ for every α, g is one-to-one and the mapping $\alpha \to g(\alpha) + r_\alpha$ is one-to-one.*

Proof. Let g be a function defined on a set $\{\alpha \mid 0 \leqq \alpha < \delta\}$, $\delta \leqq \omega$ which fulfils $g(\alpha) \in Z$ and

$$g(\alpha) \neq g(\beta) \quad \text{and} \quad g(\alpha) + r_\alpha \neq g(\beta) + r_\beta \tag{4.1}$$

for all α, β from its definition domain, $\alpha \neq \beta$. Let Q be the set of all such functions g. Denote by $\preceq$ a partial ordering on Q defined by $g^{(1)} \preceq g^{(2)}$ if the domain of definition $D^{(1)}$ of $g^{(1)}$ is contained in the domain of definition $D^{(2)}$ of $g^{(2)}$ and $g^{(1)}$, $g^{(2)}$ coincide on $D^{(1)}$. Let now $\Lambda \neq \emptyset$ be a set of indices, $g^{(\lambda)} \in Q$ for $\lambda \in \Lambda$ and let $\{g^{(\lambda)} \mid \lambda \in \Lambda\}$ be a linearly ordered subset of Q. Denote by $D^{(\lambda)} = \{\alpha \mid 0 \leqq \alpha < \delta^{(\lambda)}\}$ the corresponding domains of definition. Put $\delta = \sup_{\lambda \in \Lambda} \delta^{(\lambda)}$, $D = \{\alpha \mid 0 \leqq \alpha < \delta\}$, i.e. $D = \bigcup_{\lambda \in \Lambda} D^{(\lambda)}$. For every $\alpha \in D$ there exists $\lambda \in \Lambda$ such that $\alpha \in D^{(\lambda)}$. Put $g(\alpha) = g^{(\lambda)}(\alpha)$. Obviously $g \in Q$ and $g^{(\lambda)} \preceq g$ for $\lambda \in \Lambda$. This implies (by the Zorn lemma) that Q has a maximal element $\bar{g}$ with a domain of definition $\bar{D} = \{\alpha \mid 0 \leqq \alpha < \bar{\delta}\}$.

Assume $\bar{\delta} < \omega$. Denote $V = \{\bar{g}(\alpha) \mid 0 \leqq \alpha < \bar{\delta}\} \cup \{\bar{g}(\alpha) + r_\alpha - r_{\bar{\delta}} \mid 0 \leqq \alpha < \bar{\delta}\}$. Evidently $\varkappa(V) \leqq \varkappa(\bar{\delta}) + \varkappa(\bar{\delta}) < \varkappa(\omega) = \varkappa$ (see Lemma 4.4 and the definition of ω). Thus $Z - V \neq \emptyset$ and there exists $z \in Z - V$. Define $\hat{g}(\alpha) = \bar{g}(\alpha)$ for $0 \leqq \alpha < \bar{\delta}$ and $\hat{g}(\bar{\delta}) = z$.

Obviously $\hat{g} \in Q$ with the domain of definition $\hat{D} = \{\alpha \mid 0 \leqq \alpha < \bar{\delta} + 1\}$ and, moreover, $\bar{g} \neq \hat{g}$ and $\bar{g} \precsim \hat{g}$. This contradicts the fact that $\bar{g}$ is maximal. We conclude $\bar{\delta} = \omega$ and Lemma 4.5 is proved.

Denote by F the family of all closed subsets A of $[0, 1]$ fulfilling $m(A) > 0$. It can be deduced from Lemma 4.1 that $\varkappa(F) = \varkappa$ and since $\varkappa(\{\alpha \mid 0 \leqq \alpha < \omega\}) = \varkappa$ there exists a one-to-one mapping $\alpha \to F_\alpha$ defined on $\{\alpha \mid 0 \leqq \alpha < \omega\}$ which maps $\{\alpha \mid 0 \leqq \alpha < \omega\}$ onto F.

4.6. Lemma. *Let $C = \{(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha < \omega\}$. There exists a real function f defined on C such that*

$$f(\alpha, \beta) \in F_\alpha \quad \text{for} \quad 0 \leqq \beta \leqq \alpha\,, \tag{4.2}$$

$$f(\alpha, \beta) \neq f(\alpha', \beta') \quad \text{for} \quad (\alpha, \beta) \neq (\alpha', \beta')\,, \tag{4.3}$$

$$f(\alpha, \beta) + r_\beta \neq f(\alpha', \beta') + r_{\beta'} \quad \text{for} \quad (\alpha, \beta) \neq (\alpha', \beta')\,. \tag{4.4}$$

Proof. Denote $C^{(\delta)} = \{(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha < \delta\}$ for all $\delta \leqq \omega$ and let G be the set of all real functions defined on sets $C^{(\delta)}$ fulfilling (4.2) to (4.4). As in the proof of Lemma 4.5 we introduce a partial ordering on G. We write $f^{(1)} \precsim f^{(2)}$ if the corresponding domains of definition satisfy $C_1 \subset C_2$ and $f^{(1)} = f^{(2)}$ on C_1.

The existence of a maximal element of G can be proved analogously as in the proof of Lemma 4.5. Denote the maximal element by $\bar{f}$ and the corresponding domain of definition by $\bar{C} = \{(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha < \bar{\delta}\}$.

Assume $\bar{\delta} < \omega$. Put $W_\alpha = \{\bar{f}(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha\}$ for $\alpha < \bar{\delta}$. Obviously $\varkappa(W_\alpha) = \varkappa(\{\beta \mid 0 \leqq \beta \leqq \alpha\}) = \varkappa(\alpha) + 1 \leqq \varkappa(\bar{\delta})$. Denote $W = \bigcup_{\alpha < \bar{\delta}} W_\alpha = \{\bar{f}(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha < \bar{\delta}\}$. Since $\varkappa(\{\alpha \mid 0 \leqq \alpha < \bar{\delta}\}) = \varkappa(\bar{\delta})$ we obtain $\varkappa(W) \leqq \varkappa^2(\bar{\delta}) = \varkappa(\bar{\delta})$ (see Lemma 4.4; notice that $\varkappa(\bar{\delta})$ cannot be finite). Let $W' = \{\bar{f}(\alpha, \beta) + r_\beta - r_\gamma \mid 0 \leqq \beta \leqq \alpha < \bar{\delta},\ 0 \leqq \gamma \leqq \bar{\delta}\}$. Analogously as for W we can estimate

$$\varkappa(W') \leqq \varkappa(\{(\alpha, \beta) \mid 0 \leqq \beta \leqq \alpha < \bar{\delta}\} \times \{\gamma \mid 0 \leqq \gamma \leqq \bar{\delta}\}) = \varkappa^2(\bar{\delta})\, \varkappa(\bar{\delta}) = \varkappa(\bar{\delta})\,.$$

Denote

$$Z = F_{\bar{\delta}} - W - W'\,. \tag{4.5}$$

Using the above estimates for $\varkappa(W)$ and $\varkappa(W')$, we obtain due to Lemma 4.2 $\varkappa(Z) = \varkappa$.

Let g be a function from Lemma 4.5. Denote $\hat{f}(\alpha, \beta) = \bar{f}(\alpha, \beta)$ for $0 \leqq \beta \leqq \alpha < \bar{\delta}$, $\hat{f}(\bar{\delta}, \beta) = g(\beta)$ for $0 \leqq \beta \leqq \bar{\delta}$. The function $\hat{f}$ is defined on $\hat{C} = \{(\alpha, \beta) \mid 0 \leqq$

$\leqq \beta \leqq \alpha < \delta + 1\}$, $\hat{f}(\alpha, \beta) \in F_\alpha$ and $\hat{f}$ fulfils (4.3) and (4.4) due to (4.1) and (4.5). Evidently $\hat{f} \in G$, $\bar{f} \leqq \hat{f}$ and $\bar{f} \neq \hat{f}$. This contradiction implies $\delta = \omega$ and Lemma 4.6 is proved.

Given $M \subset [0, 1]$, then $m^*(M)$, $m_*(M)$ stand respectively for the outer and the inner Lebesgue measure of M.

4.7. Lemma. *To every ordinal type α, $0 \leqq \alpha < \omega$ there exists a set M_α and a real number r_α such that $M_\alpha \subset [0, 1]$, $r_\alpha \in [0, 1]$, $m^*(M_\alpha) = 1$, $m_*(M_\alpha) = 0$, $\{r_\alpha \mid 0 \leqq \leqq \alpha < \omega\} = [0, 1]$ and $M_\alpha \cap M_\beta = \emptyset$, $(M_\alpha + r_\alpha) \cap (M_\beta + r_\beta) = \emptyset$ for $\alpha \neq \beta$.*

Proof. Let $\alpha \to r_\alpha$ be the mapping defined above. Since this mapping is onto $[0, 1]$ we have $\{r_\alpha \mid 0 \leqq \alpha < \omega\} = [0, 1]$. Put $M_\beta = \{f(\alpha, \beta) \mid \beta \leqq \alpha < \omega\}$ where f is the function from Lemma 4.6. Obviously $M_\beta \subset [0, 1]$ and the last two disjointness properties follow from (4.3) and (4.4), respectively.

We shall prove $m^*(M_\beta) = 1$. Let β be a given ordinal type, $\beta < \omega$, and assume $m^*(M_\beta) < 1$. Then there exists an open set U, $M_\beta \subset U$, $m(U) < 1$. Denote $F^{(+)} = = [0, 1] - U$. Obviously $m(F^{(+)}) > 0$. Put $F^{(\lambda)} = F^{(+)} \cap [0, \lambda]$ and $m(\lambda) = = m(F^{(\lambda)})$. Then $m(\lambda)$ is a continuous function, $m(0) = 0$, $m(1) > 0$. Let $0 < \xi \leqq \leqq m(1)$. Define $\lambda(\xi) = \min\{\lambda \mid m(\lambda) = \xi\}$. Evidently $\lambda(\xi) \in F^{(+)}$ for every $\xi \in \in (0, m(1)]$ and $\lambda(\xi_1) \neq \lambda(\xi_2)$ for $\xi_1 \neq \xi_2$. Denote $\Lambda = \{\lambda(\xi) \mid \xi \in (0, m(1)]\}$. Evidently $\varkappa(\Lambda) = \varkappa$ and $\varkappa(\{\gamma \mid 0 \leqq \gamma < \beta\}) = \varkappa(\beta) < \varkappa(\omega) = \varkappa$. Hence there exists a couple μ, δ such that $\mu \in \Lambda$, δ is an ordinal type, $\beta \leqq \delta < \omega$ and $F^{(\mu)} = F_\delta$. Thus the set $F^{(\mu)}$ contains the point $f(\delta, \beta)$ and consequently $F^{(\mu)} \cap M_\beta \neq \emptyset$, i.e. $F^{(+)} \cap M_\beta \neq \emptyset$.

We conclude $M_\beta \not\subset U$. This contradiction proves $m^*(M_\beta) = 1$. Since M_β are disjoint we obtain

$$m_*(M_\beta) = 1 - m^*([0, 1] - M_\beta) \leqq 1 - m^*(M_\sigma) = 0\,, \quad \sigma \neq \beta\,.$$

Since the mapping $\beta \to r_\beta$ is one-to-one and onto $[0, 1]$ there exists to every r, $0 \leqq r \leqq 1$ an ordinal type $\beta < \omega$ such that $r = r_\beta$. Put $M_r = M_\beta$. Using this new notation, Lemma 4.7 can be reformulated.

4.8. Lemma. *To every r, $0 \leqq r \leqq 1$ there exists a set M_r such that $M_r \subset [0, 1]$, $m^*(M_r) = 1$, $m_*(M_r) = 0$, $M_r \cap M_g = \emptyset$, $(M_r + r) \cap (M_g + g) = \emptyset$ for $r \neq g$.*

4.9. Theorem. *There exists a set S in $[0, 1] \times R$ such that*

(4.6) $\{x \mid (t, x) \in S\}$ *contains at most one point for every t,*

(4.7) $\{t \mid (t, x) \in S\}$ *contains at most one point for every x,*

(4.8) $m^*\{t \mid (t, t + x) \in S\} = 1\,, \quad m_*\{t \mid (t, t + x) \in S\} = 0$ *for every* x, $0 \leqq x \leqq 1\,.$