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COMPACT ELEMENTS OF THE LATTICE OF CONGRUENCES IN AN ALGEBRA

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In [5] I, a basic information about partitions in a set and congruences in an algebra can be found. Here, only necessary concepts will be introduced. A *partition* A *in* a set G is a system of pairwise disjoint nonempty subsets of G. These subsets will be called *blocks* of the partition A, its union $\bigcup A$ the *domain* of A. Of course, A is a partition *on* the set $\bigcup A$. Partitions in G are in a 1-1-correspondence with the symmetric and transitive binary relations (ST-relations) in G, analogously as partitions *on* G correspond to equivalence relations in G. For this reason, we shall sometimes not distinguish partitions and ST-relations. If (G, F) is a partial algebra then the ST-relations in the set G which are stable with respect to F are called *congruences in* (G, F). For the sake of completeness we give the definition of a stable binary relation A in a partial algebra (G, F): Let $f \in F$ be n-ary ($n \geqq 1$) and $a_i A b_i$ ($i = 1, 2, \ldots, n$), let $f(a_1, \ldots, a_n)$ and $f(b_1, \ldots, b_n)$ exist. Then $f(a_1, \ldots, a_n)\, A\, f(b_1, \ldots, b_n)$.

The theory of partitions in a set and of congruences in an algebra has been an object of systematic study only recently even though the concepts appeared in the literature not less than forty years ago [2, 3, 4, 5, 7]. Nonetheless, the congruences "in" actually acted latently much earlier, already in the classical group theory, e.g. in connection with the Schreier-Zassenhaus theorem in which congruences on subgroups are considered and not only those on the whole group. It was in this domain where "in" approach yielded formal as wel as matter-of-fact means for generalizing this theorem to algebras [7].

The sets $R(G)$ of all binary relations in a set G, $P(G)$ of all partitions in G and $\mathscr{K}(G, F)$ of all congruences in a partial algebra (G, F) are complete lattices under set inclusion. In all these cases the infimum of a system of relations – elements of the corresponding lattice – is equal to their set-intersection [4, 5]. Also the lattices $\Pi(G)$ of all partitions on a set G and $\mathscr{C}(G, F)$ of all congruences on a partial algebra (G, F) are complete, the latter being a closed sublattice of the former which is not true in the situation "in". $\Pi(G)$ is a closed sublattice of $P(G)$. The lattices $\Pi(G)$ and $\mathscr{C}(G, F)$, are algebraic. In Section 1, we shall prove the same property for the lattices $R(G)$

$P(G)$ and $\mathscr{K}(G, F)$ (1.3, 1.4, 1.6, 1.13). It is shown that the compact elements of $\mathscr{K}(G, F)$ are precisely the upper $\mathscr{K}$-modifications (see Def. 1.5) of compact elements of $P(G)$ (or of $R(G)$) and the compact elements of $P(G)$ and $R(G)$ are exactly the finite relations in G (1.3, 1.4, 1.6, 1.14).

In Section 2, we construct the upper $\mathscr{K}$-modification Ψ_A of a binary relation A in a partial algebra (G, F) (2.7). The construction is similar to that of the upper $\mathscr{C}$-modification Θ_A of a relation A given in [6] 5.3, 5.4. It is identical with it if we replace the algebra (G, F) in the construction of Θ_A by its subalgebra $(\bigcup \Psi_A, F)$ (2.14). For this purpose we need to know the set $\bigcup \Psi_A$; this is established in 2.11.

1. PROPERTIES OF LATTICES $R(G)$, $P(G)$ AND $\mathscr{K}(G, F)$

1.1. ([5] I 1.2). *Let (G, F) be an algebra, and $\{A_n\} \subseteq \mathscr{K}(G, F)$. Then $\bigvee_{\mathscr{K}\alpha} A_\alpha = \bigvee_{P\beta} B_\beta$, where B_β stands for the congruence $A_{\alpha_1} \vee_{\mathscr{K}} \ldots \vee_{\mathscr{K}} A_{\alpha_n}$ for an arbitrary finite choice $A_{\alpha_1}, \ldots, A_{\alpha_n}$ in $\{A_\alpha\}$.*

In general, the theorem does not hold for partial algebras.

1.2 ([5] I 1.2.0). *Let (G, F) be an algebra and $\{A_\alpha\}$ an up-directed subset of $\mathscr{K}(G, F)$. Then $\bigvee_{\mathscr{K}\alpha} A_\alpha = \bigvee_{P\alpha} A_\alpha = \bigcup_\alpha A_\alpha$.*

Proof. The first equality is proved in [5]. The other is obvious.

1.3. Theorem. *The set $R(G)$ of all binary relations in a set G is an algebraic lattice with respect to inclusion. The compact elements of $R(G)$ are exactly the finite relations in G.*

Proof. Evidently, $R(G)$ is a complete lattice. Infima are intersections and suprema are unions.

Let $T \in R(G)$, $T = \{x_1, \ldots, x_n\}$ and let n be a positive integer. Suppose that a system $\{T_\alpha : \alpha \in I\}$ satisfies $\bigcup_{\alpha \in I} T_\alpha \supseteq T$. For each $x_i \in T$ there exists $\alpha_i \in I$ with $x_i \in T_{\alpha_i}$ $(i = 1, \ldots, n)$. Thus $\bigcup_{i=1}^{n} T_{\alpha_i} \supseteq T$.

Let $T \in R(G)$ be an infinite relation. Define $T_x = \{x\}$ for each $x \in T$. Then $T = \bigcup_{x \in T} T_x$ and $T \nsupseteq \bigcup_{x \in T_1} T_x$ for all $T_1 \subseteq T$ and $T_1 \neq T$.

Finally, let $T \in R(G)$. Then $T = \bigcup_{x \in T} \{x\}$ and $\{x\}$ is compact in $R(G)$ for all $x \in T$.

1.4. Theorem. *The lattice $P(G)$ of all partitions in a set G is algebraic. A partition is compact in $P(G)$ if and only if it contains only finitely many blocks, each of them being a finite set.*

Proof. $P(G)$ is a complete lattice by [2]. First, we shall prove that a partition $A = \{A^1\}$ with one finite block $A^1 = \{x_1, \dots, x_n\}$ is compact in $P(G)$. If $\mathfrak{A} = \{A_\delta : \delta \in \Delta\} \subseteq P(G)$ and $\bigvee \mathfrak{A} \geqq A$ then a certain block $B^1 \in \bigvee \mathfrak{A}$ contains A^1. Given $x_i, x_j \in A^1$ there exist elements $y_1, \dots, y_{m-1}$ of G and indices $\delta_1, \dots, \delta_m$ of Δ with $x_i A_{\delta_1} y_1 \dots y_{m-1} A_{\delta_m} x_j$.

Denote $\mathfrak{A}_{i,j} = \{A_{\delta_k} : k = 1, \dots, m\}$ and $\mathfrak{B}_1 = \bigcup_{i,j=1}^{n} \mathfrak{A}_{i,j}$. Then $\bigvee \mathfrak{B}_1 \geqq A$ and $\mathfrak{B}_1$ is a finite subsystem of $\mathfrak{A}$. If the partition A consists of finite blocks $A^1, \dots, A^k$ (k a positive integer) we construct a (finite) system $\mathfrak{B}_t \subseteq \mathfrak{A}$ for every A^t ($1 \leqq t \leqq k$) in the described manner; then $\bigvee \mathfrak{B} \geqq A$ for $\mathfrak{B} = \bigcup_{t=1}^{k} \mathfrak{B}_t$ and $\mathfrak{B}$ is a finite subsystem of $\mathfrak{A}$.

Next, we shall prove that a partition A 1) with at least one infinite block or 2) with infinite many blocks fails to be compact.

1) Let A^1 be infinite, $A^1 \in A$, $x, y \in A^1$. Denote by $A_{x,y}$ the partition in G which we obtain from A taking the block $\{x, y\}$ instead of A^1 (the other blocks of A remain unchanged). The join of the system $\mathfrak{A}$ of all partitions $A_{x,y}$ ($x, y \in A^1$) equals A. The blocks of the join of an arbitrary finite subsystem $\mathfrak{A}_1$ of $\mathfrak{A}$ are all blocks of the partition A except A^1 and in addition some blocks which together cover only a finite part of A^1. Thus $\bigvee \mathfrak{A}_1 > A$.

2) Let $A = \{A^\delta : \delta \in \Delta\}$, card $\Delta \geqq \aleph_0$. Define one-block partitions $A_\delta = \{A^\delta\}$, $\delta \in \Delta$. Then $\bigvee \{A_\delta : \delta \in \Delta\} = A$. It is evident that none of the finite subsystems of $\{A_\delta : \delta \in \Delta\}$ has supremum $\geqq A$.

It remains to prove that an arbitrary element of $P(G)$ is the join of compact ones. Given $A \in P(G)$, $A^1 \in A$ and $x, y \in A^1$ we construct a one-block partition $A_{x,y} = \{\{x, y\}\}$. All these partitions are compact elements of $P(G)$ and its supremum is equal to A. The theorem is proved.

1.5. Definition. Let L be a partially ordered set, $\emptyset \neq K \subseteq L$ and $a \in L$. An element $b \in K$ is said to be *an upper K-modification of a* if b is the least element of K containing a.

1.6. Theorem. *Let (G, F) be an algebra. Then $\mathscr{K}(G, F)$ is an algebraic lattice. The upper $\mathscr{K}$-modifications of compact elements of $P(G)$ are compact in $\mathscr{K}(G, F)$.*

1.7. Remark. In 1.14 we shall prove that all compact elements of $\mathscr{K}(G, F)$ are of the above mentioned form.

Proof. Let T be a compact element of $P(G)$ and K the upper $\mathscr{K}$-modification of T. Let $\{K_\alpha : \alpha \in I\} \subseteq \mathscr{K}(G, F)$ and $\bigvee_{\mathscr{K}, \alpha \in I} K_\alpha \geqq K$. By 1.1, $\bigvee_{P, \beta \in J} L_\beta = \bigvee_{\mathscr{K}, \alpha \in I} K_\alpha$, where L_β runs through the $\mathscr{K}$-suprema of all finite subsets of $\{K_\alpha : \alpha \in I\}$. We have $\bigvee_{P, \beta \in J} L_\beta = \bigvee_{\mathscr{K}, \alpha \in I} K_\alpha \geqq K \geqq T$. There exists a finite subset J_1 of J with $\bigvee_{P, \beta \in J_1} L_\beta \geqq T$. Therefore

$\bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq \bigvee_{P, \beta \in J_1} L_\beta \geqq T$ and thus $\bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq K$. For each $\beta \in J_1$ there exists a finite subset $I(\beta)$ of I such that L_β is a $\mathscr{K}$-supremum of the system $\{K_\delta\}$ $(\delta \in I(\beta))$. Let I_1 be the join of all sets $I(\beta)$ with β running over J_1. Then I_1 is finite and $\bigvee_{\mathscr{K}, \gamma \in I_1} K_\gamma \geqq$ $\geqq \bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq K$. Consequently, K is a compact element of $\mathscr{K}(G, F)$.

The lattice $\mathscr{K}(G, F)$ is complete by [5] I 1.1. It remains to prove that it is compactly generated. An arbitrary congruence K is a partition, hence it is $\bigvee_P$ of a set of compact elements of $P(G)$, say $\mathfrak{B}$. For $B \in \mathfrak{B}$ let A be the upper $\mathscr{K}$-modification of B; let $\mathfrak{A}$ be the set of these modifications A. Evidently $K = \bigvee_P \mathfrak{B} \leqq \bigvee_P \mathfrak{A} \leqq \bigvee_{\mathscr{K}} \mathfrak{A} \leqq K$. Thus $K = \bigvee_{\mathscr{K}} \mathfrak{A}$.

1.8. In what follows we shall need some known concepts definitions of which will be introduced now for convenience of the reader (see e.g. [1], [6]).

The closure operation on a partially ordered set L is a mapping $\lambda : L \to L$ with the following properties: 1) $a \leqq \lambda a$ $(a \in L)$, 2) $a \leqq b \Rightarrow \lambda a \leqq \lambda b$, 3) $\lambda\lambda a = \lambda a$ $(a \in L)$, 4) $\lambda 0 = 0$ (provided 0 exists). The set of all compact elements of L will be denoted by L^*. The closure operation λ of L will be called *algebraic* if every $a \in L^*$ satisfies the following condition: If $a \leqq \lambda x$ then there exists $x' \in L^*$ with $x' \leqq x$ and $a \leqq \lambda x'$.

1.9 ([6] 4.7). *A closure operation λ of an algebraic lattice L is algebraic if and only if it fulfils $\bigvee_L S \in \lambda L$ for every directed subset S of λL.*

1.10. Definition. Let G be a set. Then $\lambda_1 : R(G) \to P(G)$ is defined as follows: $\lambda_1(A)$ is the upper P-modification of $A \in R(G)$. If (G, F) is an algebra we define the mappings $\lambda_2 : P(G) \to \mathscr{K}(G, F)$ and $\lambda_3 : R(G) \to \mathscr{K}(G, F)$ analogously.

1.11. Theorem. *The maps λ_i $(i = 1, 2, 3)$ from Definition 1.10 are algebraic closure operations.*

Proof. It is clear that λ_i $(i = 1, 2, 3)$ is a closure operation. Further, by 1.9, it is enough to fulfil the condition $\bigcup \mathfrak{A} \in P(G)$ (as for λ_1) or $\bigcup \mathfrak{A} \in \mathscr{K}(G, F)$ (as for λ_2 and λ_3), for an arbitrary directed subset $\mathfrak{A}$ of $P(G)$ (as for λ_1) or of $\mathscr{K}(G, F)$ (as for λ_2 and λ_3), respectively.

λ_1: Let $\mathfrak{A} = \{A_\alpha : \alpha \in I\}$ be a directed subset of $P(G)$. It suffices to prove that $\bigcup_\alpha A_\alpha$ is symmetric and transitive. The first property is evident, the other follows from the fact that for $x, y \in G$ we have $x(\bigcup \mathfrak{A})\, y$ if and only if xAy for some $A \in \mathfrak{A}$ (since $\mathfrak{A}$ is directed). The assertions for λ_2 and λ_3 follow from 1.2.

1.12 ([6] 4.3). *If λ is an algebraic closure operation of an algebraic lattice L then λL is again an algebraic lattice, and it holds $\lambda(L^*) = (\lambda L)^*$.*

1.13. *Now, the property to be algebraic for $P(G)$ (G a set) and $\mathscr{K}(G, F)$ ((G, F) an algebra) follows by virtue of* 1.11 *and* 1.12. In fact, $P(G) = \lambda_1 R(G)$ and λ_1 is algebraic by 1.11. Thus by 1.12, $P(G)$ is algebraic. Analogously for $\mathscr{K}(G, F)$ with aid of λ_2 or λ_3.

In the following theorem the characterization of $\mathscr{K}(G, F)^*$ will be completed. Simultaneously, we discover the structure of $P(G)^*$.

1.14. Theorem. *Let G be a set. Compact elements of the lattice $P(G)$ are exactly the upper P-modifications of compact elements of $R(G)$ (i.e. of finite subsets of $G \times G$). Analogously for $\mathscr{K}(G, F)$ if (G, F) is an algebra.*

Also, compact elements of $P(G)$ (G a set) are exactly the finite partitions whose blocks are finite sets and if (G, F) is an algebra then compact elements of $\mathscr{K}(G, F)$ are exactly the upper $\mathscr{K}$-modifications of compact elements of $P(G)$.

Proof. According to 1.12, the first assertion follows from the fact that λ_1 and λ_3 are algebraic (1.11) and that the compact elements of $R(G)$ are precisely the finite subsets of $G \times G$ (1.3).

To obtain the other description of compact elements of $P(G)$ it suffices to verify that the upper P-modification B of a finite relation A in G is finite again. It holds $A \subseteq C \times C$, where $C = \bigcup A \cup \bigcup A^{-1}$, so that $B \subseteq C \times C$ (as $C \times C$ is a partition in G) and $C \times C$ is finite.

The last assertion follows from 1.12 since λ_2 is algebraic (1.11).

2. DETERMINATION OF THE UPPER $\mathscr{K}$-MODIFICATION OF AN ARBITRARY BINARY RELATION IN A PARTIAL ALGEBRA

The aim of this section is the determination of the upper $\mathscr{K}$-modification Ψ_A of an arbitrary relation A in a partial algebra (G, F). The construction is similar to that of the upper $\mathscr{C}$-modification Θ_A of A given in [6] 5.3 and 5.4. It is identical with it if we replace the algebra (G, F) in the construction of Θ_A by its subalgebra $(\bigcup\Psi_A, F)$ (2.14). Therefore we need to know the set $\bigcup \Psi_A$; this is established in 2.11.

2.1. Definition. (See [6] 2 and 5) *Let (G, F) be a partial algebra and X a non-empty set. For every pair of positive integers i, n ($i \leqq n$) we define the n-ary operation $e^{n,i}(x_1, \ldots, x_n)$ on G by*

$$e^{n,i}(a_1, \ldots, a_n) = a_i \quad \text{for all} \quad a_1, \ldots, a_n \in G .$$

Further, we put $F^* = F \cup \{e^{n,i}\}_{n,i}$.

If $w = w(x_1, \ldots, x_n)$ is a word over X generated by F^* and if we substitute k ($0 \leqq k \leqq n$) of its variables (e.g. $x_{n-k+1}, \ldots, x_n$) by fixed elements $a_{n-k+1}, \ldots, a_n$ of G then the resulting symbol

$$w(x_1, \ldots, x_{n-k}, a_{n-k+1}, \ldots, a_n) =: p(x_1, \ldots, x_{n-k})$$

defines an $(n - k)$-ary operation in G. It will be called an algebraic function in (G, F). For $k = n - 1$, $p(x)$ is a unary operation which is said to be a unary algebraic function.

2.2. *The symmetric-transitive hull A^T of a binary relation A in a set G is, evidently, $A^T = \bigcup_{n=1}^{\infty} B^n$, where $B = A \cup A^{-1}$.*

2.3. Definition. [6] Let (G, F) be a partial algebra and $A \in R(G)$. We define the following relations in G: A^H, A^F, A^U as follows:

A^H is the set of all $(u, v) \in G \times G$ to which there exist a word $w(x_1, \ldots, x_n)$ generated by F^* and elements $(a_i, b_i) \in A$ $(i = 1, \ldots, n)$ such that $u = w(a_1, \ldots, a_n)$, $v = w(b_1, \ldots, b_n)$.

A^F and A^U is obtained by replacing the term "word" in the above definition of A^H by "an algebraic function" and "a unary algebraic function", respectively.

Remark. *If $A \neq \emptyset$ then A^F is a reflexive relation. (If $a \in G$ and $a_1 A b_1$ then $a = e^{2,2}(a_1, a)\, A^F\, e^{2,2}(b_1, a) = a$.)*

2.4. Proposition. [6] *If S denotes any of the symbols T, H, F and U then the map $\lambda : R(G) \to R(G)$, defined by $\lambda A = A^S$, is a closure operation in $R(G)$.*

2.5. Denote

$$A_0 = A,\ A_1 \doteq A_0^H,\ A_2 = A_1^T,\ A_3 = A_2^H, \ldots, A_{2i} = A_{2i-1}^T,$$

$$A_{2i-1} = A_{2i-2}^H \quad (i = 1, 2, \ldots).$$

Evidently, it holds $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$.

Denote

$$A' = \bigcup_{n=0}^{\infty} A_n .$$

2.6. Definition. Let (G, F) be a partial algebra and $A \in R(G)$. Then Ψ_A and Θ_A, denote the upper $\mathscr{K}$-modification and the upper $\mathscr{C}$-modification of A, respectively.

2.7. Theorem. *If (G, F) is a partial algebra and $A \in R(G)$ then $\Psi_A = A'$.*

Proof. By induction, let us prove $A' \subseteq \Psi_A$. Evidently $A \subseteq \Psi_A$. Now, we shall show $A_{2i-1} \subseteq \Psi_A \Rightarrow A_{2i}$, $A_{2i+1} \subseteq \Psi_A$. The first inclusion is evident because of $A_{2i} = A_{2i-1}^T \subseteq \Psi_A$. Let $A_{2i} \subseteq \Psi_A$ and $(u, v) \in A_{2i+1} = A_{2i}^H$. There exist a word $w(x_1, \ldots, x_n)$ generated by F^* and elements $(a_j, b_j) \in A_{2i}$ $(j = 1, \ldots, n)$ such that $u = w(a_1, \ldots, a_n)$, $v = w(b_1, \ldots, b_n)$. Hence the congruence Ψ_A contains $(u, v) = (w(a_1, \ldots, a_n), w(b_1, \ldots, b_n))$ because of $(a_j, b_j) \in \Psi_A$ $(j = 1, \ldots, n)$. So $A' \subseteq \Psi_A$.

The equality will follow if we prove that A' is a congruence. A' is symmetric since every $(u, v) \in A'$ belongs to A_{2i} $(= A^T_{2i-1})$ for some i and this is symmetric. By a similar argument, A' is transitive. Analogously, A' is stable since A_{2i+1} $(= A^H_{2i})$ – – for all i – is stable.

2.8 ([6] 5.3). *Let (G, F) be a partial algebra and $A \in R(G)$. Then Θ_A is the union of the sequence of relations $A \subseteq A^F \subseteq A^{FT} \subseteq A^{FTF} \subseteq \ldots$.*

2.9. Proposition. *Let A be a congruence in a partial algebra (G, F). Then $(\bigcup A, F)$ is a subalgebra of (G, F) and A is a congruence on $(\bigcup A, F)$.*

Proof. Evidently, A is a partition on the set $\bigcup A$. Let $(a_1, \ldots, a_n) \in D(f, G) \cap \cap (\bigcup A)^n$.*) It is $a_i A a_i$ $(i = 1, \ldots, n)$ hence $f(a_1, \ldots, a_n)\, A\, f(a_1, \ldots, a_n)$ and therefore $f(a_1, \ldots, a_n) \in \bigcup A$.

2.10. Definition [2] 2.3. Let A be a binary relation in a set G and $B \subseteq G$. The intersection of the relation A and the subset B is the relation $B \sqcap A = \{(a, b) \in A: a, b \in B\}$.

2.11. Theorem. *Let (G, F) be a partial algebra and $A \in R(G)$. Then $\bigcup \Psi_A$ is the subalgebra $\langle \bigcup A \cup \bigcup A^{-1} \rangle$ **) of (G, F) generated by the set $\bigcup A \cup \bigcup A^{-1}$.*

Proof. From the symmetry of Ψ_A it follows that $\bigcup A \cup \bigcup A^{-1} \subseteq \Psi_A$ and consequently $\langle \bigcup A \cup \bigcup A^{-1} \rangle \subseteq \bigcup \Psi_A$ by 2.9. Conversely, the intersection $\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A$ is a congruence containing A, hence $\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A \supseteq \supseteq \Psi_A$. The reverse inclusion is evident so that $\bigcup(\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A) = \bigcup \Psi_A$. Thus $\bigcup \Psi_A = \langle \bigcup A \cup \bigcup A^{-1} \rangle \cap \bigcup \Psi_A = \langle \bigcup A \cup \bigcup A^{-1} \rangle$.

2.12 [6] 5.5 and 5.4. *Let (G, F) be an algebra and $A \in R(G)$. Then $A^{UT} = A^{FT}$ and $A^{UT} = A^{UTU}$. Consequently, $\Theta_A = A^{UT}$ if $A \neq \emptyset$.*

2.13. Let (G, F) be a partial algebra, (B, F) a subalgebra of (G, F) and $A \subseteq B \times B$. We need to distinguish the least congruence in (G, F) containing A from the least congruence in (B, F) containing A. We shall denote the latter by $\Psi_A(B)$ and the former by $\Psi_A(G)$. Similarly, we distinguish $\Theta_A(B)$ from $\Theta_A(G)$ and $A^{S(B)}$ from $A^{S(G)}$ for $S = H, F$ and U.

2.14. Theorem. *Let (G, F) be a partial algebra, $A \in R(G)$, and $B = \bigcup \Psi_A(G)$ $(= \langle \bigcup A \cup \bigcup A^{-1} \rangle)$. Then $\Psi_A(G) = \Theta_A(B) = A \cup A^{F(B)} \cup A^{F(B)T} \cup A^{F(B)TF(B)} \cup \ldots$. If (G, F) is an algebra then $\Psi_A(G) = \Theta_A(B) = A^{U(B)T}$.*

Proof follows from 2.8, 2.9 and 2.12.

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*) By $D(f, G)$ the set of all $(a_1, \ldots, a_n) \in G^n$ is denoted for which $f(a_1, \ldots, a_n)$ exists.

**) $\bigcup A = \{y \in G: \exists x \in G,\ yAx\}$, [5] III Df. 3.5.