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UNIVERSAL SIMULTANEOUS APPROXIMATIONS OF THE COEFFICIENT FUNCTIONALS

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Universal approximations, a concept which appeared in numerical mathematics some years ago [1], [2], had resulted from the attempt to avoid problems connected with the choice of the space over which the given functional should be (best) approximated. BABUŠKA and SOBOLEV [3] pointed out that the dependence of the best approximation on the space can have unpleasant numerical consequences. The information at our disposal is usually not sufficient to determine a unique space over which the given functional should be approximated optimally and the conclusions on the advantage of optimum methods are thus "unstable" in practice. This implies the importance of finding approximations the error of which does not differ "too much" from those of the best approximations in a wide class of spaces. Such approximations are then called universal.

In an earlier paper of the author the universal approximations of Fourier coefficients in a particular class of Hilbert spaces were studied [5]. Later, the author announced some results valid for general classes of Hilbert spaces [6]. This paper treats the approximations of the coefficient functionals associated with a basis of Banach spaces and the conclusions of [6] are here contained as a special case. Since the fundamental ideas here are similar to those of [5] we proceed rather briefly in this paper, referring to [5] whenever convenient*).

1. BEST APPROXIMATIONS: SOME LOWER BOUNDS

We shall deal with classes $\mathfrak{B}$ of Banach spaces (B-spaces) E over the field of complex numbers, generated by a common Schauder basis $\{x_j\}$. Let $E \in \mathfrak{B}$ and assume that we want to compute the values $F_j(x)$, $j = 1, 2, \ldots, r$ where

$$x = \sum_{j=1}^{\infty} F_j(x)\, x_j$$

and $r > 1$.

*) The results of this paper were presented at the Third Conference on Basic Problems of Numerical Mathematics (Prague 1973) (cf. [7]).

We shall approximate the vector $[F_j]$ of functionals from E^* by another vector $[G_j]$, $G_j \in E^*$. The approximating functionals are assumed to be of the form

$$G_j(x) = \sum_{k=1}^{n} a_k(x)\, g_k(j)\,, \tag{1.1}$$

where $1 \leqq n < r$, $a_k \in E^*$ and g_k are complex-valued functions of an integer argument j. Thus instead of calculating r values $F_j(x)$ we compute n values $a_k(x)$, $n < r$. The main question we ask in this paper is how to choose the matrix $[g_k(j)]$ ($k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, r$) properly.

For a given n, denote by M_n the set of all the approximations $[G_j]$ where G_j is of the form (1.1). We define the *error of the approximation* as

$$\omega_E([G_j]) = \max_{j=1,2,\ldots,r} \|F_j - G_j\|_{E^*}\,. \tag{1.2}$$

Let $M \subset M_n$ for some n. Then the *best* (or *optimal*) *approximation* from the set M (if it exists) has the error

$$\Omega_E(M) = \inf_{[G_j]\in M} \omega_E([G_j])\,. \tag{1.3}$$

Obviously $\Omega_E(M) \geqq \Omega_E(M_n)$ for any $M \subset M_n$.

A positive lower bound can be derived for $\Omega_E(M_n)$, which is of decisive importance for further considerations.

Theorem 1.1. *Let $E \in \mathfrak{B}$ and choose $n + 1$ integers $j_1, j_2, \ldots, j_{n+1}$ in such a way that $1 \leqq j_s \leqq r$ and $j_s \neq j_t$ whenever $s \neq t$. Then*

$$\Omega_E(M_n) \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\|\right)^{-1}\,. \tag{1.4}$$

Proof. We shall make use of Lemma 4.1 of [5], which is obviously valid also in B-spaces. We reformulate it for the reader's convenience, but without proof.

Lemma 1.1. *Let $E \in \mathfrak{B}$ and denote by θ the zero element of E. If for every $x \in E$ and all approximations $[G_j] \in M_n$*

$$\inf_{\substack{a_k\\ k=1,2,\ldots,n}} \max_{j=1,2,\ldots,r} |F_j(x) - G_j(x)| \geqq C_E(x, g_1, g_2, \ldots, g_n) \tag{1.5}$$

is valid, then

$$\Omega_E(M_n) \geqq \inf_{\substack{g_k\\ k=1,2,\ldots,n}} \sup_{\substack{x\in E\\ x\neq\theta}} \left(\|x\|^{-1} \cdot C_E(x, g_1, g_2, \ldots, g_n)\right)\,. \tag{1.6}$$

Therefore, we need a lower bound for

$$\inf_{\substack{a_k \\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} |F_j(x) - G_j(x)| =$$

$$= \inf_{\substack{a_k \\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} \left|F_j(x) - \sum_{k=1}^{n} a_k(x)\, g_k(j)\right| .$$

This can be found in the same manner as in [5]. We choose $n + 1$ integers j_s, $s = 1, 2, \dots, n + 1$ satisfying the hypotheses of the theorem. Then we compose $n + 1$ n-dimensional vectors

$$[g_1(j_s), g_2(j_s), \dots, g_n(j_s)] , \quad s = 1, 2, \dots, n + 1 .$$

These vectors are linearly dependent and we can find numbers $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ such that

$$\sum_{s=1}^{n+1} \lambda_s\, g_k(j_s) = 0 , \quad k = 1, 2, \dots, n , \tag{1.7}$$

and

$$\sum_{s=1}^{n+1} |\lambda_s| = 1 . \tag{1.8}$$

Every vector $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_{n+1}]$ satisfying (1.7) and (1.8) will be called *determined by the matrix* $[g_k(j_s)]$. For any such $\boldsymbol{\lambda}$ and any $x \in E$ we have

$$\sum_{s=1}^{n+1} \lambda_s (F_{j_s}(x) - G_{j_s}(x)) = \sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x) .$$

In virtue of (1.8) we obtain

$$\max_{s=1,2,\dots,n+1} |F_{j_s}(x) - G_{j_s}(x)| \geqq \left|\sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x)\right|$$

and further

$$\inf_{\substack{a_k \\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} |F_j(x) - G_j(x)| \geqq \left|\sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x)\right| \tag{1.9}$$

for every $x \in E$ and $[G_j] \in M_n$. The right-hand side of (1.9) is independent of a_k's and this bound thus satisfies the hypothesis of Lemma 1.1. In fact, for every $[G_j] \in M_n$ (1.9) generally represents a family of bounds (we can obtain different bounds with different solutions of the problem (1.7), (1.8)). This ambiguity is, however, insignificant and we can avoid it by assigning each $[g_k(j)]$ some fixed vector determined by $[g_k(j_s)]$.

Let $N \subset E$. Using Lemma 1.1 we now get easily

$$(1.10) \qquad \Omega_E(M_n) \geqq \inf_{\substack{g_k \\ k=1,2,\ldots,n}} \sup_{\substack{x \in N \\ x \neq \theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|} \geqq \inf_{\lambda} \sup_{\substack{x \in N \\ x \neq \theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|}$$

where the second infimum is taken over all λ's satisfying (1.8)*). We shall take for N the linear subspace of E spanned by $x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}$. Put

$$\hat{x}(\lambda) = \sum_{s=1}^{n+1} \frac{\bar{\lambda}_s}{\|x_{j_s}\|} x_{j_s} ;$$

obviously $\hat{x}(\lambda) \in N$. For every λ satisfying (1.8) we have

$$(1.11) \qquad \sup_{\substack{x \in N \\ x \neq \theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|} \geqq \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(\hat{x})\right|}{\|\hat{x}\|} = \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|},$$

as $F_{j_s}(\hat{x}) = \bar{\lambda}_s \, . \, \|x_{j_s}\|^{-1}$ and $\|\hat{x}\| \leqq 1$. In view of (1.10) we have thus obtained

$$(1.12) \qquad \Omega_E(M_n) \geqq \inf_{\lambda} \left(\sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|} \right)$$

where the infimum has the same meaning as above.

The proof can now be readily finished. Firstly,

$$(1.13) \qquad \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|} \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\| \right)^{-1},$$

which can be easily verified by the Cauchy inequality, and this lower bound is actually the least one since for

$$\lambda_s = \frac{\|x_{j_s}\|}{\sum_{t=1}^{n+1} \|x_{j_t}\|}$$

(1.13) becomes equality. The theorem is proved.

Proving Theorem 1.1 we have obtained also the following result regarding the set $M_n^g \subset M_n$ of the approximations with a fixed matrix $[g_k(j)]$ (cf. (1.11)).

Theorem 1.2. *Let $E \in \mathfrak{B}$. Given a matrix $[g_k(j)]$, choose integers $j_1, j_2, \ldots, j_{n+1}$ in such a way that $1 \leqq j_s \leqq r$ and $j_s \neq j_t$ whenever $s \neq t$.*

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*) It can be shown (by assigning each λ a fixed $[g_k(j)]$ such that (1.7) holds) that even the equality sign could be written in the second part of (1.10).

Then

$$\Omega_E(M_n^g) \geqq \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|} \tag{1.14}$$

for any $\boldsymbol{\lambda}$ *determined by* $[g_k(j_s)]$.

The above bounds on Ω are generally improvable since e.g. for a special class of Hilbert spaces we obtained in [5]

$$\Omega_H(M_n) \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\|^2\right)^{-1/2}$$

and

$$\Omega_H(M_n^g) \geqq \left(\sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|^2}\right)^{1/2}.$$

Nevertheless, for our further qualitative considerations the bounds of Theorems 1.1 and 1.2 are sufficient.

We can see from Theorem 1.1 that $\Omega_E(M_n) > 0$. Hence, we can form the ratio

$$Q_E(M, [G_j]) = \frac{\omega_E([G_j])}{\Omega_E(M)}$$

and use this ratio to measure the quality of a given approximation $[G_j] \in M \subset M_n$ with respect to the set M.

2. UNIVERSAL APPROXIMATIONS

In [5], we constructed the approximation $[K_j^*]$ which was optimal in a given Hilbert space H_0 from a class $\mathfrak{H}$ and we showed that this approximation can be very "bad" in other spaces from the class $\mathfrak{H}$ considered. Namely, we proved that for any D positive a space $H_D \in \mathfrak{H}$ existed such that $Q_{H_D}(M_n, [K_j^*]) > D$ – even if $Q_{H_0}(M_n, [K_j^*]) = 1$. This effect led us to the introduction of the concept of a universal approximation.

Definition 2.1. An approximation $[G_j] \in M_n$ is said to be *universal* for a given class $\mathfrak{B}$ of B-spaces if there exists a constant D such that

$$Q_E(M_n, [G_j]) \leqq D \tag{2.1}$$

for any $E \in \mathfrak{B}$.

So, in contrast with the optimality, universality is related to some *class* of spaces.

It is clear that for a sufficiently small class $\mathfrak{B}$ (e.g. consisting of only a finite number of spaces E) every approximation from M_n would be universal. On the other hand, for wider classes of spaces a universal approximation need not exist [5]. It is therefore reasonable to search for some (as general as possible) conditions on $\mathfrak{B}$ that would guarantee the existence of a universal approximation. The concept of a conservative class of spaces will play an important role in such conditions.

Definition 2.2. We shall call the class $\mathfrak{B}$ of B-spaces E *conservative*, if the elements of the common basis can be assigned subscripts in such a way that

$$\|x_1\| \leqq \|x_2\| \leqq \ldots \leqq \|x_r\| \tag{2.2}$$

in every $E \in \mathfrak{B}$.

In the remainder of the paper we shall be concerned with conservative classes of B-spaces only and we shall assume that the basis $\{x_j\}$ has been ordered in such a way that (2.2) holds.

We now formulate some conditions on $\mathfrak{B}$ that are sufficient for a universal approximation to exist. We denote by S_n the (continuous) linear operator given by

$$S_n(x) = \sum_{j=1}^{n} F_j(x)\, x_j\,,$$

$(x \in E,\ n = 1, 2, \ldots)$. Further denote

$$v_r(E) = \sup_{1 \leqq n \leqq r} \|S_n\|$$

and

$$v(E) = \sup_{1 \leqq n < \infty} \|S_n\|$$

(the *norm* of the basis $\{x_j\}$).

Theorem 2.1. *Let $\mathfrak{B}$ be a conservative class of B-spaces. If*

$$v_r(E) \leqq K \tag{2.3}$$

for every $E \in \mathfrak{B}$ and K is independent of E, then for each n, $1 \leqq n \leqq r$, there exists an approximation $[B_j] \in M_n$ universal with respect to $\mathfrak{B}$. This approximation is defined by (1.1), where

$$a_k = F_k\,, \quad k = 1, 2, \ldots, n\,, \tag{2.4}$$

$$g_k(j) = \delta_{kj}\,, \quad k = 1, 2, \ldots, n\,, \quad j = 1, 2, \ldots, r\,.\text{*)}$$

Moreover,

$$Q_E(M_n, [B_j]) \leqq 2K(n + 1) \tag{2.5}$$

in every $E \in \mathfrak{B}$.

*) $\delta_{kj} = 0$ unless $k = j$, in which case $\delta_{kj} = 1$.

Proof. The error of $[B_j]$ is

$$\omega_E([B_j]) = \max_{j=n+1,\ldots,r} \|F_j\|_{E^*} \equiv \|F_q\|_{E^*} \tag{2.6}$$

where $n+1 \leqq q \leqq r$. Using Theorem 1.1 with $j_s = s$, $s = 1, 2, \ldots, n+1$ we obtain

$$Q_E(M_n, [B_j]) \leqq \|F_q\|_{E^*} \cdot \sum_{s=1}^{n+1} \|x_s\| \, . \tag{2.7}$$

We need to estimate $\|F_q\|_{E^*}$ by means of $\|x_q\|^{-1}$. It will prove sufficient to proceed in a very simple manner. We write

$$|F_q(x)| = \frac{\|F_q(x)\, x_q\|}{\|x_q\|} \, .$$

This yields

$$|F_q(x)| \leqq \frac{\|\sum_{s=1}^{q} F_s(x)\, x_s\| + \|\sum_{s=1}^{q-1} F_s(x)\, x_s\|}{\|x_q\|} \leqq \frac{2\, v_q(E)\, \|x\|}{\|x_q\|} \leqq \frac{2K}{\|x_q\|} \cdot \|x\| \, .$$

Hence,

$$\|F_q\|_{E^*} \leqq \frac{2K}{\|x_q\|} \, . \tag{2.8}$$

From (2.7) and (2.8) we now get

$$Q_E(M_n, [B_j]) \leqq 2K \sum_{s=1}^{n+1} \frac{\|x_s\|}{\|x_q\|} \, .$$

Since $\mathfrak{B}$ is conservative and $q \geqq n+1$, we have $\|x_s\| \leqq \|x_q\|$, $s = 1, 2, \ldots, n+1$, and

$$Q_E(M_n, [B_j]) \leqq 2K(n+1) \, ,$$

which completes the proof.

Remark 2.1. It can be seen from the proof of Theorem 2.1 that we could assume $\|x_j\| \leqq \|x_k\|$ whenever $1 \leqq j \leqq n+1 \leqq k \leqq r$ instead of the conservativeness to obtain the same result for a fixed n.

Remark 2.2. A basis $\{x_j\}$ of a B-space E is said to be *monotone* if we have

$$\|\sum_{i=1}^{n} \alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m} \alpha_i x_i\|$$

for all finite sequences of complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+m}$. Monotone bases satisfy the condition (2.3) of Theorem 2.1 trivially since their norm is $v(E) = 1$ in any B-space [8]. In Hilbert spaces, monotonicity is equivalent to orthogonality.

We now give some examples of classes $\mathfrak{B}$ satisfying the hypotheses of Theorem 2.1.

Example 2.1. *The spaces $L^p([0, 1])$ and the Haar functions.* It is well-known [8] that the sequence of equivalence classes $\{\tilde{y}_j\}$, where y_j are the Haar functions, i.e. the functions defined on $[0, 1]$ by

$$y_1(t) \equiv 1\,, \tag{2.9}$$

$$y_{2^k+l}(t) = \begin{cases} \sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-2}{2^{k+1}}, \dfrac{2l-1}{2^{k+1}}\right), \\ -\sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-1}{2^{k+1}}, \dfrac{2l}{2^{k+1}}\right), \\ 0 & \text{for the other} \quad t\,, \end{cases}$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$ constitutes a basis of the space $L^p([0, 1])$ $(p \geqq 1)$. Further, it may be shown that this basis is monotone [8]. An easy computation yields

$$\|\tilde{y}_1\|_p = 1\,, \quad \|\tilde{y}_{2^k+l}\|_p = (2^{(p-2)/2p})^k\,, \tag{2.10}$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$ where $\|\cdot\|_p$ denotes the norm in $L^p([0, 1])$. From (2.10) we see that the class $\mathfrak{B}_1$ of the spaces $L^p([0, 1])$, $p \geqq 2$, with the Haar basis is conservative. Hence, $\mathfrak{B}_1$ satisfies the assumptions of Theorem 2.1 with $K = 1$.

Example 2.2. *General separable Orlicz spaces with the Haar basis.* Let $M(u)$ be an even convex continuous function defined on $(-\infty, +\infty)$ with the following properties:

(2.11) a) $\displaystyle\lim_{n\to 0} \frac{M(u)}{u} = 0\,,$

b) $\displaystyle\lim_{n\to\infty} \frac{M(u)}{u} = \infty\,,$

c) there exist constants $k > 0$, $u_0 \geqq 0$ such that $M(2u) \leqq k\, M(u)$ for $u \geqq u_0$.

The general separable Orlicz space*) $L_M([0, 1])$ is then the space of the equivalence classes $\tilde{u}$ given by real-valued functions $u(t)$ defined on $[0, 1]$ for which

$$\int_0^1 M(u(t))\, \mathrm{d}t < \infty\,. \tag{2.12}$$

*) Proofs of the properties of Orlicz spaces and functions $M(u)$ used in this example can be found e.g. in [4].

The norm $\|\tilde{u}\|_M$ can be introduced by the relation

$$\int_0^1 M\left[\frac{u(t)}{\|\tilde{u}\|_M}\right] \mathrm{d}t = 1\,. \tag{2.13}$$

It can be proved that $L_M([0, 1])$ with this norm is a separable B-space and, moreover, the equivalence classes $\{\tilde{y}_j\}$ where $y_j(t)$ are the Haar functions defined by (2.9) constitute a monotone basis of $L_M([0, 1])$ [9].

For example, $M(u) = |u|^p$ $(p > 1)$ satisfies (2.11) and this choice yields $L_M([0, 1]) = L^p([0, 1])$. Another possible choice of $M(u)$ is

$$M(u) = |u|^p\,(|\ln|u|| + 1) \tag{2.14}$$

with $p > 1$; the resulting Orlicz spaces are different from the L^p-spaces.

According to (2.13), the norms of the Haar functions are

$$\|\tilde{y}_1\|_M = \frac{1}{M^{-1}(1)}\,, \tag{2.15}$$

$$\|\tilde{y}_{2^k+l}\|_M = \frac{\sqrt{2^k}}{M^{-1}(2^k)}\,,$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$, where $M^{-1}(v)$ is the inverse function for the function $M(u)$ considered on $[0, +\infty)$. (It may be shown that every $M(u)$ that satisfies (2.11) is increasing on $[0, +\infty)$.)

Let $\mathfrak{B}_2$ be the class of separable Orlicz spaces $L_M([0, 1])$ whose $M(u)$ satisfy

$$M(u\sqrt{2}) \geqq 2\,M(u) \tag{2.16}$$

for $u \geqq M^{-1}(1)$. The class $\mathfrak{B}_2$ contains e.g. the spaces $L^p([0, 1])$ for $p \geqq 2$ and the spaces $L_M([0, 1])$ with $M(u)$ given by (2.14) for $p \geqq 2$.

We now show that $\mathfrak{B}_2$ with the Haar basis is conservative. It is sufficient to prove that (2.16) implies

$$2^{1/2}\,M^{-1}(v) \geqq M^{-1}(2v) \tag{2.17}$$

for $v \geqq 1$. Denote $v = M(u)$. We can write (2.16) as $2v \leqq M(u\sqrt{2})$. Since $M(u)$ is increasing for $u \geqq 0$, $M^{-1}(v)$ is increasing for $v \geqq 0$ and we have

$$M^{-1}(2v) \leqq u\sqrt{2} = 2^{1/2}\,M^{-1}(v)\,,$$

which is (2.17). (2.17) yields the conservativeness immediately.

We recall that the Haar basis is monotone and conclude that the class $\mathfrak{B}_2$ satisfies the assumptions of Theorem 2.1 with $K = 1$.

It is a priori clear that e.g. in the case described by Theorem 2.1 more than one universal approximation exist. For example, the approximation $[\tilde{B}_j]$ with the same $[g_k(j)]$ as $[B_j]$ and $a_k = F_k + c_k F_{q_k}$, $q_k \geqq n + 1$, c_k arbitrary complex numbers, $k = 1, 2, \ldots, n$, is also universal with respect to the class $\mathfrak{B}$ described in the above theorem. It is reasonable, therefore, to search for the universal approximations with minimum error.

To be able to do this we need a characterization of the set $U_n \subset M_n$ of all the approximations universal with respect to a given class $\mathfrak{B}$. We shall describe U_n by means of some conditions on $[g_k(j)]$ which the universal approximations satisfy necessarily. Such results are also of interest in answering the question of the proper choice of $[g_k(j)]$. In order to find the necessary properties of matrices $[g_k(j)]$ we must suppose, however, that the class $\mathfrak{B}$ considered is sufficiently wide.

Theorem 3.1. *Let $\mathfrak{B}$ be a conservative class of B-spaces. Let $v_r(E) \leqq K$ for every $E \in \mathfrak{B}$ (K independent of E) and let n be an integer, $1 \leqq n \leqq r$, such that for any D there exists a space $E_D \in \mathfrak{B}$ in which*

$$\frac{\|x_{n+1}\|_{E_D}}{\|x_n\|_{E_D}} > D\,. \tag{3.1}$$

Then the matrices $[g_k(j)]$ of a universal approximation $[G_j] \in U_n$ have the following two properties:

$$\text{(3.2)} \qquad a)\ g_k(j) = 0\,, \quad k = 1, 2, \ldots, n\,, \quad j = n + 1, \ldots, r\,,$$
$$b)\ \operatorname{rank}\left([g_k(j)]_{k,j=1}^{n}\right) = n\,.$$

Proof is exactly parallel to that of Theorem 5.6 in [5] and will be only sketched.

For every s such that $n + 1 \leqq s \leqq r$ denote by $[g_k(j)]_s$ the $n \times (n + 1)$ submatrix of $[g_k(j)]$ consisting of the columns $1, 2, \ldots, n, s$. We shall investigate the solutions $\boldsymbol{\lambda}^{(s)} = [\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)}]^T$ of

$$[g_k(j)]_s\, \boldsymbol{\lambda}^{(s)} = 0 \tag{3.3}$$

satisfying

$$\sum_{j=1}^{n+1} |\lambda_j^{(s)}| = 1\,. \tag{3.4}$$

Denote $|\boldsymbol{\lambda}^{(s)}| = [|\lambda_1^{(s)}|, |\lambda_2^{(s)}|, \ldots, |\lambda_{n+1}^{(s)}|]^T$ and let $\mathbf{e}_k$ be the k-th unit vector.

The proof is based on Lemma 5.1 of [5], which we present in a somewhat modified form:

Lemma 3.1. *The conditions (3.2) are equivalent to the following statement:*

There exists a unique system of vectors $\{|\boldsymbol{\lambda}^{(s)}|\}_{s=n+1}^{r}$ such that $\boldsymbol{\lambda}^{(s)}$, $s = n + 1, n + 2, \ldots, r$, satisfy (3.3) and (3.4), namely $|\boldsymbol{\lambda}^{(n+1)}| = |\boldsymbol{\lambda}^{(n+2)}| = \ldots = |\boldsymbol{\lambda}^{(r)}| = \mathbf{e}_{n+1}$.

The proof of the theorem is by contradiction. If the conditions (3.2) are violated, then using Lemma 3.1 we conclude that for some s, $n+1 \leqq s \leqq r$, there exists a vector $\boldsymbol{\lambda}^{(s)}$ satisfying (3.3) and (3.4) whose p-th component, $1 \leqq p \leqq n$, is not zero. According to Theorem 1.2, we have for the approximation $[G_j]$ violating (3.2)

$$\omega([G_j]) \geqq \sum_{j=1}^{n} \frac{|\lambda_j^{(s)}|^2}{\|x_j\|} + \frac{|\lambda_{n+1}^{(s)}|^2}{\|x_s\|} \geqq \frac{|\lambda_p^{(s)}|^2}{\|x_p\|}. \tag{3.5}$$

To complete the proof we need an appropriate upper bound for $\Omega(M_n)$. It is sufficient to make use of the trivial fact that $\Omega(M_n) \leqq \omega([I_j])$ for any approximation $[I_j] \in M_n$. Choosing for $[I_j]$ the approximation $[B_j]$ from Theorem 2.1 and using (2.8) we obtain

$$\Omega(M_n) \leqq \frac{2K}{\|x_q\|}, \tag{3.6}$$

where $n+1 \leqq q \leqq r$. (3.5) and (3.6) now yield for the approximation $[G_j]$

$$Q(M_n, [G_j]) \geqq \frac{|\lambda_p^{(s)}|^2}{2K} \cdot \frac{\|x_{n+1}\|}{\|x_n\|}$$

and, in view of (3.1), $[G_j]$ is not universal.

The classes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ from Examples 2.1 and 2.2 do not satisfy (3.1). It is easy, however, to construct classes of Hilbert spaces with orthogonal bases [5], [6] satisfying the assumptions of Theorem 3.1. The strongly periodic spaces described in [5] may serve as an example.

Theorem 1.2 and Lemma 3.1 imply immediately that the error of an optimal approximation from U_n is bounded by

$$\Omega(U_n) \geqq \frac{1}{\|x_{n+1}\|} \tag{3.7}$$

in all spaces satisfying the assumptions of Theorem 3.1.

Let us consider again the approximation $[B_j]$ from Theorem 2.1. This approximation belongs to U_n and we can compare its error with $\Omega(U_n)$.

Theorem 3.2. *Let $\mathfrak{B}$ be a conservative class of B-spaces. Let $v_r(E) \leqq K$ for every $E \in \mathfrak{B}$ and let n be an integer, $1 \leqq n \leqq r$, such that for any D there exists a space $E_D \in \mathfrak{B}$ in which* (3.1) *holds. Then*

$$Q_E(U_n, [B_j]) \leqq 2K \tag{3.8}$$

in every $E \in \mathfrak{B}$.

If, moreover, the basis $\{x_j\}$ is orthogonal); then $[B_j]$ is an optimal universal approximation in every $E \in \mathfrak{B}$, i.e. $Q_E(U_n, [B_j]) = 1$ in every $E \in \mathfrak{B}$.*

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*) A basis $\{x_j\}$ of E is orthogonal, if every permutation of $\{x_j\}$ is a monotone basis of E. In Hilbert spaces this is the orthogonality in the usual sense [8].