Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON HADAMARD'S CONCEPTS OF CORRECTNESS

MIROSLAV SOVA, Praha

(Received January 21, 1976)

In the present paper, we first continue in Section 2 the study of well-posedness or correctness of the Duhamel initial value problem in the sense as introduced in [1]. In Section 3, a weakened form of correctness, called here Hadamardian correctness, is newly introduced and studied. It is characterised by the fact that the continuous dependence of solutions on the initial values is omitted, so that the Hadamardian correctness becomes of almost algebraic character. The main results concern the relations between correctness and Hadamardian correctness in Banach spaces. Finally, in Section 5, we obtain the equivalence between these both notions, naturally only under strong restrictions, i.e. for a special system of coefficient operators in Hilbert spaces.

In the text, we use the notation and definitions introduced in [1]. In particular, it is necessary to be acquainted with the points 1.10, 5.1 – 5.3, 7.1, 7.4 and 7.7 of [1]. Moreover, we need some results of [1], which will be quoted when necessary.

1. PRELIMINARIES

1.1 The complex number field will be denoted by C.

1.2 Lemma. *Let* $\varphi, \psi, \chi \in R^+ \to R$. *If the function* φ *is continuous on* R^+ *and bounded on* $(0, 1)$, *the functions* ψ, χ *are nondecreasing and*

$$|\varphi(t)| \leqq \psi(t) + \chi(t) \int_0^t |\varphi(\tau)| \, d\tau \quad \text{for every} \quad t \in R^+ ,$$

then

$$|\varphi(t)| \leqq \psi(t) \, e^{t \chi(t)} \quad \text{for every} \quad t \in R^+ .$$

Proof. See [3], p. 19.

1.3 By a Fréchet space F we mean a metrizable complete linear topological convex space.

1.4 Lemma. *Let F_1, F_2 be two Fréchet spaces and T a linear transformation from $F_1 \to F_2$. If the transformation T is closed, then it is continuous.*

2. BASIC NOTIONS AND RESULTS

The notions of definiteness, extensiveness and correctness are introduced or recapitulated and some of their properties, needed in the sequel, are discussed. This part should be regarded as a completion and extension of the paper [1].

2.1 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots \ldots, A_n$ will be called definite if every null solution for the operators $A_1, A_2, \ldots, A_n$ is identically zero.

2.2 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots \ldots, A_n$ belong to $L(E)$, then the system $A_1, A_2, \ldots, A_n$ is definite.*

Proof. Let u be an arbitrary null solution for the operators $A_1, A_2, \ldots, A_n$. By [1] 5.6

(1)
$$u^{(n-1)}(t) + A_1 \int_0^t u^{(n-1)}(\tau)\, d\tau + \ldots$$
$$\ldots + \frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} u^{(n-1)}(\tau)\, d\tau = 0 \quad \text{for every} \quad t \in R^+ .$$

Let us denote

(2)
$$K = \max\left(\|A_1\|, \|A_2\|, \ldots, \|A_n\|\right).$$

It follows from (1) and (2) that

(3)
$$\|u^{(n-1)}(t)\| \leqq K\left(\int_0^t \|u^{(n-1)}(\tau)\|\, d\tau + t \int_0^t \|u^{n-1}(\tau)\|\, d\tau + \ldots\right.$$
$$\left.\ldots + \frac{t^{n-1}}{(n-1)!} \int_0^t \|u^{n-1}(\tau)\|\, d\tau\right) \quad \text{for every} \quad t \in R^+ .$$

We can rewrite (3) in the form

(4)
$$\|u^{(n-1)}(t)\| \leqq K e^t \int_0^t \|u^{(n-1)}(\tau)\|\, d\tau \quad \text{for every} \quad t \in R^+ .$$

Using 1.2, we obtain from (4) that $u^{(n-1)}(t) = 0$ for every $t \in R^+$ which implies according to [1] 2.10 that $u(t) = 0$ for every $t \in R^+$.

The proof is complete.

2.3 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are closed and if there exists a sequence P_k, $k \in \{1, 2, \ldots\}$, of operators from $L(E)$ such that*

(α) *$P_k^2 = P_k$ for every $k \in \{1, 2, \ldots\}$,*

(β) *$P_k x \to x$ $(k \to \infty)$ for every $x \in E$,*

(γ) *$P_k x \in D(A_i)$ for every $x \in E$, $k \in \{1, 2, \ldots\}$ and $i \in \{1, 2, \ldots, n\}$,*

(δ) *$P_k A_i x = A_i P_k x$ for every $k \in \{1, 2, \ldots\}$, $i \in \{1, 2, \ldots, n\}$ and $x \in D(A_i)$,*

then the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Proof. Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, we see from (γ) by virtue of [1] 1.11 that

(1) $A_i P_k \in L(E)$ for every $k \in \{1, 2, \ldots\}$ and $i \in \{1, 2, \ldots, n\}$.

Let now u be an arbitrary null solution for the operators $A_1, A_2, \ldots, A_n$.

Let us denote $u_k(t) = P_k\, u(t)$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

It follows without difficulty from (α), (γ) and (δ) that

(2) for every $k \in \{1, 2, \ldots\}$, u_k is a null solution for the operators $A_1 P_k, A_2 P_k, \ldots$ $\ldots, A_n P_k$.

Using now 2.2 we obtain from (1) and (2) that

(3) $u_k(t) = 0$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

On the other hand, it follows from (β) that

(4) $$u_k(t) \to u(t) \quad (k \to \infty) \quad \text{for every} \quad t \in R^+ .$$

It follows from (3) and (4) that $u(t) = 0$ for every $t \in R^+$ which was to be proved.

2.4 Remark. A different criterion of definiteness (of spectral type) was given in [1] 7.3.

2.5 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are everywhere defined and bounded, then for every $x \in E$ there exists a Duhamel solution u such that $u^{(n-1)}(0_+) = x$ and for every $t \in R^+$*

$$\|u(t)\| \leqq (1 + \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|)) \frac{t^{n-1}}{(n-1)!}$$
$$[\exp(1 + \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|))t] \|x\| .$$

Proof. Let us denote

(1) $$K = \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|) .$$

Further, let us choose a fixed $x \in E$ and let us put for $t \in R^+$

(2) $$g(t) = A_1 x + t\, A_2 x + \ldots + \frac{t^{n-1}}{(n-1)!} A_n x .$$

Obviously, by (1) and (2),

(3) $$\|g(t)\| \leqq K\, e^t \|x\| \quad \text{for every} \quad t \in R^+ .$$

Let us now denote by $\mathbf{C}$ the set of all functions $v \in R^+ \to E$ which are continuous on R^+ and bounded on $(0, 1)$.

It is clear from (2) that

(4) $$g \in \mathbf{C} .$$

Further, let us take for $w \in \mathbf{C}$ and $t \in R^+$

(5) $$T\,w(t) = A_1 \int_0^t w(\tau)\,\mathrm{d}\tau + A_2 \int_0^t (t - \tau)\, w(\tau)\,\mathrm{d}\tau + \dots$$
$$\dots + \frac{1}{(n-1)!} A_n \int_0^t (t - \tau)^{n-1}\, w(\tau)\,\mathrm{d}\tau .$$

It is clear from (5) that

(6) T transforms $\mathbf{C}$ into itself.

Further, we see without difficulty that

(7) if $w_k \in \mathbf{C}$, $k \in \{1, 2, \dots\}$, $w \in R^+ \to E$ and $w_k \to w$ $(k \to \infty)$ uniformly on bounded subsets of R^+, then $w \in \mathbf{C}$ and $Tw_k \to Tw$ $(k \to \infty)$ uniformly on bounded subsets of R^+.

On the other hand, if follows from (1) and (5) that

(8) $$\|T\,w(t)\| \leqq Kt \sup_{0<\tau\leqq t} \|w(\tau)\| \quad \text{for every} \quad w \in \mathbf{C} \quad \text{and} \quad t \in R^+ .$$

By induction in virtue of [1] 1.8 and [1] 2.9 we obtain immediately from (8) that

(9) $$\|T^k\, w(t)\| \leqq \frac{K^k t^k}{k!} \sup_{0<\tau\leqq t} \|w(\tau)\|$$
$$\text{for every} \quad w \in \mathbf{C}, \quad t \in R^+ \quad \text{and} \quad k \in \{0, 1, \dots\} .$$

It follows from (9) that

(10) $\sum_{k=0}^{\infty} (-T)^k\, w$ converges uniformly on bounded subsets of R^+ for every $w \in \mathbf{C}$.

Let us now write

(11) $$v = -\sum_{k=0}^{\infty} (-T)^k\, g .$$

It follows easily from (3), (6), (7) and (9) that

(12) $$v \in \mathbf{C} ,$$

(13) $$\|v(t)\| \leqq K\, e^{(K+1)t}\|x\| \quad \text{for every} \quad t \in R^+ ,$$

(14) $$v + Tv = -g .$$

According to (2) and (5) we can write (14) in the form

(15) $$v(t) + A_1 \int_0^t v(\tau)\, d\tau + \ldots + \frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau =$$
$$= -\left[A_1 x + t\, A_2 x + \ldots + \frac{t^{n-1}}{(n-1)!} A_n x \right]$$

for every $t \in R^+$.

Let us now define for $t \in R^+$

(16) $$u(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau + \frac{t^{n-1}}{(n-1)!} x .$$

It follows from (12), (13) and (16) by means of [1] 1.7 and [1] 2.8 that

(17) the function u is n-times differentiable on R^+,

(18) $$u^{(n)} = v ,$$

(19) $$u(0_+) = u(0_+) = \ldots = u^{(n-2)}(0_+) = 0 , \quad u^{(n-1)}(0_+) = x ,$$

(20) $$u^{(n)}(t) + A_1\, u^{(n-1)}(t) + \ldots + A_n\, u(t) = v(t) + \left[A_1 \int_0^t v(\tau)\, d\tau + A_1 x \right] + \ldots$$
$$\ldots + \left[\frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau + \frac{t^{n-1}}{(n-1)!} A_n x \right] \quad \text{for every} \quad t \in R^+ ,$$

(21) $$\|u(t)\| \leqq (K+1) \frac{t^{n-1}}{(n-1)!} e^{(K+1)t} \|x\| \quad \text{for every} \quad t \in R^+ .$$

By (12) and (18) we conclude that

(22) the function $u^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$.

Further, by (15) and (20)

(23) $$u^{(n)}(t) + A_1\, u^{(n-1)}(t) + \ldots + A_n\, u(t) = 0 \quad \text{for every} \quad t \in R^+ .$$

Since $x \in E$ was chosen arbitrarily, we see that the statement of our theorem is, with regard to [1] 5.1, contained in (1), (17), (19) and (21)–(23).

2.6 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ will be called extensive if there exists a subset $Z \subseteq E$ dense in $\overline{D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)}$, such that for every $x \in Z$, we can find a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ so that $u^{(n-1)}(0_+) = x$.

2.7 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ belong to $L(E)$, then the system $A_1, A_2, \ldots, A_n$ is extensive.*

Proof. An immediate consequence of 2.5.

2.8 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are closed and if there exists a set $\mathfrak{P}$ of operators from $L(E)$ such that*

(α) *$P^2 = P$ for every $P \in \mathfrak{P}$,*

(β) *the closure of the set $\{Px : P \in \mathfrak{P}, x \in E\}$ contains $D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$,*

(γ) *$Px \in D(A_i)$ for every $P \in \mathfrak{P}$, $x \in E$ and $i \in \{1, 2, \ldots, n\}$,*

(δ) *$PA_i x = A_i P x$ for every $P \in \mathfrak{P}$, $i \in \{1, 2, \ldots, n\}$ and $x \in D(A_i)$,*

then the system of operators $A_1, A_2, \ldots, A_n$ is extensive.

Proof. Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, we see from (γ) by virtue of [1] 1.11 that

(1) $$A_i P \in L(E) \quad \text{for every} \quad P \in \mathfrak{P} \quad \text{and} \quad i \in \{1, 2, \ldots, n\}\,.$$

Using 2.5 we obtain from (1) that

(2) for every $x \in E$ and $P \in \mathfrak{P}$, there exists a Duhamel solution v_P for the operators $A_1 P, A_2 P, \ldots, A_n P$ such that $v_P^{(n-1)}(0_+) = x$.

Let us now define for $P \in \mathfrak{P}$

(3) $$u_P = P v_P\,.$$

It follows easily from (α), (γ) and (8) that

(4) for every $x \in E$ and $P \in \mathfrak{P}$, the function u_P is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u_P^{(n-1)}(0_+) = Px$.

The extensiveness of the system of operators $A_1, A_2, \ldots, A_n$ follows from (β) and (4).

2.9 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ will be called subcorrect of class m if

(A) it is extensive,

(B) there exist two nonnegative constants M, ω such that for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t - \tau)^m A_i u^{(n-i)}(\tau)\, d\tau \right\| \leq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\|\,.$$

2.10 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots$ $\ldots, A_n$ will be called subcorrect if there exists an $m \in \{0, 1, \ldots\}$ so that the system $A_1, A_2, \ldots, A_n$ is subcorrect of class m.

2.11 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$ and $m \in \{0, 1, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ is correct of class m[correct] if and only if it is subcorrect of class m[subcorrect] and the set $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ is dense in E.*

2.12 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect, then it is also definite.*

Proof. Use 2.9 (B).

2.13 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *the set $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ is dense in E,*

(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect of class m,*

then there exists a $\mathscr{W} \in R^+ \times E \to E$ such that

(a) *for every $x \in E$, the function $\mathscr{W}(., x)$ is continuous on R^+ and*

$$\frac{m!}{t^m} \mathscr{W}(t, x) \xrightarrow[t \to 0_+]{} x ,$$

(b) $\int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x) \, d\tau \in \mathsf{D}(A_i)$ *for every $x \in E$, $t \in R^+$ and $\tau \in \{1, 2, \ldots, n\}$,*

(c) *for every $x \in E$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x) \, d\tau$ is continuous on R^+ and bounded on (0, 1),*

(d)
$$\mathscr{W}(t, x) + A_1 \int_0^t \mathscr{W}(\tau, x) \, d\tau + A_2 \int_0^t (t - \tau) \mathscr{W}(\tau, x) \, d\tau + \ldots$$
$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} \mathscr{W}(\tau, x) \, d\tau = \frac{t^m}{m!} x \quad \textit{for every} \quad x \in E \quad \textit{and} \quad t \in R^+ ,$$

(e) *for every $t \in R^+$, the function $\mathscr{W}(t, \cdot)$ is a linear mapping,*

(f) *there exists two nonnegative constants M, ω so that for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$*

$$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x) \, d\tau \right\| \leq M e^{\omega t} \|x\| .$$

Proof. It follows immediately from 2.12 that

(1) the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Further, we can choose a dense linear subset $Z \subseteq E$ and two nonnegative constants M, ω so that

(2) for every $x \in Z$, there exists a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$,

(3) for every Duhamel solution u for operators $A_1, A_2, \ldots, A_n$, for every $t \in R^+$ and $i \in \{1, 2, \ldots\}$

$$\left\| \frac{1}{m!} \int_0^t (t - \tau)^m A_i u^{(n-i)}(\tau) \right\| \leqq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\| .$$

Now we see easily from the assumptions and from (1)–(3) that the hypotheses of [1] 7.10 and [1] 7.11 are fulfilled. Hence the assertion of our theorem easily follows.

2.14 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ *and* $\mathscr{W} \in R^+ \times E \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the conditions* 2.13 (a)–(d) *are fulfilled,*

then for every $l \in \{0, 1, \ldots\}$

(a) *for every* $x \in E$, *the function* $(\mathrm{d}/\mathrm{d}t) \int_0^t (t - \tau)^l \mathscr{W}(\tau, x) \,\mathrm{d}\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(b) $\int_0^t (t - \tau)^{i-1+l} \mathscr{W}(\tau, x) \,\mathrm{d}\tau \in D(A_i)$ *for every* $x \in E$, $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$,

(c) *for every* $x \in E$ *and* $i \in \{1, 2, \ldots, n\}$, *the function* $A_i \int_0^t (t - \tau)^{i-1+l} \mathscr{W}(\tau, x) \,\mathrm{d}\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(d)
$$\frac{1}{l!} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t - \tau)^l \mathscr{W}(\tau, x) \,\mathrm{d}\tau + A_1 \frac{1}{l!} \int_0^t (t - \tau)^l \mathscr{W}(\tau, x) \,\mathrm{d}\tau + $$
$$+ A_2 \frac{1}{(l + 1)!} \int_0^t (t - \tau)^{l+1} \mathscr{W}(\tau, x) \,\mathrm{d}\tau + \ldots + A_n \frac{1}{(l + n - 1)!} \int_0^t (t - \tau)^{l+n-1}$$
$$\mathscr{W}(\tau, x) \,\mathrm{d}\tau = \frac{t^{l+m}}{l + m!} x \text{ for every } x \in E \text{ and } t \in R^+,$$

(e) *for every* $t \in R^+$, *the function* $(\mathrm{d}/\mathrm{d}t) \int_0^t (t - \tau)^l \mathscr{W}(\tau, \cdot) \,\mathrm{d}\tau$ *is a linear mapping,*

(f) *there exist two nonnegative constants* M, ω *so that for every* $x \in E$, $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$

$$\left\| A_i \frac{1}{(i - 1 + l)!} \int_0^t (t - \tau)^{i-1+l} \mathscr{W}(\tau, x) \,\mathrm{d}\tau \right\| \leqq M e^{\omega t} \frac{t^l}{l!} \|x\| .$$

Proof. An easy consequence of 2.13 by means of [1] 1.8, [1] 2.4, [1] 2.7 and [1] 2.9.

2.15 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ *and* $\mathscr{W} \in R^+ \times E \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite,*

(γ) *the conditions* 2.13 (a)–(d) *are fulfilled,* *then for every* $x \in \mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ *and for every* $t \in R^+$

$$\mathscr{W}(t, x) + \int_0^t \mathscr{W}(\tau, A_1 x)\, d\tau + \int_0^t (t - \tau)\, \mathscr{W}(\tau, A_2 x)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1}\, \mathscr{W}(\tau, A_n x)\, d\tau = \frac{t^m}{m!}\, x\,.$$

Proof. Let us fix an $x \in \mathsf{D}_1(A_1, A_2, \ldots, A_n)$ and let us put for $t \in R^+$

$$w(t) = \mathscr{W}(t, x) + \int_0^t \mathscr{W}(\tau, A_1 x)\, d\tau + \int_0^t (t - \tau)\, \mathscr{W}(\tau, A_2 x)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1}\, \mathscr{W}(\tau, A_n x)\, d\tau - \frac{t^m}{m!}\, x\,.$$

A simple calculation using conditions 2.13 (a)–(d) and 2.14 (a)–(d) shows that the function w has properties [1] 7.10 (1)–(4). Hence by Lemma [1] 7.10, $w(t) = 0$ for every $t \in R^+$ and this proves our proposition.

2.16 Proposition. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite,*

(γ) *there exists a function* $\mathscr{W} \in R^+ \times E \to E$ *such that* 2.13 (a)–(f) *hold,*

then

(a) *for every* $x \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$, *there exists a Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *so that*

$$u^{(n-1)}(0_+) = x\,,$$

(b) *there exists a nonnegative constant* $\varkappa$ *such that for every Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *satisfying* $u^{(n-1)}(0_+) \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$ *and for every* $i \in \{1, 2, \ldots, n\}$, *the function* $e^{-\varkappa t} A_i\, u^{(n-i)}(t)$ *is bounded on* R^+,

(c) *there exist two nonnegative constants* M, ω *such that for every Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$, *for every* $t \in R^+$ *and every* $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t-\tau)^m A_i\, u^{(n-i)}(\tau)\, d\tau \right\| \leqq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\|.$$

(d) *the set* $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ *is dense in* E.

Proof. For the sake of simplicity we shall write

(1) $$\mathfrak{N} = \{1, 2, \ldots, n\}\,.$$

Further we choose, by assumption (γ), a fixed function $\mathscr{W} \in R^+ \times E \to E$ for which

(2) the conditions 2.13 (a)–(f) are fulfilled.

We begin with proving the assertion (a).

To this aim let us fix an arbitrary $x \in D_{m+1}(A_1, A_2, \ldots, A_n)$ and let us write for $t \in R^+$

(3)
$$u(t) = \frac{t^{n-1}}{(n-1)!} x - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{n-1+\alpha_1}}{(n-1+\alpha_1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{n-1+\alpha_1+\alpha_2}}{(n-1+\alpha_1+\alpha_2)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{n-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(n-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \int_0^t \frac{(t-\tau)^{n-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(n-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} . \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau .$$

By means of [1] 1.8 and [1] 2.8 we obtain easily from (3) that

(4) the function u is n-times differentiable on R^+,

(5)
$$u^{(n-i)}(t) = \frac{t^{i-1}}{(i-1)!} x - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{n+m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \int_0^t \frac{(t-\tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} . \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\} ,$$

(6)
$$u^{(n)}(t) = - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{\alpha_1 - 1}}{(\alpha_1 - 1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{\alpha_1+\alpha_2-1}}{(\alpha_1+\alpha_2-1)!} A_{\alpha_1} A_{\alpha_2} x - \ldots \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_m-1}}{(\alpha_1+\alpha_2+\ldots+\alpha_m-1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \frac{d}{dt} \int_0^t \frac{(t-\tau)^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} . \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ .$$

It follows from (2) and (5) that

(7)
$$u^{(n-1)}(0_+) = x .$$

With regard to the assumptions of our proposition, we see from (2) that Theorem 2.14 may be applied and therefore

(8) the conditions 2.14 (a)–(f) hold for every $l \in \{0, 1, \ldots\}$.

Using the properties 2.14 (b) and (c) with $l = i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1$, $\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}$, we see easily from (5) and (8) that

(9) $$u^{(n-i)}(t) \in D(A_i) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\},$$

(10) the functions $A_i u^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

(11) $$A_i u^{(n-i)}(t) = \frac{t^{i-1}}{(i-1)!} A_i x - \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_i A_{\alpha_1} x + \\ + \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_i A_{\alpha_1} A_{\alpha_2} x - \ldots + \\ \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + \\ + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}} A_i \int_0^t \frac{(t-\tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \cdot \\ \cdot \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Our next objective is to find out that

(12) $$u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0 \quad \text{for every} \quad t \in R^+.$$

To this aim we first consider the terms of the expressions (6) and (11) except the last ones. After a simple calculation we verify that

(13) $$\left[- \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{\alpha_1 - 1}}{(\alpha_1 - 1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{\alpha_1 + \alpha_2 - 1}}{(\alpha_1 + \alpha_2 - 1)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + \right. \\ \left. + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{N}} \frac{t^{\alpha_1 + \alpha_2 + \ldots + \alpha_m - 1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_m - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x \right] + \\ + \left[\sum_{i=1}^n \frac{t^{i-1}}{(i-1)!} A_i x - \sum_{i=1}^n \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_i A_{\alpha_1} x + \right. \\ + \sum_{i=1}^n \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_i A_{\alpha_1} A_{\alpha_2} x - \ldots \\ + \ldots (-1)^{m-1} \sum_{i=1}^n \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m-1}}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m-1})!} \cdot \\ \cdot A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m-1}} x +$$

$$+ (-1)^m \sum_{i=1}^{n} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_m)!} \,.\, A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x \Bigg] =$$

$$= (-1)^m \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x$$

$$\text{for every} \quad t \in R^+ .$$

On the other hand, using the properties 2.14 (b)–(d) with $l = \alpha_1 + \alpha_2 + \ldots \ldots + \alpha_{m+1} - m - 1$, $\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}$, we obtain from (8) that for the last term of (6) and (11) the following identity holds:

$$(14) \qquad \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \left[\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t - \tau)^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1)!} \,.\right.$$

$$. \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \,\mathrm{d}\tau + \sum_{i=1}^{n} A_i \int_0^t \frac{(t - \tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1)!} \,.$$

$$. \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \,\mathrm{d}\tau =$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x \quad \text{for every} \quad t \in R^+ .$$

Now the identity (12) follows at once from (6), (11), (13) and (14).

The above considerations, namely the points (4), (7), (9), (10) and (12), show that

(15) the function u is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$.

Since $x \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$ has been arbitrary, the property (15) shows that

(16) the statement (a) holds.

Let us now turn to the statement (b).

By (8) [2.14 (f)], we can find fixed nonnegative constants M, ω so that

$$(17) \qquad \left\| A_i \frac{1}{(i - 1 + l)!} \int_0^t (t - \tau)^{i-1+l} \mathscr{W}(\tau, x) \,\mathrm{d}\tau \right\| \leq M e^{\omega t} \frac{t^l}{l!} \|x\|$$

for every $x \in E$, $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$.

Let now u be an arbitrary Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

$$(18) \qquad u^{(n-1)}(0_+) \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n) .$$

Using the definiteness assumption, we obtain from (15) and (18) that

(19) the solution u may be expressed by the formula (3) with $x = u^{(n-1)}(0_+)$.

It follows from (11), (17) and (19) that

$$\|A_i u^{(n-i)}(t)\| \leqq \frac{t^{i-1}}{(i-1)!}\|A_i u^{(n-1)}(0_+)\| + \tag{20}$$

$$+ \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!}\|A_i A_{\alpha_1} u^{(n-1)}(0_+)\| +$$

$$+ \sum_{\alpha_1,\alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!}\|A_i A_{\alpha_1} A_{\alpha_2} u^{(n-1)}(0_+)\| + \dots$$

$$\dots + \sum_{\alpha_1,\alpha_2,\dots,\alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\dots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\dots+\alpha_m)!} \cdot \|A_i A_{\alpha_1} A_{\alpha_2} \dots A_{\alpha_m} u^{(n-1)}(0_+)\| +$$

$$+ \sum_{\alpha_1,\alpha_2,\dots,\alpha_{m+1} \in \mathfrak{N}} M e^{\omega t} \frac{t^{\alpha_1+\alpha_2+\dots+\alpha_{m+1}-m-1}}{(\alpha_1+\alpha_2+\dots+\alpha_{m+1}-m-1)!} \cdot$$

$$\cdot \|A_{\alpha_1} A_{\alpha_2} \dots A_{\alpha_{m+1}} u^{(n-1)}(0_+)\| \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \dots, n\}.$$

Let us now choose

$$\varkappa > \omega . \tag{21}$$

Since ω was chosen nonnegative, we obtain immediately from (20) and (21) that (22) the functions $e^{-\varkappa t} A_i u^{(n-i)}(t)$ are bounded on R^+ for every $i \in \{1, 2, \dots, n\}$.

Now an immediate consequence of (22) is, if we take into account the assumption on the solution u, that

(23) the assertion (b) holds.

Now we have to prove the assertion (c).

To this aim, let u be an arbitrary Duhamel solution for the operators $A_1, A_2, \dots, A_n$.

Let us write for $t \in R^+$

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-1)}(\tau)\,\mathrm{d}\tau\right). \tag{24}$$

It follows from [1] 2.9, [1] 5.6 and [1] 5.7 that

(25) the function v is continuous on R^+ and bounded on $(0, 1)$,

(26) $\int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \dots, n\}$,

(27) the functions $A_i \int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \dots, n\}$,

$$v(t) + A_1 \int_0^t v(\tau)\,\mathrm{d}\tau + A_2 \int_0^t (t-\tau)\, v(\tau)\,\mathrm{d}\tau + \dots \tag{28}$$

$$\ldots + A_n \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} v(\tau)\,d\tau = \frac{t^m}{m!} u^{(n-1)}(0_+)$$

for every $t \in R^+$,

(29) $$\frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,d\tau = A_i \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1} v(\tau)\,d\tau$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

It follows from (2) [2.13 (a)–(d)] and (25)–(28) by means of [1] 7.10 that

(30) $$v(t) = \mathscr{W}(t, u^{(n-1)}(0_+)) \quad \text{for every} \quad t \in R^+ .$$

Taking $l = 0$ in (17) we can write

(31) $$\left\| A_i \frac{1}{(i-1)!}\int_0^t (t-\tau)^{i-1} \mathscr{W}(\tau, x)\,d\tau \right\| \leqq M\, e^{\omega t}\|x\|$$

for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Now we obtain from (29)–(31) that

(32) $$\left\| \frac{1}{m!}\int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\,d\tau \right\| \leqq M\, e^{\omega t}\|u^{(n-1)}(0_+)\|$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Since the Duhamel solution u examined above was arbitrary we obtain from (32) that

(33) the assertion (c) holds.

Finally, by (2) and (8), we can apply 2.13 (a) and 2.14 (b) and we easily obtain that

(34) the assertion (d) holds.

According to (16), (23), (33) and (34), the proof is complete.

2.17 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. If*

(α) the operators $A_1, A_2, \ldots, A_n$ are closed,

(β) the set $\mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$ is dense in $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$,

then the following two statements (a) *and* (b) *are equivalent:*

(a) *the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect of class m and the set $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ is dense in E,*

(b) *the system of operators $A_1, A_2, \ldots, A_n$ is definite and there exists a function $\mathscr{W} \in R^+ \times E \to E$ such that the properties* 2.13 (a)–(f) *are fulfilled.*

Proof. An immediate consequence of 2.13 and 2.16. •

3. HADAMARDIAN CONCEPTS

In chapter two of book one of his treatise [2], J. HADAMARD introduced different concepts of correctness for partial differential equations which are mostly very general or too weak. An abstract variant of these concepts (but not so general) is defined and studied in the remaining part of this paper.

3.1 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots$ $\ldots, A_n$ will be called exponentially Hadamardian if

(A) it is definite

(B) there exists a constant $\varkappa$ such that for every $x \in D_\infty(A_1, A_2, \ldots, A_n)$ we can find a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ for which $u^{(n-1)}(0_+) = x$ and the function $e^{-\varkappa t} A_i u^{(n-i)}(t)$ is bounded on R^+ for every $i \in \{1, 2, \ldots, n\}$.

3.2 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. In the sequel, we shall consider the linear space $D_\infty(A_1, A_2, \ldots, A_n)$ as a linear topological space determined by the following system of seminorms:

$$|x|_{\alpha_1, \alpha_2, \ldots, \alpha_d} = \|x\| + \|A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x\|$$

for $x \in D_\infty(A_1, A_2, \ldots, A_n)$, $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots n\}$.

3.3 Lemma. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$ is convex and metrizable.*

3.4 Lemma. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are closed then the linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$ is a Fréchet space.*

Proof. By 3.3 it is only necessary to prove the completeness of $D_\infty(A_1, A_2, \ldots, A_n)$.

Hence, let x_l, $l \in \{1, 2, \ldots\}$, be an arbitrary Cauchy sequence in the linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$.

This implies by 3.2 that

(1) x_l, $l \in \{1, 2, \ldots\}$ is a Cauchy sequence in E,

(2) for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x_l$, $l \in \{1, 2, \ldots\}$, is a Cauchy sequence in E.

It follows from (1) that there exists an $x \in E$ such that

(3) $x_l \to x$ $(l \to \infty)$.

It is clear that it suffices to prove that

(4) $x \in D_d(A_1, A_2, \ldots, A_n)$ for every $d \in \{1, 2, \ldots\}$,

(5) for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x_l \to$ $\to A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x$ $(l \to \infty)$.

To prove this we proceed by induction on d.

First, it follows immediately from the closedness of operators $A_1, A_2, \ldots, A_n$ that

(6) $x \in D_1(A_1, A_2, \ldots, A_n)$,

(7) for every $\alpha_1 \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} x_l \to A_{\alpha_1} x$ $(l \to \infty)$.

Now we suppose that (4) and (5) are true for some fixed $d \in \{1, 2, \ldots\}$. Using this assumption and the closedness of operators $A_1, A_2, \ldots, A_n$, we obtain easily that

(8) $x \in D_{d+1}(A_1, A_2, \ldots, A_n)$,

(9) for every $\alpha_1, \alpha_2, \ldots, \alpha_{d+1}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{d+1}} x_l \to A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{d+1}} \times$ $(l \to \infty)$.

This argument implies that the assertions (4) and (5) hold for every $d \in \{1, 2, \ldots\}$ and this completes the proof.

3.5 Proposition. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots, A_n$ are closed, then the system of operators $A_1, A_2, \ldots, A_n$ is exponentially Hadamardian if and only if*

(A) *there exists a set $Z \subseteq D_\infty(A_1, A_2, \ldots, A_n)$ dense in the linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$ such that for every $x \in Z$ we can find a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ fulfilling $u^{(n-1)}(0_+) = x$,*

(B) *there exist two nonnegative constants $N, \varkappa$ and a finite sequence $q_1, q_2, \ldots, q_r \in \{1, 2, \ldots, n\}$, $r \in \{1, 2, \ldots\}$, so that for every Duhamel solution u fulfilling $u^{(k-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$*

$$\|A_i u^{(n-i)}(t)\| \leqq N e^{\varkappa t}[\|u^{(n-1)}(0_+)\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+)\|] .$$

Proof. "Only if" part.

Let us assume that the system $A_1, A_2, \ldots, A_n$ is exponentially Hadamardian and let us try to verify the properties 3.5 (A) and 3.5 (B).

The property 3.5 (A) being evident we should only prove 3.5 (B).

To this aim, let us introduce some notation.

First we choose a fixed constant $\varkappa$ such that the condition 3.1 (B) holds.

We denote by **Q** the linear space of all functions $f \in R^+ \to E$ such that

(1) f is n-times differentiable on R^+,

(2) $f^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$,

(3) $f^{(n-i)}(t) \in D(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(4) the functions $A_i f^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

(5) the functions $e^{-\varkappa t} A_i f^{(n-i)}(t)$ are bounded on R^+ for every $i \in \{1, 2, \ldots, n\}$.

The space **Q** will be equipped with the following system of seminorms:

$$|f|_0 = \sup_{t \in R^+} e^{-\varkappa t} \|A_i f^{(n-i)}(t)\| , \tag{6}$$

$$|f|_T = \sup_{0<t<T} \{\|f(t)\| + \|f'(t)\| + \ldots + \|f^{(n)}(t)\| + \tag{7}$$

$$+ \|A_1 f^{(n-1)}(t)\| + \|A_2 f^{(n-2)}(t)\| + \ldots + \|A_n f(t)\|\} \quad \text{for} \quad T > 0 .$$

Clearly

(8) **Q** is a linear topological space.

Moreover, it is almost evident that

(9) the linear topological space **Q** is convex and metrizable.

Now, utilizing the assumed closedness of the operators $A_1, A_2, \ldots, A_n$ we obtain easily that

(10) the linear topological space **Q** is complete.

Hence, by (8)–(10), we can state that

(11) the space **Q** is a Fréchet space.

After these preparatory constructions, we can define, in virtue of the properties 3.1 (A), (B), a linear transformation $U \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n) \to \mathbf{Q}$ in the following way:

(12) for $x \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$, we denote by Ux the unique Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ fulfilling

$$u^{(n-1)}(0_+) = x .$$

Using the assumed closedness of the operators $A_1, A_2, \ldots, A_n$ we deduce easily from the properties defining the spaces $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ and **Q** that

(13) the operator U is closed as a transformation of the linear topological space $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ into the linear topological space **Q**.

Applying now the closed graph theorem 1.4 we get from 3.4 and from (11) and (13) that

(14) the operator U is continuous as a transformation of the linear topological space $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ into the linear topological space **Q**.

The required property (B) is an immediate consequence of (14).

The proof of the "only if" part is complete.

The "if" part.

Now we suppose that the conditions 3.5 (A), 3.5 (B) hold and we try to prove 3.1 (A), (B).

Since the property 3.1 (A) is an immediate consequence of 3.5 (B), it remains in fact to prove only 3.1 (B).

To this aim let us choose

$$x \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n) . \tag{15}$$

Further, we choose fixed nonnegative constants $N, \varkappa$, a number $r \in \{1, 2, \ldots\}$ and a finite sequence $q_1, q_2, \ldots, q_r \in \{1, 2, \ldots, n\}$ so that 3.5 (B) holds.

Now it is easy to conclude from 3.5 (A), 3.5 (B) that there exists a sequence $u_l \in R^+ \to E$, $l \in \{1, 2, \ldots\}$ so that

(16) for every $l \in \{1, 2, \ldots\}$, the function u_l is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$,

$$(17) \qquad u_l^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n) \quad \text{for every} \quad l \in \{1, 2, \ldots\},$$

$$(18) \qquad u_l^{(n-1)}(0_+) \to x \ (l \to \infty),$$

$$(19) \qquad A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} u_l^{(n-1)}(0_+) \to A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x \ (l \to \infty)$$

for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$,

$$(20) \qquad \|A_i u_{l_1}^{(n-i)}(t) - A_i u_{l_2}^{(n-i)}(t)\| \leqq$$

$$\leqq N e^{\varkappa t} [\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} (u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|]$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$ and $l_1, l_2 \in \{1, 2, \ldots\}$.

It follows from (20) that

$$(21) \qquad \|u_{l_1}^{(n)}(t) - u_{l_2}^{(n)}(t)\| \leqq n N e^{\varkappa t} [\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| +$$

$$+ \|A_{q_1} A_{q_2} \ldots A_{q_r} (u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|] \quad \text{for every} \quad t \in R^+ \text{ and } l_1, l_2 \in \{1, 2, \ldots\}.$$

Now using [1] 2.10, we obtain from (21) that

$$(22) \qquad \|u_{l_1}^{(j)}(t) - u_{l_2}^{(j)}(t)\| \leqq$$

$$\leqq \left[n N e^{\varkappa t} \frac{t^{n-j}}{(n-j)!} + 1 \right] [\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| +$$

$$+ \|A_{q_1} A_{q_2} \ldots A_{q_r} (u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|]$$

for every $t \in R^+$, $j \in \{0, 1, \ldots, n\}$ and $l_1, l_2 \in \{1, 2, \ldots\}$.

It follows from (18), (19) and (22) that there exists a function $u \in R^+ \to E$ such that

$$(23) \qquad u_l(t) \to u(t) \ (l \to \infty) \quad \text{for every} \quad t \in R^+.$$

Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, it is easy to obtain from (18), (19), (22) and (23) by means of [1] 2.6 that u is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$ and this was to prove.

The proof of "if" part is complete.

3.6 Remark. The exponential Hadamardian property is related with the Hadamard notion of "correctly set" problem (cf. [2], p. 4). Since it does not involve the class of correctness, it is interesting to study its relations with the notion of correctness.

In the sequel, we prove that correctness always implies exponential Hadamardian property, but as to the converse we are able to get it only under strong a priori restrictions on operators $A_1, A_2, \ldots, A_n$ as shown in the section 5.

It should be said that a more general property would be adequate to the (roughly described) Hadamardian notion of correctly set problem, i.e. it would be necessary to replace the property 3.1 (B) by

(B′) for every $x \in D_\infty(A_1, A_2, \ldots, A_n)$, there exists a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$.

Such systems will be called Hadamardian.

It is easy to see that every exponentially Hadamardian system is also Hadamardian.

3.7 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *the set $D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$ is dense in E,*

(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect,*

then this system is also exponentially Hadamardian.

Proof. An immediate consequence of 2.9, 2.10, 2.12, 2.13 and 2.16.

3.8 Example. *There exist a Banach space E and an operator $A \in L^+(E)$ so that*

(a) *the operator A is closed,*

(b) *the system of operators $0, -A$ is subcorrect of class zero,*

(c) *the system of operators $0, A$ is definite,*

(d) *the system of operators $0, A$ is extensive,*

(e) *the system of operators $0, A$ is not exponentially Hadamardian and consequently also not subcorrect.*

Proof. Let

(1) $$E = L_2(0, \pi)$$

and assume that the operator A is defined as follows:

(2) $x \in D(A)$ if and only if $x \in E$, $x(0_+) = x(1_-) = 0$, x is differentiable on $(0, \pi)$ and there exists a $y \in E$ so that for every $0 < \xi_1 < \xi_2 < \pi$, there is $x'(\xi_2) - x'(\xi_1) = \int_{\xi_1}^{\xi_2} y(\eta)\, d\eta$; then $Ax = y$.

It is easy to prove by elementary means that the assertion (a) holds.

Let us now denote

(3) $e_k(\xi) = (2/\pi)^{1/2} \sin k\xi$ for every $0 < \xi < \pi$ and $k \in \{1, 2, \ldots\}$,

(4) $Z = \{\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_k e_k;\ \alpha_1, \alpha_2, \ldots, \alpha_k \in C,\ k \in \{1, 2, \ldots\}\}$.

It is easy to prove that

(5) $e_k \in D(A)$ and $Ae_k = k^2 e_k$ for every $k \in \{1, 2, \ldots\}$,

(6) the sequence e_k, $k \in \{1, 2, \ldots\}$, is orthonormal,

(7) the set Z is dense in E.

Now the assertion (b) can be derived easily from (5)–(7) by means of Fourier series developments.

The assertions (c) and (d) follow from (5)–(7) by means of 2.3 and 2.8 or simply by direct verification.

It remains to prove (e).

Since $D_\infty(0, A) = \bigcap_{r=1}^{\infty} D(A^r)$, we prove easily that

(8) Z is a dense subset of the Fréchet space $D_\infty(0, A)$.

Further, let us denote

(9) $u_k(t) = \sinh kt\, e_k$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

It is obvious that

(10) for every $k \in \{1, 2, \ldots\}$, the function u_k is a Duhamel solution for the operators $0, A$ such that $u_k^{(n-1)}(0_+) = e_k$.

We see from (4) and (8)–(10) that the condition 3.5 (A) is fulfilled. But it is an easy matter to show by means of the sequence u_k, $k \in \{1, 2, \ldots\}$, that 3.5 (B) cannot be fulfilled due to the exponential growth of hyperbolic sinus.

Hence the system $0, A$ cannot be exponentially Hadamardian and, by 3.8, not even subcorrect.

But this says that (e) holds.

The proof is complete.

3.9 Remark. The above example 3.8 is a somewhat elaborated version of the famous example of a non-correctly set problem, given for the first time by Hadamard in 1917 (cf. [2], pp. 33 and 37).

4. SOME AUXILIARY RESULTS

This section collects some mostly known results on polynomials, on solutions of ordinary differential equations with constant coefficients and on normal operators in Hilbert spaces which will be necessary in Section 5.

4.1 Let $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, and $\varphi \in R^+ \to C$. The fuuction φ will be called a standard solution for the numbers $a_1, a_2, \ldots, a_n$ if

(1) the function φ is n-times differentiable on R^+,

(2) the function $\varphi^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$,

(3) $\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_n \varphi(t) = 0$ for every $t \in R^+$,

(4) $\varphi(0_+) = \varphi'(0_+) = \ldots = \varphi^{(n-2)}(0_+) = 0$, $\varphi^{(n-1)}(0_+) = 1$.

4.2 Lemma. *For every $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, there exists a unique standard solution φ for the numbers $a_1, a_2, \ldots, a_n$.*

Proof. Well-known result which is also an immediate consequence of 2.2 and 2.5.

4.3 Lemma. *Let $a_1, a_2, \ldots, a_n \in C$, $z_1, z_2, \ldots, z_n \in C$, $n \in \{1, 2, \ldots\}$, ω a real constant and $\varphi \in R^+ \to C$. If*

(α) $z^n + a_1 z^{n-1} + \ldots + a_n = (z - z_1)(z - z_2)\ldots(z - z_n)$ *for every* $z \in C$,

(β) $\operatorname{Re} z_i \leqq \omega$ *for every* $i \in \{1, 2, \ldots, n\}$,

(γ) *the function* φ *is a standard solution for the numbers* $a_1, a_2, \ldots, a_n$,

then

(a) $|\varphi(t)| \leqq 3^n(1 + t)^n e^{\omega t}$ *for every* $t \in R^+$,

(b) $\left|\dfrac{a_i}{(i-1)!}\displaystyle\int_0^t (t - \tau)^{i-1} \varphi(\tau)\,d\tau\right| \leqq 3^n(1 + t)^n e^{\omega t}$ *for every* $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$.

Proof. We proceed by induction on n.

The case $n = 1$ is verified by a simple calculation.

Now let us assume the estimates (a), (b) take place for $n - 1$, $n > 1$ and try to prove them for n.

To this aim, we need some preparatory considerations.

By Fundamental Theorem of Algebra, we can find numbers $\alpha \in C$ and $b_1, b_2, \ldots, b_{n-1}$ so that

(1) $$z^n + a_1 z^{n-1} + \ldots + a_n = (z - \alpha)(z^{n-1} + b_1 z^{n-2} + \ldots + b_{n-1}) \quad \text{for every} \quad z \in C\,.$$

For the sake of simplicity we shall write

(2) $$b_0 = 1\,.$$

It is easy to see from (1) and (2) that

(3) $$a_1 = b_1 - \alpha b_0,\ a_2 = b_2 - \alpha b_1, \ldots, a_{n-1} = b_{n-1} - \alpha b_{n-2},\ a_n = -\alpha b_{n-1}\,.$$

Let now

(4) ψ be a standard solution for the numbers $b_1, b_2, \ldots, b_{n-1}$.

It is an easy matter to prove using (1) and (4) that

(5) $$\varphi(t) = \int_0^t e^{\alpha(t-\tau)} \psi(\tau)\,d\tau \quad \text{for every} \quad t \in R^+\,.$$

Using (5), we obtain easily the following identities:

(6) $$\frac{1}{p!}\int_0^t (t - \tau)^p \varphi(\tau)\,d\tau = \int_0^t e^{\alpha(t-\tau)} \frac{1}{p!}\int_0^\tau (\tau - \sigma)^p \psi(\sigma)\,d\sigma\,d\tau$$
$$\text{for every} \quad t \in R^+ \quad \text{and} \quad p \in \{0, 1, \ldots\}\,,$$

(7) $$\alpha \int_0^t \varphi(\tau)\,d\tau = \int_0^t (e^{\alpha(t-\tau)} - 1)\,\psi(\tau)\,d\tau \quad \text{for every} \quad t \in R^+\,,$$

$$(8) \qquad \alpha \frac{1}{(p+1)!} \int_0^t (t-\tau)^{p+1} \varphi(\tau)\, d\tau = \int_0^t (e^{\alpha(t-\tau)} - 1) \frac{1}{p!} \int_0^\tau (\tau - \sigma)^p \psi(\sigma)\, d\sigma\, d\tau$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad p \in \{0, 1, \ldots\}.$$

On the other hand, we have by induction hypothesis that

$$(9) \qquad |\psi(t)| \leqq 3^{n-1}(1+t)^{n-1} e^{\omega t} \quad \text{for every} \quad t \in R^+,$$

$$(10) \qquad \left| b_j \frac{1}{(j-1)!} \int_0^t (t-\tau)^{j-1} \psi(\tau)\, d\tau \right| \leqq 3^{n-1}(1+t)^{n-1} e^{\omega t}$$

$$\text{for every} \quad t \in R^+ \quad \text{and} \quad j \in \{1, 2, \ldots, n-1\}.$$

The desired estimates are now simple consequences of (2), (3) and (5)–(10).

4.4 Lemma. *Let* $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, *and* $\varphi \in R^+ \to C$. *If the function* φ *is a standard solution for the numbers* $a_1, a_2, \ldots, a_n$, *then*
(a) $|\varphi^{(j)}(t)| \leqq e^{[1+\max(|a_1|,|a_2|,\ldots,|a_n|)]t}$ *for every* $t \in R^+$ *and* $j \in \{0, 1, \ldots, n-1\}$,
(b) $|\varphi^{(n)}(t)| \leqq |a_1|\, e^{[1+\max(|a_1|,|a_2|,\ldots,|a_n|)]t}$ *for every* $t \in R^+$.

Proof. Using the properties 4.1 (1)–(4) we obtain easily the following two identities:

$$(1) \qquad \varphi^{(n-1)}(t) = 1 - a_1 \int_0^t \varphi^{(n-1)}(\tau)\, d\tau - a_2 \int_0^t \varphi^{(n-2)}(\tau)\, d\tau - \ldots - a_n \int_0^t \varphi(\tau)\, d\tau,$$

$$(2) \qquad \varphi^{(n)}(t) = -a_1 - a_1 \int_0^t \varphi^{(n)}(\tau)\, d\tau - a_2 \int_0^t \varphi^{(n-1)}(\tau)\, d\tau - \ldots - a_n \int_0^t \varphi'(\tau)\, d\tau$$

$$\text{for every} \quad t \in R^+.$$

The identities (1) and (2) give the estimates

$$(3) \quad |\varphi^{(n-1)}(t)| \leqq 1 + \max(|a_1|, a_2|, \ldots, |a_n|) \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)|)\, d\tau,$$

$$(4) \quad |\varphi^{(n)}(t)| \leqq |a_1| + \max(|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n)}(\tau)|)\, d\tau$$

for every $t \in R^+$.

Using the inequalities (3) and (4) we see easily that

$$(5) \qquad |\varphi(t)| + |\varphi'(t)| + \ldots + |\varphi^{(n-2)}(t)| + |\varphi^{(n-1)}(t)| =$$

$$= \left| \int_0^t \varphi'(\tau)\, d\tau \right| + \left| \int_0^t \varphi''(\tau)\, d\tau \right| + \ldots + \left| \int_0^t \varphi^{(n-1)}(\tau)\, d\tau \right| + |\varphi^{(n-1)}(t)| \leqq$$

$$\leqq \int_0^t |\varphi'(\tau)| + \int_0^t |\varphi''(\tau)| \, d\tau + \ldots + \int_0^t |\varphi^{(n-2)}(\tau)| \, d\tau + \int_0^t |\varphi^{(n-1)}(\tau)| \, d\tau +$$

$$+ 1 + \max (|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)|) \, d\tau \leqq$$

$$\leqq 1 + [1 + \max (|a_1|, |a_2|, \ldots, |a_n|)] \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-2)}(\tau)| +$$

$$+ |\varphi^{(n-1)}(\tau)|) \, d\tau ,$$

(6)
$$|\varphi'(t)| + |\varphi''(t)| + \ldots + |\varphi^{(n-1)}(t)| + |\varphi^{(n)}(t)| =$$

$$= \left|\int_0^t \varphi''(\tau) \, d\tau\right| + \left|\int_0^t \varphi'''(\tau) \, d\tau\right| + \ldots + \left|\int_0^t \varphi^{(n)}(\tau) \, d\tau\right| + |\varphi^{n}(t)| \leqq$$

$$\leqq \int_0^t |\varphi''(\tau)| \, d\tau + \int_0^t |\varphi'''(\tau)| \, d\tau + \ldots + \int_0^t |\varphi^{(n)}(\tau)| \, d\tau +$$

$$+ |a_1| + \max (|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n)}(\tau)|) \, d\tau \leqq$$

$$\leqq |a_1| + [1 + \max (|a_1|, |a_2|, \ldots, |a_n|)] \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)| +$$

$$+ |\varphi^{(n)}(\tau)| \, d\tau \quad \text{for every} \quad t \in R^+ .$$

The inequalities (a), (b) follow immediately from (5), (6) by means of 1.2.

4.5 Lemma. *Let $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, $z \in C$ and $\varphi \in R^+ \to C$. If*
(α) $z^n + a_1 z^{n-1} + \ldots + a_n = 0$,
(β) *the function φ is a standard solution for the numbers $a_1, a_2, \ldots, a_n$, then for every $t \in R^+$*

$$e^{zt} = [\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-1} \varphi(t)] +$$
$$+ z[\varphi^{(n-2)}(t) + a_1 \varphi^{(n-3)}(t) + \ldots + a_{n-2} \varphi(t)] + \ldots +$$
$$+ z^{n-2}[\varphi'(t) + a_1 \varphi(t)] + z^{n-1} \varphi(t) .$$

Proof. Let us denote the right hand side of the identity to be proved by $\psi(t)$. Now we have

(1)
$$\psi'(t) - z \psi(t) = [\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t)] -$$
$$- z[\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-1} \varphi(t)] +$$
$$+ z[\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-2} \varphi'(t)] -$$

$$- z^2[\varphi^{(n-2)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-2} \varphi(t)] + \ldots +$$
$$+ z^{n-2}[\varphi''(t) + a_1 \varphi'(t)] - z^{n-1}[\varphi'(t) + a_1 \varphi(t)] +$$
$$+ z^{n-1} \varphi'(t) - z^n \varphi(t) =$$
$$= [\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t)] -$$
$$- [z^n + a_1 z^{n-1} + \ldots + a_{n-1} z] \varphi(t) \quad \text{for every} \quad t \in R^+ .$$

Since by assumptions (α) and (β)

$$\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t) = -a_n \varphi(t),$$
$$z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

we see immediately from (1) that

(2) $$\psi'(t) - z \psi(t) = 0 \quad \text{for every} \quad t \in R^+ .$$

On the other hand, it is easily verified that

(3) $$\psi(0_+) = 1 .$$

Now we prove without difficulty from (2) and (3) that for $t \in R^+$

$$e^{zt} - \psi(t) - z \int_0^t (e^{z\tau} - \psi(\tau)) \, d\tau = 0$$

and consequently

(4) $$|e^{zt} - \psi(t)| \leqq |z| \int_0^t |e^{z\tau} - \psi(\tau)| \, d\tau \quad \text{for every} \quad t \in R^+ .$$

Now it suffices to apply 1.2 and it follows from (4) that $e^{zt} - \psi(t) = 0$ for every $t \in R^+$ which was to prove.

4.6 Lemma. *Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n \in C$ and $\varphi, \psi \in R^+ \to C$. If φ is a standard solution for the numbers $a_1, a_2, \ldots, a_n$ and ψ for the numbers $b_1, b_2, \ldots \ldots, b_n$, then, writing*

$$K = \max(|a_1|, |a_2|, \ldots, |a_n|),$$
$$L = \max(|b_1|, |b_2|, \ldots, |b_n|),$$
$$\delta = \max(|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|),$$

we have for every $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$

$$|\varphi^{(j)}(t) - \psi^{(j)}(t)| \leqq \delta(K + 1) e^{(3+K+Le^t)t} .$$

Proof. By [1] 2.10 we can write for every $t \in R^+$

$$(1) \quad \varphi^{(n)}(t) + a_1 \int_0^t \varphi^{(n)}(\tau)\,d\tau + \ldots + \frac{a_n}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,\varphi^{(n)}(\tau)\,d\tau = -a_1\,,$$

$$(2) \quad \psi^{(n)}(t) + b_1 \int_0^t \psi^{(n)}(\tau)\,d\tau + \ldots + \frac{b_n}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,\psi^{(n)}(\tau)\,d\tau = -b_1\,.$$

It follows from (1) and (2) that for every $t \in R^+$

$$(3) \quad \varphi^{(n)}(t) - \psi^{(n)}(t) = -(a_1 - b_1) -$$

$$- \left[(a_1 - b_1)\int_0^t \varphi^{(n)}(\tau)\,d\tau + \ldots + \frac{a_n - b_n}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,\varphi^{(n)}(\tau)\,d\tau\right] -$$

$$- \left[b_1 \int_0^t (\varphi^{(n)}(\tau) - \psi^{(n)}(\tau))\,d\tau + \ldots + \frac{b_n}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,(\varphi^{(n)}(\tau) - \psi^{(n)}(\tau))\,d\tau\right].$$

Moreover, we have by 4.4 (b) for every $t \in R^+$

$$(4) \quad |\varphi^{(n)}(t)| \leqq K e^{(1+K)t}\,.$$

It follows from (3) and (4) that for every $t \in R^+$

$$(5) \quad |\varphi^{(n)}(t) - \psi^{(n)}(t)| \leqq$$

$$\leqq \delta + \delta\left[\int_0^t |\varphi^{(n)}(\tau)|\,d\tau + \ldots + \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,|\varphi^{(n)}(\tau)|\,d\tau\right] +$$

$$+ L\left[\int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau + \ldots + \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}\,|\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\right] \leqq$$

$$\leqq \delta + \delta\left[t \max_{0<\tau<t} |\varphi^{(n)}(\tau)| + \ldots + \frac{t^n}{n!}\max_{0<\tau<t} |\varphi^{(n)}(\tau)|\right] +$$

$$+ L\left[\int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau + \ldots + \frac{t^{n-1}}{(n-1)!}\int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\right] \leqq$$

$$\leqq \delta + \delta e^t \max_{0<\tau<t} (|\varphi^{(n)}(\tau)|) + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau \leqq$$

$$\leqq \delta + \delta e^t K e^{(1+K)t} + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau \leqq$$

$$\leqq \delta(K+1)\,e^{(2+K)t} + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\,.$$

Applying now 1.2 to the inequality (5) we obtain immediately for every $t \in R^+$

$$(6) \quad |\varphi^{(n)}(t) - \psi^{(n)}(t)| \leqq \delta(K+1)\,e^{(2+K+Le^t)t}\,.$$

Now the desired inequality follows easily from (6).

4.7 The system of all Borel subsets of C is denoted by $\mathscr{B}(C)$.

4.8 Lemma. *Let* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in C \to C$, $n \in \{1, 2, \ldots\}$, *and* $\boldsymbol{m} \in R^+ \times C \to C$. *If*
(α) *the functions* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ *are Borel measurable,*
(β) *for every* $s \in C$, *the function* $\boldsymbol{m}(\cdot, s)$ *is a standard solution for the numbers* $\boldsymbol{a}_1(s), \boldsymbol{a}_2(s), \ldots, \boldsymbol{a}_n(s)$,
then for every $t \in R^+$ *and* $j \in \{0, 1, \ldots, n\}$, *the functions* $\boldsymbol{m}_t^{(j)}(t, \cdot)$ *are Borel measurable.*

Proof. It follows from (α) that there exist a sequence X_k, $k \in \{1, 2, \ldots\}$, of Borel subsets and a sequence K_k, $k \in \{1, 2, \ldots\}$, of nonnegative constants such that

(1) $$\bigcup_{k=1}^{\infty} X_k = C,$$

(2) $$|a_i(s)| \leqq K_k \quad \text{for every} \quad s \in X_k \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Let us now fix $t \in R^+$ and $\varepsilon > 0$.
We take for $k \in \{1, 2, \ldots\}$

(3) $$\delta_k = \frac{\varepsilon}{(K_k + 1)\, e^{(3 + K_k + K_k e^t)t}}.$$

By (α), there exists for every $k \in \{1, 2, \ldots\}$ a subset $\Delta_k \subseteq \mathscr{B}(C)$ such that

(4) $$\bigcup \Delta_k = X_k,$$

(5) $$\text{for every} \quad i \in \{1, 2, \ldots, n\}, \quad X \in \Delta_k \quad \text{and} \quad s_1, s_2 \in X,$$
$$\text{we have} \quad |\boldsymbol{a}_i(s_1) - \boldsymbol{a}_i(s_2)| \leqq \delta_k.$$

Now we use 4.6 with $\delta = \delta_k$, $K = L = K_k$ for every $k \in \{1, 2, \ldots\}$ and we obtain from (β), (2), (3) and (5) that
(6) for every $j \in \{0, 1, \ldots, n\}$, $X \in \Delta_k$ and $s_1, s_2 \in X$, we have
$$|\boldsymbol{m}_t^{(j)}(t, s_1) - \boldsymbol{m}_t^{(j)}(t, s_2)| \leqq \delta_k(K_k + 1)\, e^{(3 + K_k + K_k e^t)t} = \varepsilon.$$

Let us now denote $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$.
Then by (1), (4) and (6)

(7) $$\bigcup \Delta = C,$$

(8) $$\text{for every} \quad j \in \{0, 1, \ldots, n\}, \quad X \in \Delta \quad \text{and} \quad s_1, s_2 \in X, \quad \text{we have}$$
$$|\boldsymbol{m}_t^{(j)}(t, s_1) - \boldsymbol{m}_t^{(j)}(t, s_2)| \leqq \varepsilon.$$

Since $t \in R^+$ and $\varepsilon > 0$ have been arbitrary, the assertion of our lemma follows immediately from (7) and (8).

4.9 A Banach space E will be called Hilbert space if $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for every $x, y \in E$. In a Hilbert space E we introduce the so-called scalar product $\langle x, y\rangle$ for every $x, y \in E$ in the following way: $\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2]$ in the real case, $\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2]$ in the complex case. This scalar product has the usual well-known properties. The notion of the adjoint operator A^* to an operator $A \in L^+(E)$ is introduced in the usual way.

4.10. In the sequel we always suppose that E is a complex Hilbert space.

4.11. An operator $A \in L^+(E)$ is called normal if $AA^* = A^*A$.

4.12. Let $\mathscr{E} \in \mathscr{B}(C) \to L(E)$. The function $\mathscr{E}$ is called a spectral measure if $\mathscr{E}(C) = I$, $\mathscr{E}(X)$ is an orthogonal (symmetric) projector for every $X \in \mathscr{B}(C)$, $\mathscr{E}(X \cup Y) = \mathscr{E}(X) + \mathscr{E}(Y) - \mathscr{E}(X \cap Y)$ for every $X, Y \in \mathscr{B}(C)$ and $\mathscr{E}(X_k)\, x \underset{k\to\infty}{\to} 0$ for every $x \in E$ and every nondecreasing sequence $X_k \in \mathscr{B}(C)$, $k \in \{1, 2, \ldots\}$, such that $\bigcap_{k=1}^{\infty} X_k = \emptyset$.

4.13. Lemma. *For every spectral measure $\mathscr{E}$ in E, an integral calculus can be developed* (see [4, Chap. VII] and [5, Chap. XVIII]). *The elementary rules of this calculus will be frequently applied in Section 5 and we refer to them by quoting this point.*

The following facts are particularly important

(a) $\|\mathscr{E}(\cdot)\, x\|^2$ *is a nonnegative measure on* $\mathscr{B}(C)$ *for every* $x \in E$,

(b) *if f is a Borel measurable function from* $C \to C$, *then for some* $x \in E$ *and* $X \in \mathscr{B}(C)$:

$$\int_X f(s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \textit{exists if and only if} \quad \int_X |f(s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2 \quad \textit{and}$$

$$\left\|\int_X f(s)\, \mathscr{E}(\mathrm{d}s)\, x\right\|^2 = \int_X |f(s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2 .$$

4.14. Lemma. *Let $A \in L^+(E)$. If the operator A is normal, then there is a unique spectral measure $\mathscr{E}$ such that*

(I) $x \in D(A)$ *if and only if* $\int_C s\mathscr{E}(\mathrm{d}s)\, x$ *exists,*

(II) $Ax = \int_C s\, \mathscr{E}(\mathrm{d}s)\, x$ *for every* $x \in D(A)$.

Proof. See [4, Chap. VIII].

4.15. Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, be normal operators. This system is called abelian if the corresponding spectral measures $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ (cf. 4.14) are commutative, i.e. $\mathscr{E}_i(X_i)\,\mathscr{E}_j(X_j) = \mathscr{E}_j(X_j)\,\mathscr{E}_i(X_i)$ for every $X_1, X_2 \in \mathscr{B}(C)$, $i, j \in \{1, 2, \ldots, n\}$.

4.16. Lemma. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are normal and this system is abelian, then there exists a spectral measure $\mathscr{E} \in \mathscr{B}(C) \to \mathsf{L}(E)$ and Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in C \to C$ so that for every $i \in \{1, 2, \ldots, n\}$*

(I) *$x \in \mathsf{D}(A_i)$ if and only if $\int_C \boldsymbol{a}_i(s)\,\mathscr{E}(\mathrm{d}s)\,x$ exists,*

(II) *$A_i(x) = \int_C \boldsymbol{a}_i(s)\,\mathscr{E}(\mathrm{d}s)\,x$ for every $x \in \mathsf{D}(A_i)$.*

Proof. See [4, Chap. X, especially § 3].

5. ABELIAN SYSTEMS OF NORMAL OPERATORS IN HILBERT SPACES

In this section, we shall study linear differential equations in a Hilbert space over C whose coefficients form an abelian system of normal operators. In particular, we show that in this class of operators, the exponentially Hadamardian systems are correct.

5.1 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If*

(α) *E is a Hilbert space over C,*

(β) *the operators $A_1, A_2, \ldots, A_n$ are normal,*

(γ) *the system of operators, $A_1, A_2, \ldots, A_n$ is abelian,*

then the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Proof. Let us choose by 4.16 Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and a spectral measure $\mathscr{E}$ so that 4.16 (I), (II) hold.

Let us now define for $k \in \{1, 2, \ldots\}$

$$S_k = \{s : |\boldsymbol{a}_i(s)| \leqq k \text{ for every } i \in \{1, 2, \ldots, n\}\}\,. \tag{1}$$

It is clear that the sets S_k are Borel measurable for every $k \in \{1, 2, \ldots\}$ and hence we can take

$$P_k = \mathscr{E}(S_k) \quad \text{for} \quad k \in \{1, 2, \ldots\}\,. \tag{2}$$

It follows from 4.13 that the assumptions of 2.3 are fulfilled and hence the statement is true.

5.2 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the assumptions 5.1 (α)–(γ) are fulfilled, then the system of operators $A_1, A_2, \ldots, A_n$ is extensive.*

Proof. Let us choose by 4.16 Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and a spectral measure $\mathscr{E}$ so that 4.16 (I), (II) hold.

Let us now define

(1) $\mathscr{S} = \{S : S \in \mathscr{B}(C),$ the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are bounded on $S\}$.

Now we take

(2) $$\mathfrak{P} = \{\mathscr{E}(S) : S \in \mathscr{S}\} .$$

It follows from 4.13 that the assumptions of 2.8 are fulfilled and consequently the statement is true.

5.3 Remark. We see immediately that the preceding Theorem 5.2 gives more, namely, under the assumptions of 5.2, the Duhamel solutions exist in fact for initial data from a dense subset of E.

5.4 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If*
(α) E is a Hilbert space over C,
(β) the operators $A_1, A_2, \ldots, A_n$ are normal,
(γ) the system of operators $A_1, A_2, \ldots, A_n$ is abelian,
(δ) the system of operators $A_1, A_2, \ldots, A_n$ is exponentially Hadamardian,
then this system is correct (of class $n - 1$).

Proof. It follows from 5.1 that

(1) the system of operators $A_1, A_2, \ldots, A_n$ is definite.
Further, by 5.2

(2) the system of operators $A_1, A_2, \ldots, A_n$ is extensive.
With regard to (2), it suffices to prove that

(3) the condition 2.9 (B) is satisfied.
To prove (3), we need a series of preparatory considerations.
First, using 4.16, we obtain from (α)–(γ) that there exist functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in$ $\in C \to C$ and a function $\mathscr{E} \in \mathscr{B}(C) \to L(E)$ so that

(4) the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are Borel measurable,

(5) the function $\mathscr{E}$ is a spectral measure,

(6) for every $i \in \{1, 2, \ldots, n\}$, $x \in D(A_i)$ if and only if $\int_C \boldsymbol{a}_i(s)\, \mathscr{E}(\mathrm{d}s)\, x$ exists ,

(7) $$A_i x = \int_C \boldsymbol{a}_i(s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every} \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad x \in D(A_i) .$$

By 4.13, we obtain from (4)–(7) that

(8) $$\mathscr{E}(X)\, A_i \leqq A_i\, \mathscr{E}(X) \quad \text{for every} \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad X \in \mathscr{B}(C) .$$

Let us now denote

(9) $\mathscr{S} = \{X : X \in \mathscr{B}(C),$ the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are bounded on $X\}$.

By 4.13, we obtain from (4)–(7) and (9) that

(10) $\mathscr{E}(X)\, x \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ for every $x \in E$ and $X \in \mathscr{S}$,

(11) $$\|A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d}\, \mathscr{E}(X)\, x\| = \left[\int_X |\boldsymbol{a}_{\alpha_1}(s)\, \boldsymbol{a}_{\alpha_2}(s) \ldots \boldsymbol{a}_{\alpha_d}(s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2\right]^{1/2}$$

for every $x \in E$, $X \in \mathscr{S}$, $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$,

(12) there exists a sequence $X_\nu \in \mathscr{S}$, $\nu = \{1, 2, \ldots\}$ such that $X_\nu \subseteq X_{\nu+1}$ for every $\nu \in \{1, 2, \ldots\}$ and $\bigcup\{X_\nu : \nu \in \{1, 2, \ldots\}\} = C$.

On the other hand, by 4.2 there exists a unique function $\boldsymbol{m} \in R^+ \times C \to C$ such that

(13) for every $s \in C$, the function $\boldsymbol{m}(\cdot, s)$ is a standard solution for the numbers $\boldsymbol{a}_1(s), \boldsymbol{a}_2(s), \ldots, \boldsymbol{a}_n(s)$.

Using 4.8, we obtain from (4) and (13) that

(14) the functions $\boldsymbol{m}_t^j(t, \cdot)$ are Borel measurable for every $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$.

Further, using 4.4 we obtain from (9) and (13) that

(15) for every $X \in \mathscr{S}$, there exists a constant K so that for every $t \in R^+$, $s \in X$ and $j \in \{0, 1, \ldots, n\}$

$$|\boldsymbol{m}_t^{(j)}(t, s)| \leqq K e^{Kt}.$$

By 4.13, we obtain from (13)–(15) that

(16) for every $x \in E$ and $X \in \mathscr{S}$, the function $\int_X \boldsymbol{m}(\cdot, s)\, \mathscr{E}(\mathrm{d}s)\, x$ is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

$$\left(\int_X \boldsymbol{m}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x\right) \xrightarrow[t \to 0_+]{} \mathscr{E}(X)\, x,$$

(17) $$\frac{\mathrm{d}^j}{\mathrm{d}t^j} \int_X \boldsymbol{m}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x = \int_X \boldsymbol{m}_t^{(j)}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every}$$

$t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$,

(18) $$\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i \frac{\mathrm{d}^{n-i}}{\mathrm{d}\tau^{n-i}} \left(\int_X \boldsymbol{m}(\tau, s)\, \mathscr{E}(\mathrm{d}s)\, x\right) \mathrm{d}\tau =$$

$$= \int_X \boldsymbol{a}_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1}\, \boldsymbol{m}(\tau, s)\, \mathrm{d}\tau\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every}$$

$t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $i \in \{1, 2, \ldots, n\}$,

(19) $$\left\|\int_X \boldsymbol{m}_t^{(j)}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x\right\| = \left[\int_X |\boldsymbol{m}_t^{(j)}(t, s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2\right]^{1/2}$$

for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$,

(20)
$$\left\| \int_X a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \, \mathscr{E}(ds) \, x \right\| =$$
$$= \left[\int_X \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \right|^2 \|\mathscr{E}(ds) \, x\|^2 \right]^{1/2}$$
for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $i \in \{1, 2, \ldots, n\}$.

Our next purpose is to establish some estimates of growth of the function m.

It follows from Theorem 3.5 that we can fix two nonnegative constants $N, \varkappa$, a number $r \in \{1, 2, \ldots\}$ and a finite sequence $q_1, q_2, \ldots, q_r$ so that

(21) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, for every $t \in R^+$ and every $i \in \{1, 2, \ldots, n\}$
$$\|A_i u^{(n-i)}(t)\| \leqq N e^{\varkappa t} [\|u^{(n-1)}(0_+)\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+)\|].$$

Since for every $t \in R^+$
$$u^{(n)}(t) = -[A_1 u^{(n-1)}(t) + A_2 u^{(n-2)}(t) + \ldots + A_n u(t)],$$
for every $t \in R^+$ and $k \in \{0, 1, \ldots, n-1\}$
$$u^{(k)}(t) = \frac{1}{(n-1-k)!} \int_0^t (t-\tau)^{n-1-k} u^{(n)}(\tau) \, d\tau + \frac{t^{n-1-k}}{(n-1-k)!} u^{(n-1)}(0_+)$$
and for every $t \in R^+$, $\delta > 0$ and $l \in \{0, 1, \ldots\}$
$$\frac{t^l}{l!} \leqq \frac{1}{\delta^l} e^{\delta t},$$
we deduce from (21) after a simple calculation that

(22) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, for every $t \in R^+$, $j \in \{0, 1, \ldots, n\}$ and $\delta > 0$
$$\|u^{(j)}(t)\| \leqq \frac{1}{\delta^{n-j}} (nN + \delta) \, e^{(\varkappa+\delta)t} [\|u^{(n-1)}(0_+)\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+)\|].$$

It follows from (1), (10), (16), (17) and (22) that

(23)
$$\left\| \int_X m_t^{(j)}(t, s) \, \mathscr{E}(ds) \, x \right\| \leqq$$
$$\leqq \frac{1}{\delta^{n-j}} (nN + \delta) \, e^{(\varkappa+\delta)t} [\|\mathscr{E}(X) \, x\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} \mathscr{E}(X) \, x\|]$$
for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$.

Now (11), (19) and (23) give, with regard to the inequality $(a^{1/2} + b^{1/2})^2 \leqq \leqq 2(a + b)$ for $a \geqq 0$, $b \geqq 0$

(24) $$\int_X |m_t^{(j)}(t, s)|^2 \|\mathscr{E}(ds)\, x\|^2 \leqq \frac{1}{\delta^{2(n-j)}} (nN + \delta)^2\, e^{2(\varkappa+\delta)t} \Big[\Big(\Big(\int_X \|\mathscr{E}(ds)\, x\|^2\Big)^{1/2} +$$

$$+ \Big(\int_X |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|^2 \|\mathscr{E}(ds)\, x\|^2\Big)^{1/2}\Big]^2 \leqq$$

$$\leqq \frac{2}{\delta^{2(n-j)}} (nN + \delta)^2\, e^{2(\varkappa+\delta)t} \int_X (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|^2) \|\mathscr{E}(ds)\, x\|^2 \leqq$$

$$\leqq \frac{2}{\delta^{2(n-j)}} (nN + \delta)^2\, e^{2(\varkappa+\delta)t} \int_X (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)^2 \|\mathscr{E}(ds)\, x\|^2$$

for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$.

Let us now define for $\delta > 0$, $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$

(25) $$N_{\delta,t,j} = \Big\{s : s \in C,\ |m_t^{(j)}(t, s)| > \frac{\sqrt{2}}{\delta^{n-j}} (nN + \delta)\, e^{(\varkappa+\delta)t}(1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)\Big\}.$$

It is clear from (14) and (25) that

(26) the set $N_{\delta,t,j}$ is Borel measurable for every $\delta > 0$, $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$.

Let us now put for $\delta > 0$

(27) $$N_\delta = \bigcup\{N_{\delta,t,j} : t \in R^+,\ t \text{ rational},\ j \in \{0, 1, \ldots, n\}\}.$$

We see from (26) and (27) that

(28) the set N_δ is Borel measurable for every $\delta > 0$.

It follows from (13) (the continuity of $m_t^{(j)}(\cdot, s)$ follows by 4.1), (25) and (27) that

(29) $$C \setminus N_\delta = \Big\{s : |m_t^{(j)}(t, s)| \leqq \frac{\sqrt{2}}{\delta^{(n-j)}} (nN + \delta)\, e^{(\varkappa+\delta)t}(1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)$$
$$\text{for every } t \in R^+ \text{ and } j \in \{0, 1, \ldots, n\}\Big\} \text{ for every } \delta > 0.$$

Now we need to prove that

(30) $$\mathscr{E}(N_\delta) = 0 \quad \text{for every} \quad \delta > 0.$$

It is seen from (12) that it is sufficient for the validity of (30) to prove that $\mathscr{E}(N_\delta)\, \mathscr{E}(X)\, x = 0$ for every $\delta > 0$, $x \in E$ and $X \in \mathscr{S}$, i.e. that

(31) $$\mathscr{E}(N_\delta \cap X)\, x = 0 \quad \text{for every} \quad \delta > 0,\ x \in E \text{ and } X \in \mathscr{S}.$$

On the contrary, suppose that (31) is not true. Then there exist $\delta > 0$, $x \in E$ and $X \in \mathscr{S}$ so that $\mathscr{E}(N_\delta \cap X)\, x \neq 0$. Consequently, by (27) we can find $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$ so that

$$\mathscr{E}(N_{\delta,t,j} \cap X)\, x \neq 0\,.$$

Hence by (25)

$$\int_{N_{\delta,t,j} \cap X} |m_t^{(j)}(t, s)|^2 \, \|\mathscr{E}(\mathrm{d}s)\, x\|^2 >$$

$$> \frac{2}{\delta^{2(n-j)}} (nN + \delta)^2 \, e^{(\varkappa+\delta)t} \int_{N_{\delta,t,j} \cap X} (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)^2 \, \|\mathscr{E}(\mathrm{d}s)\, x\|^2\,.$$

Since $N_{\delta,t,j} \cap X \in \mathscr{S}$ by (9) and (26), the last inequality obviously contradicts (24) and this proves (31).

The statements (29) and (30) represent the needed growth properties of the function m and will now be used to estimate the roots of the characteristic polynomial.

By Fundamental Theorem of Algebra, there exist functions $z_1, z_2, \ldots, z_n \in C \to C$ such that

(32) $$z^n + a_1(s)\, z^{n-1} + \ldots + a_n(s) =$$

$$= (z - z_1(s))\,(z - z_2(s)) \ldots (z - z_n(s)) \quad \text{for every} \quad s, z \in C\,.$$

Applying 4.5 to (32) we obtain easily

(33) $$|e^{z_i(s)t}| \leqq (1 + |z_i(s)|)^{n-1}\, (1 + |a_1(s)| + |a_2(s)| + \ldots$$

$$+ |a_{n-1}(s)|)\, (|m(t, s)| + |m'(t, s)| + \ldots + |m_t^{(n-1)}(t, s)|)$$

$$\text{for every} \quad t \in R^+\,, \quad s \in C \quad \text{and} \quad i \in \{1, 2, \ldots, n\}\,.$$

We get from (29) and (33)

(34) $$e^{\operatorname{Re} z_i(s)t} = |e^{z_i(s)t}| \leqq (1 + |z_i(s)|)^{n-1}\, (1 + |a_1(s)| +$$

$$+ |a_2(s)| + \ldots + |a_{n-1}(s)|)\, \sqrt{2} \left(\frac{1}{\delta^n} + \frac{1}{\delta^{n-1}} + \ldots + \frac{1}{\delta}\right) (nN + \delta)\,.$$

$$.\, e^{(\varkappa+\delta)t} (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|) =$$

$$= e^{(\varkappa+\delta)t} \Bigg[\sqrt{2} \left(\frac{1}{\delta} + \frac{1}{\delta^2} + \ldots + \frac{1}{\delta^n}\right) (nN + \delta)\, (1 + |z_i(s)|)^{n-1}\,.$$

$$.\, (1 + |a_1(s)| + |a_2(s)| + \ldots + |a_n(s)|)\, (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)\Bigg]$$

$$\text{for every} \quad t \in R^+\,, \quad \delta > 0 \quad \text{and} \quad s \in C \setminus N_\delta\,.$$

Since the member in the last brackets does not depend on t, it follows immediately from (34) that

$$\text{(35)} \qquad \operatorname{Re} z_i(s) \leqq \varkappa + \delta \quad \text{for every} \quad \delta > 0 \quad \text{and} \quad s \in C \setminus N_\delta .$$

Let us now put

$$\text{(36)} \qquad N = \bigcup\{N_{1/k} : k \in \{1, 2, \ldots\}\} .$$

It follows from (30) that

$$\text{(37)} \qquad \mathscr{E}(N) = 0 .$$

On the other hand, by (35) and (36)

$$\text{(38)} \qquad \operatorname{Re} z_i(s) \leqq \varkappa \quad \text{for every} \quad s \in C \setminus N .$$

The last results (37) and (38) allow us to estimate the growth of a general Duhamel solution which is our task from (3).

However, to this aim we need still an auxiliary result, namely

(39) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u^{(n-i)}(\tau) \, d\tau \right\| \leqq$$

$$\leqq \left[\int_C \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \right|^2 \| \mathscr{E}(ds) \, u^{(n-1)}(0_+) \|^2 \right]^{1/2} .$$

To prove (39) let u be an arbitrary Duhamel solution for the operators $A_1, A_2, \ldots, A_n$.

By (12), we can choose a sequence X_ν. $\nu \in \{1, 2, \ldots\}$ such that

(40) $X_\nu \in \mathscr{S}$ for every $\nu \in \{1, 2, \ldots\}$, $X_\nu \subseteq X_{\nu+1}$ for every $\nu \in \{1, 2, \ldots\}$ and $\bigcup\{X_\nu : \nu \in \{1, 2, \ldots\}\} = C$,

$$\text{(41)} \qquad \mathscr{E}(X_\nu) \, x \to x \ (\nu \to \infty) \quad \text{for every} \quad x \in E .$$

By (16), (18), (20), (40) and (41), there exists a sequence u_ν, $\nu \in \{1, 2, \ldots\}$, such that

(42) for every $\nu \in \{1, 2, \ldots\}$, the function u_ν is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

$$u_\nu^{(n-1)}(0_+) = \mathscr{E}(X_\nu) \, u^{(n-1)}(0_+) .$$

$$\text{(43)} \qquad \left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u(\tau) \, d\tau \right\| =$$

$$= \left[\int_{X_\nu} \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \right|^2 \| \mathscr{E}(ds) \, u^{(n-1)}(0_+) \|^2 \right]^{1/2}$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

On the other hand, we establish easily by means of (8) that
(44) for every $\nu \in \{1, 2, \ldots\}$, the function $\mathscr{E}(X_\nu) u$ is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $(\mathscr{E}(X_\nu) u)^{(n-1)}(0_+) = \mathscr{E}(X_\nu) u^{(n-1)}(0_+)$.

Now we get from (1), (42) and (44) that

$$(45) \qquad u_\nu = \mathscr{E}(X_\nu) u \quad \text{for every} \quad \nu \in \{1, 2, \ldots\}.$$

It follows from (43) and (45) that

$$(46) \qquad \left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i \mathscr{E}(X_\nu) u^{(n-i)}(\tau) \, d\tau \right\| =$$

$$= \left[\int_{X_\nu} \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \right|^2 \|\mathscr{E}(ds) u^{(n-1)}(0_+)\|^2 \right]^{1/2}$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

By (8)

$$(47) \qquad \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i \mathscr{E}(X_\nu) u^{(n-i)}(\tau) \, d\tau =$$

$$= \mathscr{E}(X_\nu) \left[\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u^{(n-i)}(\tau) \, d\tau \right]$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

Letting $\nu \to \infty$, we see easily from (40), (41), (46) and (47) that (39) is valid.

Using Lemma 4.3 we see from (13), (32), (38) and (39) that (48) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u^{(n-i)}(\tau) \, d\tau \right\| \leqq 3^n (1+t)^n e^{\varkappa t} \|u^{(n-1)}(0_+)\|.$$

But (48) clearly yields (3) if we take $M = 3^n$, $\omega = \varkappa + 1$.

The proof is complete.

5.5 Remark. The preceding theorem shows that *the system of operators* $A_1, A_2, \ldots$ $\ldots, A_n$ *with the properties* 5.4 (α), (β), (γ) *is correct if and only if it is exponentially Hadamardian.*

Moreover, in the course of the proof, we have shown that *the system of operators* $A_1, A_2, \ldots, A_n$ *with the properties* 5.4 (α), (β), (γ) *is correct if and only if it is correct of class* $n - 1$.

For Hadamardian systems, Theorem 5.4 does not hold and certain additional restrictive assumptions on the operators $A_1, A_2, \ldots, A_n$ must be introduced.