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ON A CLASS OF NONLINEAR EVASION GAMES

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In this paper we shall consider a differential game described by the system of differential equations

$$(1) \qquad z^{(n)} + A_1 z^{(n-1)} + \ldots + A_{n-1} z' + A_n z = \\ = f(u, v) + \mu g(z, z', \ldots, z^{(n-1)}, u, v),$$

where $z \in R^m, f \in R^m, A_i, i = 1, 2, \ldots, n$ are constant matrices, $f(u, v)$ is a continuous function of the point $(u, v) \in U \times V, U \subset R^p, V \subset R^q$ are compact sets, $\mu \in (-\infty, \infty)$ is a parameter. We shall suppose that the function $g(z_1, z_2, \ldots, z_n, u, v)$ is continuous and bounded on $R^{mn} \times U \times V$.

In the paper [1] a sufficient condition for existence of evasion strategy for a differential game described by equation (1) for $\mu = 0$ is given. In the paper [2] a sufficient condition for existence of such strategy for a game described by a first order system of differential equations of type (1) is given. That condition is different from the condition given in our paper. Our condition is similar to that given in [1]. Similarly to [1] we shall use the technique of convolutions in the formulation of results as well as in the proof.

A mapping $V_u(t, Z_0)$ defined on the set of measurable controls $u(\tau)$, $0 \leqq \tau < \infty$, $u(\tau) \in U$ depending on $t \geqq 0$ and on the vector of initial conditions $Z_0 = (z_0, z_0', \ldots \ldots, z_0^{(n-1)})$ is said to be a strategy, if it possesses the following properties:

(1) For an arbitrary measurable control $u(\tau)$, $0 \leqq \tau < \infty$ and for an arbitrary fixed Z_0, the mapping $V_u(t, Z_0)$ is measurable as a function of t and has values in V.

(2) If $u_1(\tau), u_2(\tau), 0 \leqq \tau < \infty$ are two controls and $u_1(\tau) = u_2(\tau)$ almost everywhere on $[0, T]$, where T is arbitrary, then $V_{u_1}(t, Z_0) = V_{u_2}(t, Z_0)$ almost everywhere on $[0, T]$ for every Z_0.

Let M be a subspace of R^m of a dimension $\leqq m - 2$. Our problem is to choose a strategy $V_u(t, Z_0)$ such that the solution $z(t)$, $0 \leqq t < \infty$ of the equation

$$z^{(n)} + A_1 z^{(n-1)} + \ldots + A_n z = \\ = f(u(t), V_u(t, Z_0)) + \mu g(z(t), \ldots, z^{(n-1)}, u(t), V_u(t, Z_0))$$

with the initial condition

$$Z(0) = (z(0), z'(0), \ldots, z^{(n-1)}(0)) = Z_0 , \quad z(0) \notin M$$

does not intersect the subspace M for any $t \geqq 0$, for an arbitrary control $u(t)$ and for an arbitrary vector Z_0. We shall call this strategy an evasion strategy.

Now, using the convolution symbolism (cf. [1]) we can rewrite the equation (1) in the form

$$z^{(n)} + \hat{A}_1 * z^{(n-1)} + \ldots + \hat{A}_n * z = f(u, v) + \mu g(z, z', \ldots, z^{(n-1)}, u, v)$$

and express the solution of this equation by the following formula:

(2) $$z_\mu = z_0 + S * z_0' + \ldots + S^{n-1} * z_0^{(n-1)} + \\ + S^n * (\Phi_0 * z_0 + \ldots + \Phi_{n-1} * z_0^{(n-1)}) + S^n * R(S) * f(u, v) + \\ + \mu S^n * R(S) * g(z, z', \ldots, z^{(n-1)}, u, v) ,$$

where $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$ are certain entire matrices over the Mikusiński ring $\mathcal{M}$ (cf. [1]),

$$R(S) = \hat{I} + C(S) + C^2(S) + \ldots ,$$

$$C(S) = -(S * \hat{A}_1 + S^2 * \hat{A}_2 + \ldots + S^n * \hat{A}_n) ,$$

$\hat{I} = \operatorname{diag}(\delta, \delta, \ldots, \delta)$ is the unit matrix, δ is the unit element in the ring $\mathcal{M}$, $\hat{A}_i$, $i = 1, 2, \ldots, n$ are constant matrices, i.e. the functions identically equal to A_i. It was shown in [1] that the series for $R(S)$ converges uniformly in a disc with center at the origin of an arbitrary large radius ϱ.

Let L be a subspace of R^m of a dimension $k \geqq 2$ which lies in the orthogonal complement of $M \subset R^m$ and let $\pi : R^m \to R^k$ be a linear mapping corresponding to the orthogonal projection of R^m onto L.

We assume that

(3) $$\hat{\pi} * R(S) * f(u, v) = H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + \chi(t) ,$$

where

(a) $\Psi_i(u, v)$ are continuous in $(u, v) \in U \times V$, $i = 0, 1, 2, \ldots$.

(b) $|\Psi_i(u, v)| \leqq \lambda_i$ for all $(u, v) \in U \times V$, $|\cdot|$ being the Euclidean norm in R^k and the series $\hat{\lambda}_0 + S * \hat{\lambda}_1 + S^2 * \hat{\lambda}_2 + \ldots$ is an entire function of the variable t.

(c) $H(S)$ is an entire matrix over the ring $\mathcal{M}$ and $\det^* H(S) \neq 0$. ($\det^* H(S)$ is calculated as a determinant in the ordinary formal way using the ring multiplication).

(d) The function $\chi(t)$ does not depend on u, v.

(e) Denote by $[\Psi_0(u, v)]$ the smallest linear subspace of R^k containing all points $\Psi_0(u, v)$, $(u, v) \in U \times V$. Let us suppose that the subspace $[\Psi_0(u, v)]$ has the largest possible dimension among all representations (3).

We shall say that the parameter v in the expression $\hat{\pi} * R(S) * f(u, v)$ has complete maneuverability, if the set

$$(4) \qquad \bigcap_{u \in U} \operatorname{co}_v \Psi_0(u, v) \subset R^k$$

contains interior points, where $\operatorname{co}_v \Psi_0(u, v)$ denotes the convex hull of the set of all points $\Psi_0(u, v)$, $v \in V$ for fixed $u \in U$.

Now, we can formulate a sufficient condition for evasion.

Theorem 1. *If the parameter v in the expression $\hat{\pi} * R(S) * f(u, v)$ has complete maneuverability, then there exists a number $\mu_1 > 0$ such that for all μ, $|\mu| < \mu_1$ there exists an evasion strategy. Moreover, there exist numbers $\lambda, \nu, \theta > 0$ and an integer l such that*

$$(5) \qquad \varrho(z_\mu(t), M) \geqq \frac{1}{2} \theta \left(\frac{(z_\mu(0), M)}{\lambda \nu} \right)^{n+1} \frac{1}{(1 + |z_\mu(t)|)^{n+1}}$$

for $0 \leqq t < \infty$, where $\varrho(z_\mu(t), M)$ is the distance of the point $z_\mu(t)$ from the subspace M ($z_\mu(t)$ denotes the solution of (1) corresponding to a value μ of the parameter).

Remark. The number l in Theorem 1 is equal to the number l_k, where

$$H(S) = H^{(1)}(S) * \operatorname{diag}(S^{l_1}, \ldots, S^{l_k}) * H^{(2)}(S),$$

$l_1 \leqq l_2 \leqq \ldots \leqq l_k$, $H^{(i)}(S)$, $i = 1, 2$ are entire invertible matrices. It was shown in [1] that an arbitrary entire matrix $H(S)$ has such a representation.

For the sake of simplicity of computations, we can assume that the origin of R^k is an interior point of the set (4). Denote by Q the closed k-dimensional cube with the center at the origin and with sides parallel to the axes and such that $Q \subset \subset \operatorname{int} \bigcap_{u \in U} \operatorname{co}_v \psi_0(u, v)$ ($\operatorname{int} P$ denotes the interior of P).

For the proof of Theorem 1 we need the following lemma, which was proved in [1].

Lemma 1. *For sufficiently small Q there exists a number $T > 0$ such that for any $\varepsilon > 0$ there exists a measurable function $v(t) \in V$, $0 \leqq t \leqq T$ such that*

$$(6) \qquad \|S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots + \chi(t)] + t^{n+1}\xi\| \leqq \varepsilon$$

for $0 \leqq t \leqq T$ and for an arbitrary preassigned $u(t) \in U$, $\xi \in Q$. For the calculation of $v(t)$ we need the values $u(t)$ on the interval $[0, t]$ and the point ξ only.

Remark. $\|p(t)\| = \sup_{t \in [0, T]} \left|\int_0^t p(\tau) \, d\tau\right|$, where $|\cdot|$ is the Euclidean norm in R^k.

Proof of Theorem 1. From (2), (3) we get

$$\hbar * z_\mu(t) = \varphi(t, Z_0) + S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \dots) + \chi(t)] +$$
$$+ \mu S^n * R(S) * g(z_\mu, z'_\mu, \dots, z_\mu^{(n-1)}, u, v),$$

where

$$\varphi(t, Z_0) = \hbar * [z_0 + S * z_0 + \dots + S^{n-1} * z_0^{(n-1)} +$$
$$+ S^n * (\Phi_0 * z_0 + \dots + \Phi_{n-1} * z_0^{(n-1)})].$$

Sublemma 1. *If $\mu_1 > 0$ is a given number and $\varrho(z_\mu(0), M) > 0$ for $|\mu| < \mu_1$, then* (a) *for a sufficiently large number λ*

(7) $$\varrho(z_\mu(t), M) \geqq \frac{\varrho(z_\mu(0), M)}{2} \quad \text{for} \quad 0 \leqq t \leqq \frac{\varrho(z_\mu(0), M)}{\lambda(1 + |Z_\mu(0)|)},$$
$$|\mu| < \mu_1, \quad Z_\mu(0) = (z_\mu(0), z'_\mu(0), \dots, z_\mu^{(n-1)}(0)) = Z_0.$$

(b) *If T is sufficiently small, then there exists a number $\nu > 0$ such that for an arbitrary Z_0 and for $|\mu| < \mu_1$*

(8) $$\nu(1 + |Z_\mu(t)|) \geqq 1 + |Z_0|, \quad 0 \leqq t \leqq T,$$
$$(Z_\mu(t) = (z_\mu(t), z'_\mu(t), \dots, z_\mu^{(n-1)}(t))).$$

The proof of Sublemma 1 is analogous to the proof of inequalities (5.4), (5.5) in [1].

Sublemma 2. *There exists a $\theta > 0$ so small that for an arbitrary initial vector Z_0, there exists a point $\xi(Z_0) \in Q$ satisfying the condition*

(9) $$|\varphi(t, Z_0) - S * t^{n+l-1} \xi(Z_0)| \geqq \theta t^{n+1}, \quad 0 \leqq t \leqq T.$$

Proof. By [1, Lemma 5.1] there exist a point $\xi(Z_0) \in Q$ and a number $\theta' > 0$ such that

$$\left| \frac{(n+1)\, \varphi(t, Z_0)}{t^{n+1}} - \xi(Z_0) \right| \geqq \theta'.$$

This implies

$$\left| \varphi(t, Z_0) - \frac{t^{n+1}}{n+1} \xi(Z_0) \right| = |\varphi(t, Z_0) - S * t^{n+l-1} \xi(Z_0)| \geqq \theta t^{n+1},$$

where $\Theta = \Theta'/(n+1)$.

Now, we choose a number $\sigma > 0$, which satisfies the following inequalities:

(10) $$\sigma < \tfrac{1}{2}\theta T^{n+1}, \quad \sigma < \lambda T, \quad \frac{\sigma}{2} > \theta \left(\frac{\sigma}{\lambda}\right)^{n+1},$$

where λ can be chosen arbitrarily large.

Let us suppose that at the beginning of the game at time $t = 0$ it is $\varrho(z_\mu(0), M) > \sigma$. Choose the control $v(t)$ arbitrarily. If for some $t = t_1$, $\varrho(z_\mu(t_1), M) = \sigma$, then define a control $v(t)$ on the interval $[t_1, t_1 + T]$ in the following way

$$v(t) = w(t - t_1, u, \xi(Z_\mu(t_1)), \varepsilon)\,, \tag{11}$$

where $w(t, u, \xi, \varepsilon)$ is a control satisfying the inequality (9) for given $\varepsilon > 0$, $u(t) \in U$ and $\xi \in Q$.

Sublemma 3. *If $v(t)$ is a control defined by the equality* (11), *then there exists a number $\mu_1 > 0$ such that for $|\mu| < \mu_1$*

(12) (a) $$\varrho(z_\mu(t), M) \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t)|)^{n+1}}\,, \quad t_1 \leqq t \leqq t_1 + T$$

(b) $$\varrho(z_\mu(t_1 + T), M) \geqq \sigma\,.$$

Proof. From (7), (8) it follows that for

$$0 \leqq t - t_1 \leqq \frac{\varrho(z_\mu(t_1), M)}{\lambda(1 + |Z_\mu(t_1)|)} = \frac{\sigma}{\lambda(1 + |Z_\mu(t_1)|)}\,,$$

$$\varrho(z_\mu(t), M) \geqq \frac{\sigma}{2} \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t_1)|)^{n+1}} - \varepsilon\,. \tag{13}$$

$$\begin{aligned}
\varrho(z_\mu(t), M) &= |\hat{\pi} * z_\mu(t)| = |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1)) + \\
&+ S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1)) + S^n * [H(S) * (\Psi_0 + S * \Psi_1 + \ldots) + X(t)] + \\
&+ \mu S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \geqq \\
&\geqq |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \\
&- |S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + X(t)] + \\
&+ S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \mu |S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \geqq \\
&\geqq |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \\
&- \|S^{n-1} * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + X(t)] + \\
&+ (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))\| - \mu |S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)|\,.
\end{aligned}$$

Since $|g(z_1, z_2, \ldots, z_n, u, v)| \leqq c$ for all $(z_1, z_2, \ldots, z_n, u, v) \in R^{mn} \times U \times V$, where $c > 0$ is constant, there exists a constant $c_1 > 0$ such that for $|\mu| \leqq \mu_1$, $0 \leqq t \leqq T + t_1$ it is $|S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \leqq c_1$. Therefore, using Sublemma 1 and Sublemma 2 we conclude

$$\varrho(z_\mu(t), M) \geqq \theta(t - t_1)^{n+1} - \varepsilon - \mu c_1\,.$$

Choose ε and μ_1 so small that

$$0 < \varepsilon + \mu c_1 < \min\left(\tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t_1)|)^{n+1}},\ \tfrac{1}{2}\theta T^{n+1}\right).$$

Then for $t_1 \leqq t \leqq t_1 + T$, $|\mu| < \mu_1$ we get

$$\varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t_1)|)^{n+1}},$$

$$\varrho(z_\mu(t_1 + T), M) \geqq \tfrac{1}{2}\theta T^{n+1} > \sigma.$$

Inequalities (8) and (13) imply

$$\varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda\nu}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t)|)^{n+1}}, \quad t_1 \leqq t \leqq t_1 + T \tag{14}$$

and

$$\varrho(z_\mu(t_1 + T), M) \geqq \sigma$$

which proves Sublemma 3.

Since at the end of the evasion maneuver the solution $z_\mu(t)$ is outside of the σ-neighborhood of M and the number T is fixed, it is possible to continue the game for an arbitrarily long time, provided the conditions (14) are fulfilled. Theorem 1 is proved.

Example. Let the game be described by the following system of differential equations

$$\begin{aligned} x^{(p)} + A_1 x^{(p-1)} + \ldots + A_p x &= u + \mu g_1(x, y, x', y', \ldots, x^{(s)}, y^{(s)}, u, v) \\ y^{(q)} + B_1 y^{(q-1)} + \ldots + B_q y &= v + \mu g_2(x, y, x', y', \ldots, x^{(s)}, y^{(s)}, u, v) \end{aligned} \tag{15}$$

where $x, y \in R^m$, $m \geqq 2$, A_i, $i = 1, 2, \ldots, p$, B_i, $i = 1, 2, \ldots, q$ are constant matrices, $s < \min(p, q)$, $g_i(z_1, z_2, \ldots, z_{2m(s+1)}, u, v)$, $i = 1, 2$ are continuous and bounded on $R^{2m(s+1)} \times U \times V$, U, V are compact sets, $\mu \in (-\infty, \infty)$ is a parameter. Let $M = \{z = (x, y) \in R^m \times R^m \mid x - y = 0\}$. The orthogonal complement of M is $M^\perp = \{z = (x, y) \in R^m \times R^m \mid x + y = 0\}$. The matrix of the projection on $M^\perp$ is

$$\pi = \tfrac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad \text{and} \quad \hat{\pi} = \tfrac{1}{2}\begin{pmatrix} \hat{I} & -\hat{I} \\ -\hat{I} & \hat{I} \end{pmatrix},$$

where I is the unit $m \times m$ matrix.

(1) Suppose $q < p$. Then the system (15) has the following form

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} F(t) & 0 \\ 0 & G(t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \begin{pmatrix} S^p * P(S) & 0 \\ 0 & S^q * Q(S) \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$P(S) = \hat{I} + C_1(S) + C_1^2(S) + \ldots, \quad C_1(S) = -(S * A_1 + \ldots + S^p * A_p),$$