ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 102 * PRAHA 22. 8. 1977 * ČÍSLO 3

PERMUTABLE TOLERANCES

IVAN CHAJDA, Přerov, and BOHDAN ZELINKA, Liberec
(Received October 29, 1975)

1. In the papers [1] and [2] the so-called compatible tolerances on algebras are introduced. In the papers [2] and [3], existence conditions for compatible tolerances which are not congruences are investigated. In the paper [4] it is proved that the set of all compatible tolerances on a given algebra forms a lattice, some of whose properties are the same as those of the lattice of all congruences on this algebra or are analogous to them.

In [6] the importance of the permutability of congruences at investigating the lattice of all congruences of a given algebra is shown. For example, if all congruences on an algebra $\mathfrak{A}$ are permutable, then this lattice is modular, and if $\mathfrak{A}$ has a one-element sublagebra, then a generalization of Schreier's theorem on refinements (see Theorem 88 in [6]) holds for the congruences. Thus it is a natural problem to study which analoga hold between permutable congruences and permutable compatible tolerances on a given algebra.

2. By the symbol $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ we denote an algebra $\mathfrak{A}$ with the support A and with the set $\mathscr{F}$ of fundamental operations. If $\mathfrak{A}$ is a lattice, then we shall not distinguish an algebra and its support, i.e. for a lattice L, the symbol L denotes also the support of this lattice.

Definition 1. Let A be a set. Each reflexive and symmetric binary relation on A is called a *tolerance* on A. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra and let T be a tolerance on A. The tolerance T is called *compatible with* $\mathfrak{A}$, if each n-ary operation $f \in \mathscr{F}$ and arbitrary $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of A for which $a_i T b_i$ for $i = 1, \ldots, n$ satisfy $f(a_1, \ldots, a_n) \, T f(b_1, \ldots, b_n)$.

In the paper [4] it is proved that the set of all compatible tolerances on an algebra $\mathfrak{A}$ forms a complete lattice with respect to the set inclusion. By $LT(\mathfrak{A})$ we denote the lattice of all compatible tolerances on the algebra $\mathfrak{A}$.

The lattice operations in $LT(\mathfrak{A})$ will be denoted by the symbols $\vee$ (join), $\wedge$ (meet). Further, $K(\mathfrak{A})$ denotes the lattice of all congruences on the algebra $\mathfrak{A}$ and the symbol $\uplus$ denotes the join in the lattice $K(\mathfrak{A})$. By the symbol $\cup$ we shall denote the set union of two tolerances (taken as subsets of the Cartesian power of the corresponding set).

From the definition it is evident that each congruence on an algebra $\mathfrak{A}$ is a tolerance compatible with $\mathfrak{A}$.

Definition 2. Let A be a set, let R_1, R_2 be two binary relations on A. The relations R_1, R_2 are called *permutable,* if $R_1 . R_2 = R_2 . R_1$, where the symbol "$\cdot$" denotes the product of relations.

3. In [6], supplement of Theorem 86, it is proved that if C_1, C_2 are permutable congruences on an algebra $\mathfrak{A}$, then $C_1 . C_2 = C_1 \uplus C_2$. We shall study the interrelation between $T_1 . T_2$ and $T_1 \vee T_2$ for compatible tolerances T_1, T_2 on $\mathfrak{A}$.

Lemma. *Let A be a set, let T_1, T_2 be two tolerances on A. Then $T_1 . T_2$ is a tolerance on A if and only if $T_1 . T_2 = T_2 . T_1$, i.e. if T_1 and T_2 are permutable.*

Proof is straightforward.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let T_1, T_2 be two permutable tolerances from $LT(\mathfrak{A})$. Then $T_1 . T_2 \in LT(\mathfrak{A})$ and $T_1 \cup T_2 \subseteq T_1 \vee T_2 \subseteq T_1 . T_2 \subseteq$ $\subseteq (T_1 \cup T_2)^2 \subseteq (T_1 \vee T_2)^2$.*

Proof. By Theorem 3 from [5] we have $T_1 . T_2 \in LT(\mathfrak{A})$. Since $U \subseteq V$ and $R \subseteq S$ implies $R . U \subseteq S . V$ for any binary relation on A, the proof of the assertion concerning the inclusions is immediate.

Corollary. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $T_1 \in LT(\mathfrak{A})$, $T_2 \in LT(\mathfrak{A})$. Let T_1, T_2 be permutable, let $T_1 \vee T_2$ be a congruence. Then $T_1 . T_2 = T_1 \vee T_2$.*

Proof. As $T_1 \vee T_2$ is a congruence, we have $(T_1 \vee T_2)^2 = T_1 \vee T_2$. Thus $T_1 \vee T_2 \subseteq T_1 . T_2 \subseteq T_1 \vee T_2$, which means $T_1 \vee T_2 = T_1 . T_2$.

4. In the paper [7] it is proved that for every distributive lattice L the lattice $K(L)$ is a sublattice of $LT(L)$, if and only if $K(L) = LT(L)$, i.e. if each compatible tolerance on L is a congruence (see Corollary 2 in [7]). Now we shall show that this condition follows from the condition of equality of product and join for congruences from $LT(L)$.

Theorem 2. *Let L be a distributive lattice. For any two congruences C_1, C_2 on L let $C_1 . C_2 = C_1 \vee C_2$. Then each compatible tolerance on L is a congruence.*

Proof. Let the condition be fulfilled. Then any two congruences on L are permutable, because $C_1 . C_2 = C_1 \vee C_2 = C_2 \vee C_1 = C_2 . C_1$ for any two congruences C_1, C_2 on L. The product $C_1 . C_2$ is a congruence on L (see for example [8]). But $C_1 . C_2 = C_1 \vee C_2$ is the least compatible tolerance on L which contains C_1 and C_2. As it is a congruence, we have $C_1 \vee C_2 = C_1 \uplus C_2$. The meet in both $K(L)$ and $LT(L)$ is the set intersection, thus $K(L)$ is a sublattice of $LT(L)$. By Corollary 2 from [7] this assertion is equivalent to the assertion of the theorem.

Remark. The assertion can be proved also by using the result from [9] saying that if all congruences on an algebra $\mathfrak{A}$ from a given variety are permutable, then each compatible reflexive relation on $\mathfrak{A}$ is a congruence.

5. Now we shall prove some theorems on permutable compatible tolerances and their transitive hulls. In [4] it is proved that the transitive hull of a tolerance compatible with an algebra $\mathfrak{A}$ is a congruence on $\mathfrak{A}$.

Theorem 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let T_1, T_2 be two permutable tolerances from $LT(\mathfrak{A})$. Let C_1, C_2 be the transitive hulls of T_1, T_2 respectively. Then C_1 and C_2 are permutable and $C_1 . C_2$ is the transitive hull of $T_1 . T_2$.*

Proof. We have $C_1 = \bigcup_{n=1}^{\infty} T_1^n$, $C_2 = \bigcup_{n=1}^{\infty} T_2^n$. Let $a\, C_1 . C_2\, b$ for some $a \in A$, $b \in A$. This means that there exists $c \in A$ such that $a\, C_1\, c$, $c\, C_2\, b$. Now there exist positive integers m, n such that $a\, T_1^m\, c$, $c\, T^n\, b$. Thus $a\, T_1^m . T_2^n\, b$. As T_1, T_2 are permutable, so are T_1^m, T_2^n and we have $a T_2^n . T_1^m b$. There exists $d \in A$ such that $a\, T_2^n\, d$, $d\, T_1^m\, b$. But $T_2^n \subseteq C_2$, $T_1^m \subseteq C_1$ and we have $a\, C_2\, d$, $d\, C_1\, b$, which means $a\, C_2 . C_1\, b$. As a, b were chosen arbitrarily, we have $C_1 . C_2 \subseteq C_2 . C_1$. Analogously we can prove the inverse inclusion and thus $C_1 . C_2 = C_2 . C_1$. Now as $T_1 \subseteq C_1$, $T_2 \subseteq C_2$, we have $T_1 . T_2 \subseteq C_1 . C_2$, thus also the transitive hull of $T_1 . T_2$ is contained in $C_1 . C_2$. Let $x\, C_1 . C_2\, y$ for $x \in A$, $y \in A$. Then there exists $z \in A$ such that $x\, C_1\, z$, $z\, C_2\, y$. This means that there exist positive integers r, s such that $x\, T_1^r\, z$, $z\, T_2^s\, y$. Let $t = \max(r, s)$. As T_1, T_2 are reflexive, the inequalities $r \leqq t$, $s \leqq t$ imply the inclusions $T_1^r \subseteq T_1^t$, $T_2^s \subseteq T_2^t$. We have $x\, T_1^t\, z$, $z\, T_2^t\, y$, which means $x T_1^t . T_2^t y$. As T_1, T_2 are permutable, we have $T_1^t . T_2^t = (T_1 . T_2)^t$. Thus $C_1 . C_2 \subseteq \bigcup_{n=1}^{\infty} (T_1 . T_2)^n$, but the right-hand side of this inclusion is the transitive hull of $T_1 . T_2$. We have proved that this transitive hull is equal to $C_1 . C_2$.

Theorem 4. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $T \in LT(\mathfrak{A})$. Let C be the transitive hull of T. Then $C . T = T . C = C$.*

Proof. Let I denote the identity relation on A. Then $I \subseteq T \subseteq C$ and, by the remark above, we have

$$T = I . T \subseteq C . T \subseteq C . C = C = C . I \subseteq C . T .$$

Hence $C = C . T$ and, similarly, $C = T . C$.

Corollary 2. *Let* $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ *be an algebra. Let* $\mathfrak{S}$ *be the maximal (with respect to set inclusion) set of compatible tolerances on* $\mathfrak{A}$ *such that any two tolerances from this set are permutable. Then* $\mathfrak{S}$ *is a commutative semigroup with the property that each monogenous subsemigroup of* $\mathfrak{S}$ *either is infinite, or has the period one. The unit element of* $\mathfrak{S}$ *is the identity relation on* A*, the zero element of* $\mathfrak{S}$ *is the universal relation on* A.

References

[1] *B. Zelinka:* Tolerance in algebraic structures. Czech. Math. J. *20* (1970), 179—183.
[2] *B. Zelinka:* Tolerance in algebraic structures II. Czech. Math. J. *25* (1975), 175—178.
[3] *I. Chajda* and *B. Zelinka:* Tolerance relations on lattices. Čas. pěstov. mat. *99* (1974), 394 to 399.
[4] *I. Chajda* and *B. Zelinka:* Lattices of tolerances. Čas. pěstov. mat. *102* (1977), 10—24.
[5] *I. Chajda:* Systems of equations and tolerance relations. Czech. Math. J. *25* (*100*) 1975, 302—308.
[6] *G. Szász:* Introduction to lattice theory. Akad. Kiadó Budapest 1963.
[7] *I. Chajda* and *B. Zelinka:* Minimal compatible tolerances on lattices. Czech. Math. J. (to appear).
[8] *А. Г. Курош:* Лекции по общей алгебре. Москва 1962.
[9] *H. Werner:* A Mal'cev condition or admissible relations. Algebra Univ. *3* (1973), p. 263.

Authors' addresses: I. Chajda, 750 00 Přerov, tř. Lidových milicí 290. B. Zelinka, 460 01 Liberec 1, Komenského 2 (katedra matematiky VŠST).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON A CLASS OF NONLINEAR EVASION GAMES

MILAN MEDVEĎ, Bratislava

(Received October 29, 1975)

In this paper we shall consider a differential game described by the system of differential equations

$$(1)\qquad z^{(n)} + A_1 z^{(n-1)} + \ldots + A_{n-1} z' + A_n z = $$
$$= f(u, v) + \mu g(z, z', \ldots, z^{(n-1)}, u, v),$$

where $z \in R^m$, $f \in R^m$, A_i, $i = 1, 2, \ldots, n$ are constant matrices, $f(u, v)$ is a continuous function of the point $(u, v) \in U \times V$, $U \subset R^p$, $V \subset R^q$ are compact sets, $\mu \in (-\infty, \infty)$ is a parameter. We shall suppose that the function $g(z_1, z_2, \ldots, z_n, u, v)$ is continuous and bounded on $R^{mn} \times U \times V$.

In the paper [1] a sufficient condition for existence of evasion strategy for a differential game described by equation (1) for $\mu = 0$ is given. In the paper [2] a sufficient condition for existence of such strategy for a game described by a first order system of differential equations of type (1) is given. That condition is different from the condition given in our paper. Our condition is similar to that given in [1]. Similarly to [1] we shall use the technique of convolutions in the formulation of results as well as in the proof.

A mapping $V_u(t, Z_0)$ defined on the set of measurable controls $u(\tau)$, $0 \leqq \tau < \infty$, $u(\tau) \in U$ depending on $t \geqq 0$ and on the vector of initial conditions $Z_0 = (z_0, z_0', \ldots$ $\ldots, z_0^{(n-1)})$ is said to be a strategy, if it possesses the following properties:

(1) For an arbitrary measurable control $u(\tau)$, $0 \leqq \tau < \infty$ and for an arbitrary fixed Z_0, the mapping $V_u(t, Z_0)$ is measurable as a function of t and has values in V.

(2) If $u_1(\tau)$, $u_2(\tau)$, $0 \leqq \tau < \infty$ are two controls and $u_1(\tau) = u_2(\tau)$ almost everywhere on $[0, T]$, where T is arbitrary, then $V_{u_1}(t, Z_0) = V_{u_2}(t, Z_0)$ almost everywhere on $[0, T]$ for every Z_0.

Let M be a subspace of R^m of a dimension $\leqq m - 2$. Our problem is to choose a strategy $V_u(t, Z_0)$ such that the solution $z(t)$, $0 \leqq t < \infty$ of the equation

$$z^{(n)} + A_1 z^{(n-1)} + \ldots + A_n z =$$
$$= f(u(t), V_u(t, Z_0)) + \mu g(z(t), \ldots, z^{(n-1)}, u(t), V_u(t, Z_0))$$

with the initial condition

$$Z(0) = (z(0), z'(0), \ldots, z^{(n-1)}(0)) = Z_0\,, \quad z(0) \notin M$$

does not intersect the subspace M for any $t \geqq 0$, for an arbitrary control $u(t)$ and for an arbitrary vector Z_0. We shall call this strategy an evasion strategy.

Now, using the convolution symbolism (cf. [1]) we can rewrite the equation (1) in the form

$$z^{(n)} + \hat{A}_1 * z^{(n-1)} + \ldots + \hat{A}_n * z = f(u, v) + \mu g(z, z', \ldots, z^{(n-1)}, u, v)$$

and express the solution of this equation by the following formula:

(2)
$$z_\mu = z_0 + S * z_0' + \ldots + S^{n-1} * z_0^{(n-1)} +$$
$$+ S^n * (\Phi_0 * z_0 + \ldots + \Phi_{n-1} * z_0^{(n-1)}) + S^n * R(S) * f(u, v) +$$
$$+ \mu S^n * R(S) * g(z, z', \ldots, z^{(n-1)}, u, v)\,,$$

where $\Phi_0, \Phi_1, \ldots, \Phi_{n-1}$ are certain entire matrices over the Mikusiński ring $\mathscr{M}$ (cf. [1]),

$$R(S) = \hat{I} + C(S) + C^2(S) + \ldots\,,$$
$$C(S) = -(S * \hat{A}_1 + S^2 * \hat{A}_2 + \ldots + S^n * \hat{A}_n)\,,$$

$\hat{I} = \mathrm{diag}\,(\delta, \delta, \ldots, \delta)$ is the unit matrix, δ is the unit element in the ring $\mathscr{M}$, $\hat{A}_i$, $i = 1, 2, \ldots, n$ are constant matrices, i.e. the functions identically equal to A_i. It was shown in [1] that the series for $R(S)$ converges uniformly in a disc with center at the origin of an arbitrary large radius ϱ.

Let L be a subspace of R^m of a dimension $k \geqq 2$ which lies in the orthogonal complement of $M \subset R^m$ and let $\pi : R^m \to R^k$ be a linear mapping corresponding to the orthogonal projection of R^m onto L.

We assume that

(3)
$$\hat{\pi} * R(S) * f(u, v) = H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + \chi(t)\,,$$

where

(a) $\Psi_i(u, v)$ are continuous in $(u, v) \in U \times V$, $i = 0, 1, 2, \ldots$.

(b) $|\Psi_i(u, v)| \leqq \lambda_i$ for all $(u, v) \in U \times V$, $|\cdot|$ being the Euclidean norm in R^k and the series $\hat{\lambda}_0 + S * \hat{\lambda}_1 + S^2 * \hat{\lambda}_2 + \ldots$ is an entire function of the variable t.

(c) $H(S)$ is an entire matrix over the ring $\mathscr{M}$ and $\det^* H(S) \neq 0$. ($\det^* H(S)$ is calculated as a determinant in the ordinary formal way using the ring multiplication).

(d) The function $\chi(t)$ does not depend on u, v.

(e) Denote by $[\Psi_0(u, v)]$ the smallest linear subspace of R^k containing all points $\Psi_0(u, v)$, $(u, v) \in U \times V$. Let us suppose that the subspace $[\Psi_0(u, v)]$ has the largest possible dimension among all representations (3).

We shall say that the parameter v in the expression $\hat{\pi} * R(S) * f(u, v)$ has complete maneuverability, if the set

$$(4) \qquad \bigcap_{u \in U} \mathrm{co}_v \, \Psi_0(u, v) \subset R^k$$

contains interior points, where $\mathrm{co}_v \, \Psi_0(u, v)$ denotes the convex hull of the set of all points $\Psi_0(u, v)$, $v \in V$ for fixed $u \in U$.

Now, we can formulate a sufficient condition for evasion.

Theorem 1. *If the parameter v in the expression $\hat{\pi} * R(S) * f(u, v)$ has complete maneuverability, then there exists a number $\mu_1 > 0$ such that for all μ, $|\mu| < \mu_1$ there exists an evasion strategy. Moreover, there exist numbers λ, ν, $\theta > 0$ and an integer l such that*

$$(5) \qquad \varrho(z_\mu(t), M) \geqq \frac{1}{2} \theta \left(\frac{(z_\mu(0), M)}{\lambda \nu} \right)^{n+1} \frac{1}{(1 + |z_\mu(t)|)^{n+1}}$$

for $0 \leqq t < \infty$, where $\varrho(z_\mu(t), M)$ is the distance of the point $z_\mu(t)$ from the subspace M ($z_\mu(t)$ denotes the solution of (1) corresponding to a value μ of the parameter).

Remark. The number l in Theorem 1 is equal to the number l_k, where

$$H(S) = H^{(1)}(S) * \mathrm{diag}\,(S^{l_1}, \ldots, S^{l_k}) * H^{(2)}(S),$$

$l_1 \leqq l_2 \leqq \ldots \leqq l_k$, $H^{(i)}(S)$, $i = 1, 2$ are entire invertible matrices. It was shown in [1] that an arbitrary entire matrix $H(S)$ has such a representation.

For the sake of simplicity of computations, we can assume that the origin of R^k is an interior point of the set (4). Denote by Q the closed k-dimensional cube with the center at the origin and with sides parallel to the axes and such that $Q \subset \subset \mathrm{int} \bigcap_{u \in U} \mathrm{co}_v \, \psi_0(u, v)$ ($\mathrm{int}\, P$ denotes the interior of P).

For the proof of Theorem 1 we need the following lemma, which was proved in [1].

Lemma 1. *For sufficiently small Q there exists a number $T > 0$ such that for any $\varepsilon > 0$ there exists a measurable function $v(t) \in V$, $0 \leqq t \leqq T$ such that*

$$(6) \qquad \|S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots + \chi(t)] + t^{n+1}\xi\| \leqq \varepsilon$$

for $0 \leqq t \leqq T$ and for an arbitrary preassigned $u(t) \in U$, $\xi \in Q$. For the calculation of $v(t)$ we need the values $u(t)$ on the interval $[0, t]$ and the point ξ only.

Remark. $\|p(t)\| = \sup_{t \in [0,T]} |\int_0^t p(\tau)\, \mathrm{d}\tau|$, where $|\cdot|$ is the Euclidean norm in R^k.

Proof of Theorem 1. From (2), (3) we get

$$\hat{\kappa} * z_\mu(t) = \varphi(t, Z_0) + S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + \chi(t)] + + \mu S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v),$$

where

$$\varphi(t, Z_0) = \hat{\kappa} * [z_0 + S * z_0 + \ldots + S^{n-1} * z_0^{(n-1)} + + S^n * (\Phi_0 * z_0 + \ldots + \Phi_{n-1} * z_0^{(n-1)})].$$

Sublemma 1. *If $\mu_1 > 0$ is a given number and $\varrho(z_\mu(0), M) > 0$ for $|\mu| < \mu_1$, then* (a) *for a sufficiently large number λ*

(7) $$\varrho(z_\mu(t), M) \geqq \frac{\varrho(z_\mu(0), M)}{2} \quad for \quad 0 \leqq t \leqq \frac{\varrho(z_\mu(0), M)}{\lambda(1 + |Z_\mu(0)|)},$$

$$|\mu| < \mu_1, \quad Z_\mu(0) = (z_\mu(0), z'_\mu(0), \ldots, z_\mu^{(n-1)}(0)) = Z_0.$$

(b) *If T is sufficiently small, then there exists a number $\nu > 0$ such that for an arbitrary Z_0 and for $|\mu| < \mu_1$*

(8) $$\nu(1 + |Z_\mu(t)|) \geqq 1 + |Z_0|, \quad 0 \leqq t \leqq T,$$

$$(Z_\mu(t) = (z_\mu(t), z'_\mu(t), \ldots, z_\mu^{(n-1)}(t))).$$

The proof of Sublemma 1 is analogous to the proof of inequalities (5.4), (5.5) in [1].

Sublemma 2. *There exists a $\theta > 0$ so small that for an arbitrary initial vector Z_0, there exists a point $\xi(Z_0) \in Q$ satisfying the condition*

(9) $$|\varphi(t, Z_0) - S * t^{n+l-1} \xi(Z_0)| \geqq \theta t^{n+1}, \quad 0 \leqq t \leqq T.$$

Proof. By [1, Lemma 5.1] there exist a point $\xi(Z_0) \in Q$ and a number $\theta' > 0$ such that

$$\left| \frac{(n+1)\, \varphi(t, Z_0)}{t^{n+1}} - \xi(Z_0) \right| \geqq \theta'.$$

This implies

$$\left| \varphi(t, Z_0) - \frac{t^{n+1}}{n+1} \xi(Z_0) \right| = |\varphi(t, Z_0) - S * t^{n+l-1} \xi(Z_0)| \geqq \theta t^{n+1},$$

where $\Theta = \Theta'/(n+1)$.

Now, we choose a number $\sigma > 0$, which satisfies the following inequalities:

(10) $$\sigma < \tfrac{1}{2}\theta T^{n+1}, \quad \sigma < \lambda T, \quad \frac{\sigma}{2} > \theta \left(\frac{\sigma}{\lambda} \right)^{n+1},$$

where λ can be chosen arbitrarily large.

Let us suppose that at the beginning of the game at time $t = 0$ it is $\varrho(z_\mu(0), M) > \sigma$. Choose the control $v(t)$ arbitrarily. If for some $t = t_1$, $\varrho(z_\mu(t_1), M) = \sigma$, then define a control $v(t)$ on the interval $[t_1, t_1 + T]$ in the following way

$$v(t) = w(t - t_1, u, \xi(Z_\mu(t_1)), \varepsilon), \tag{11}$$

where $w(t, u, \xi, \varepsilon)$ is a control satisfying the inequality (9) for given $\varepsilon > 0$, $u(t) \in U$ and $\xi \in Q$.

Sublemma 3. *If $v(t)$ is a control defined by the equality* (11), *then there exists a number $\mu_1 > 0$ such that for $|\mu| < \mu_1$*

(12) (a) $$\varrho(z_\mu(t), M) \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t)|)^{n+1}}, \quad t_1 \leqq t \leqq t_1 + T$$

(b) $$\varrho(z_\mu(t_1 + T), M) \geqq \sigma.$$

Proof. From (7), (8) it follows that for

$$0 \leqq t - t_1 \leqq \frac{\varrho(z_\mu(t_1), M)}{\lambda(1 + |Z_\mu(t_1)|)} = \frac{\sigma}{\lambda(1 + |Z_\mu(t_1)|)},$$

$$\varrho(z_\mu(t), M) \geqq \frac{\sigma}{2} \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \geqq \theta \left(\frac{\sigma}{\lambda}\right)^{n+1} \frac{1}{(1 + |Z_\mu(t_1)|)^{n+1}} - \varepsilon. \tag{13}$$

$$\begin{aligned}
\varrho(z_\mu(t), M) &= |\hat{\pi} * z_\mu(t)| = |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1)) + \\
&+ S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1)) + S^n * [H(S) * (\Psi_0 + S * \Psi_1 + \ldots) + X(t)] + \\
&+ \mu S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \geqq \\
&\geqq |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \\
&- |S^n * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + X(t)] + \\
&+ S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \mu |S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \geqq \\
&\geqq |\varphi(t - t_1, Z_\mu(t_1)) - S * (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))| - \\
&- \|S^{n-1} * [H(S) * (\Psi_0(u, v) + S * \Psi_1(u, v) + \ldots) + X(t)] + \\
&+ (t - t_1)^{n+l-1}\, \xi(Z_\mu(t_1))\| - \mu |S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)|.
\end{aligned}$$

Since $|g(z_1, z_2, \ldots, z_n, u, v)| \leqq c$ for all $(z_1, z_2, \ldots, z_n, u, v) \in R^{mn} \times U \times V$, where $c > 0$ is constant, there exists a constant $c_1 > 0$ such that for $|\mu| \leqq \mu_1$, $0 \leqq t \leqq T + t_1$ it is $|S^n * R(S) * g(z_\mu, z'_\mu, \ldots, z_\mu^{(n-1)}, u, v)| \leqq c_1$. Therefore, using Sublemma 1 and Sublemma 2 we conclude

$$\varrho(z_\mu(t), M) \geqq \theta(t - t_1)^{n+1} - \varepsilon - \mu c_1.$$

Choose ε and μ_1 so small that

$$0 < \varepsilon + \mu c_1 < \min\left(\tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{n+1}\frac{1}{(1+|Z_\mu(t_1)|)^{n+1}},\ \tfrac{1}{2}\theta T^{n+1}\right).$$

Then for $t_1 \leqq t \leqq t_1 + T$, $|\mu| < \mu_1$ we get

$$\varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda}\right)^{n+1}\frac{1}{(1+|Z_\mu(t_1)|)^{n+1}},$$

$$\varrho(z_\mu(t_1+T), M) \geqq \tfrac{1}{2}\theta T^{n+1} > \sigma .$$

Inequalities (8) and (13) imply

$$(14)\qquad \varrho(z_\mu(t), M) \geqq \tfrac{1}{2}\theta\left(\frac{\sigma}{\lambda\nu}\right)^{n+1}\frac{1}{(1+|Z_\mu(t)|)^{n+1}},\quad t_1 \leqq t \leqq t_1 + T$$

and

$$\varrho(z_\mu(t_1+T), M) \geqq \sigma$$

which proves Sublemma 3.

Since at the end of the evasion maneuver the solution $z_\mu(t)$ is outside of the σ-neighborhood of M and the number T is fixed, it is possible to continue the game for an arbitrarily long time, provided the conditions (14) are fulfilled. Theorem 1 is proved.

Example. Let the game be described by the following system of differential equations

$$(15)\qquad \begin{aligned} x^{(p)} + A_1x^{(p-1)} + \ldots + A_px &= u + \mu g_1(x, y, x', y', \ldots, x^{(s)}, y^{(s)}, u, v)\\ y^{(q)} + B_1y^{(q-1)} + \ldots + B_qy &= v + \mu g_2(x, y, x', y', \ldots, x^{(s)}, y^{(s)}, u, v)\end{aligned}$$

where $x, y \in R^m$, $m \geqq 2$, A_i, $i = 1, 2, \ldots, p$, B_i, $i = 1, 2, \ldots, q$ are constant matrices, $s < \min(p, q)$, $g_i(z_1, z_2, \ldots, z_{2m(s+1)}, u, v)$, $i = 1, 2$ are continuous and bounded on $R^{2m(s+1)} \times U \times V$, U, V are compact sets, $\mu \in (-\infty, \infty)$ is a parameter. Let $M = \{z = (x, y) \in R^m \times R^m \mid x - y = 0\}$. The orthogonal complement of M is $M^\perp = \{z = (x, y) \in R^m \times R^m \mid x + y = 0\}$. The matrix of the projection on $M^\perp$ is

$$\pi = \tfrac{1}{2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix} \quad\text{and}\quad \hat{\pi} = \tfrac{1}{2}\begin{pmatrix} \hat{I} & -\hat{I}\\ -\hat{I} & \hat{I}\end{pmatrix},$$

where I is the unit $m \times m$ matrix.

(1) Suppose $q < p$. Then the system (15) has the following form

$$z(t) = \begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} F(t) & 0\\ 0 & G(t)\end{pmatrix}\begin{pmatrix} x(0)\\ y(0)\end{pmatrix} + \begin{pmatrix} S^p * P(S) & 0\\ 0 & S^q * Q(S)\end{pmatrix} * \begin{pmatrix} u\\ v\end{pmatrix},$$

where

$$P(S) = \hat{I} + C_1(S) + C_1^2(S) + \ldots,\quad C_1(S) = -(S * A_1 + \ldots + S^p * A_p),$$

$$Q(S) = \hat{I} + C_2(S) + C_2^2(S) + \dots, \quad C_2(S) = -(S * B_1 + \dots + S^q + B_q),$$

$$\hat{\pi} * \begin{pmatrix} S^p * P(S) & 0 \\ 0 & S^q * Q(S) \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} = S^q * \hat{\pi} * \begin{pmatrix} S^{p-q} * P(S) & 0 \\ 0 & Q(S) \end{pmatrix} * \begin{pmatrix} u \\ v \end{pmatrix} =$$

$$= S^q * \hat{\pi} * \left[\begin{pmatrix} 0 & 0 \\ 0 & Q(S) \end{pmatrix} + S^{p-q} * \begin{pmatrix} P(S) & 0 \\ 0 & 0 \end{pmatrix} \right] * \begin{pmatrix} u \\ v \end{pmatrix} =$$

$$= S^q * \left[\begin{pmatrix} -Q(S) * v \\ Q(S) * v \end{pmatrix} + S^{p-q} * \begin{pmatrix} P(S) * u \\ -P(S) * u \end{pmatrix} \right] =$$

$$= S^q * \left[\begin{pmatrix} -v \\ v \end{pmatrix} + S * \Psi_1(u, v) + S^2 * \Psi_2(u, v) + \dots \right],$$

i.e. $\Psi_0(u, v) = \begin{pmatrix} -v \\ v \end{pmatrix}$. Therefore, if the convex hull of the set V contains an interior point, then the set

$$\bigcap_{u \in U} \operatorname{co}_v \Psi_0(u, v) = \operatorname{co}_v \begin{pmatrix} -v \\ v \end{pmatrix}$$

contains an interior point as well. The conditions of Theorem 1 are fulfilled and so for sufficiently small μ there exists an evasion strategy and

$$\varrho(z_\mu(t), M) \geqq \frac{1}{2} \left(\frac{\varrho(z_\mu(0), M)}{\lambda \nu} \right)^q \frac{1}{(1 + |Z_\mu(t)|)^q}$$

for λ, ν sufficiently large, where θ is a positive constant.

(2) It is possible to compute that for $p = q$ the vector $\Psi_0(u, v) = \begin{pmatrix} u - v \\ v - u \end{pmatrix}$. To satisfy the condition $\operatorname{int} \bigcap_{u \in U} \operatorname{co}_v \Psi_0(u, v) \neq \emptyset$ it suffices to satisfy the condition: $\operatorname{int} \operatorname{co} V \neq \emptyset$ and $U \subset^* \operatorname{int} \operatorname{co} V$, where $\operatorname{co} V$ is the convex hull of V and $U \subset^* \operatorname{int} \operatorname{co} V$ means that there exists a vector $a \in R^k$ such that $U + a = \{u + a \mid u \in U\} \subset \subset \operatorname{int} \operatorname{co} V$.

This example for $\mu = 0$ was shown by R. V. GAMKRELIDZE in his lecture during the semester on optimal control theory held in the S. Banach International Mathematical Center in Warsaw in 1973.

References

[1] *R. V. Gamkrelidze* and *G. L. Kharatishvili:* A Differential Game of Evasion With Nonlinear Control, SIAM Journal on Control, Vol. *12*, Number 2 (1974).

[2] *Н. Сатимов:* Задача убегания для одного класса нелинейных дифференциальных игр, Дифф. урав. Т. *XI.*, Но. 4 (1975).

Author's address: 886 25 Bratislava, ul. Obrancov mieru 50 (Matematický ústav SAV).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

REMARK ON THE THEOREM OF EGOROFF

WŁADYSŁAW WILCZYŃSKI, ŁÓDŹ

(Received October 29, 1975)

I. VRKOČ in [1] has proved the following theorem: *there exists a real function f defined and measurable on* $[0, 1]$ *such that there does not exist a countable family* $\{A_n\}$ *of sets fulfilling* $\bigcup_n A_n = [0, 1]$ *such that the restricted function* $f \mid A_n$ *is continuous for every* n. The theorem of Vrkoč is a refinement of the well known theorem of Lusin. In this short note we shall prove the theorem which can be considered as a similar refinement of the theorem of Egoroff.

Before stating the theorem we shall prove the following lemma:

Lemma. *Let* $\{n(k, i)\}$ *be a double sequence of natural numbers, which for every* k *is increasing with respect to the variable* i. *There exists an increasing sequence* $\{n(i)\}$ *of natural numbers such that for every* k *and for every* $i \geqq k$

$$n(i) > n(k, i) .$$

Proof. Put $n(i) = 1 + \max(n(1, i), n(2, i), \ldots, n(i, i))$ for every natural i. It is easy to see that the sequence $\{n(i)\}$ fulfills all required conditions.

Theorem. *For every set* A *of the power of continuum there exists a sequence of real functions* $\{f_n\}$ *defined on* A *such that* $f_n(x)$ *tends to zero for every* $x \in A$ *and there does not exist a countable family* $\{A_k\}$ *of sets fulfilling* $\bigcup_k A_k = A$ *such that the restricted sequence* $\{f_n \mid A_k\}$ *is uniformly convergent for every* k.

Proof. Let N be a set of all increasing sequences of natural numbers. Of course, N is a set of the power of continuum. Let $\Phi : A \to_{\text{onto}} N$ be a one-to-one correspondence.

For $x \in A$ let us put $f_{n(1)}(x) = 1^{-1}$, $f_{n(2)}(x) = 2^{-1}, \ldots, f_{n(i)}(x) = i^{-1}, \ldots$ and $f_j(x) = 0$ for remaining natural j, where $\{n(1), n(2), \ldots, n(i), \ldots\} = \Phi(x)$.

So we have defined a sequence of real functions $\{f_n\}$ and it is easy to verify that $f_n(x) \to 0$ for every $x \in A$.

Suppose that there exists a sequence $\{A_k\}$ of sets such that $\bigcup_k A_k = A$ and $\{f_n \mid A_k\}$ tends uniformly to 0 for every k.

Let for fixed k the sequence $\{n(k, i)\}$ of variable i be a sequence of natural numbers corresponding to $\varepsilon = 1^{-1}, \varepsilon = 2^{-1}, \ldots, \varepsilon = i^{-1}, \ldots$ and to uniform convergence of $\{f_n\}$ on A_k, i.e. for every i, for every $j > n(k, i)$ and for every $x \in A_k$ we have $|f_j(x)| < i^{-1}$. Obviously we can choose $\{n(k, i)\}$ to be increasing with respect to i. If k changes in the set of natural numbers, we obtain a double sequence $\{n(k, i)\}$. In virtue of the lemma there exists an increasing sequence $\{n(i)\}$ such that for every k and for every $i \geqq k$ $n(i) > n(k, i)$. Let $x = \Phi^{-1}(\{n(i)\})$. There exists a natural number k_0 such that $x \in A_{k_0}$. So for $i \geqq k_0$ we have $n(i) > n(k_0, i)$ and $|f_{n(i)}(x)| < < i^{-1}$ and simultaneously from the definition we have $f_{n(i)}(x) = i^{-1}$, a contradiction. The theorem is proved.

Corollary. *There exists a sequence of measurable real functions $\{f_n\}$ defined on $[0, 1]$, which tends to zero at every point and such that there does not exist a sequence $\{A_k\}$ of sets fulfilling $\bigcup_k A_k = [0, 1]$ such that the restricted sequence $\{f_n \mid A_k\}$ is uniformly convergent for every k.*

Proof. It suffices to take in the theorem the set $A \subset [0, 1]$ of the power of continuum and of measure zero and to define additionally $f_n(x) = 0$ for every n and for every $x \notin A$. Then we obtain a sequence of functions which are equal almost everywhere to zero and hence measurable.

References

[1] *Vrkoč, Ivo:* Remark about the relation between measurable and continuous functions, Čas. pro pěst. mat. *96* (1971) p. 225—228.

Author's address: 91-464 Łódź, ul. Zgierska 75/81, m. 226, Poland.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

A REMARK ON ISOTOPIES OF DIGRAPHS AND PERMUTATION MATRICES

BOHDAN ZELINKA, Liberec

(Received January 6, 1976)

In [1], [2], [3] the concepts of isotopy and autotopy of a digraph are studied. Here we shall make a remark on applications of permutation matrices in investigating these concepts.

Let G and G' be two digraphs, let V be the vertex set of G, let V' be the vertex set of G'. The isotopy of G onto G' is an ordered pair $\langle f_1, f_2 \rangle$ of bijections of V onto V' with the property that for any two vertices u, v of G the edge $\overrightarrow{f_1(u) f_2(v)}$ exists in G' if and only if the edge $\overrightarrow{uv}$ exists in G. Two digraphs G and G' are called isotopic, if there exists an isotopy of G onto G'. An autotopy of a digraph is an isotopy of G again onto G.

Here we shall consider digraphs in which loops may exist as well as various edges with the same initial vertex and the same terminal vertex. For these graphs we adapt the definition of the isotopy so that the number of edges going from $f_1(u)$ into $f_2(v)$ in G' is equal to the number of edges going from u into v in G.

If G is a finite digraph with n vertices $u_1, u_2, \ldots, u_n$, then its adjacency matrix $\mathbf{A}_G$ is the $n \times n$ matrix in which the term in the i-th row and the j-th column is equal to the number of edges going from u_i into u_j in G.

Now consider a permutation π of the set of numbers $\{1, 2, \ldots, n\}$. The matrix of the permutation π is the $n \times n$ matrix $\mathbf{P}(\pi)$ in which the term in the i-th row and the j-th column is equal to the Kronecker delta $\delta_i^{p(j)}$. Each matrix which is the matrix of a certain permutation is called a permutation matrix.

We shall recall some well-known properties of permutation matrices.

Proposition 1. *A square matrix $\mathbf{M}$ is a permutation matrix, if and only if exactly one term in each row and exactly one term in each column of $\mathbf{M}$ is equal to* 1 *and all other terms of $\mathbf{M}$ are equal to* 0.

Proposition 2. *Let π_1 and π_2 be two permutations of the set of numbers $\{1, 2, \ldots, n\}$. Then*

$$\mathbf{P}(\pi_1)\,\mathbf{P}(\pi_2) = \mathbf{P}(\pi_2\pi_1)\,.$$

Proposition 3. *Let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. Then the transposed matrix to the matrix $\mathbf{P}(\pi)$ is the inverse matrix to $\mathbf{P}(\pi)$ and is equal to $\mathbf{P}(\pi^{-1})$.*

Now let $\mathbf{M}$ be a matrix with n rows, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. To perform π on the rows of $\mathbf{M}$ means to construct a matrix with n rows in which $\pi(i)$-th row is equal to the i-th row of $\mathbf{M}$. For a matrix $\mathbf{M}$ with n columns we define analogously the meaning of "to perform a permutation on the columns of $\mathbf{M}$".

Proposition 4. *Let $\mathbf{M}$ be a matrix with n rows, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. The product $\mathbf{P}(\pi^{-1})\,\mathbf{M}$ is the matrix obtained from $\mathbf{M}$ by performing the permutation π on its rows.*

Proposition 5. *Let $\mathbf{M}$ be a matrix with n columns, let π be a permutation of the set of numbers $\{1, 2, \ldots, n\}$. The product $\mathbf{M}\,\mathbf{P}(\pi)$ is the matrix obtained from $\mathbf{M}$ by performing the permutation π on its columns.*

Now consider the adjacency matrix $\mathbf{A}_G$ of a digraph G with n vertices.

Theorem 1. *Let G and G' be two finite digraphs with n vertices, let $\mathbf{A}_G$ and $\mathbf{A}_{G'}$ be their adjacency matrices, respectively. The graphs G and G' are isotopic, if and only if there exist permutation $n \times n$ matrices $\mathbf{P}$ and $\mathbf{Q}$ such that*

$$\mathbf{A}_G\mathbf{P} = \mathbf{Q}\mathbf{A}_{G'}\,.$$

Proof. Let G and G' be isotopic, let $\langle f_1, f_2 \rangle$ be an isotopy of G onto G'. The vertices of G are $u_1, \ldots, u_n$, the vertices of G' are $u'_1, \ldots, u'_n$ in the notation corresponding to the adjacency matrices $\mathbf{A}_G$, $\mathbf{A}_{G'}$. The mappings f_1, f_2 are bijections of the vertex set V of G onto the vertex set V' of G'. Let π_1, π_2 be such permutations of the set of numbers $\{1, 2, \ldots, n\}$ that $f_1(u_i) = u'_{\pi_1(i)}, f_2(u_i) = u'_{\pi_2(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then the term of $\mathbf{A}_{G'}$ in the $\pi_1(i)$-th row and the $\pi_2(j)$-th column is equal to the term of $\mathbf{A}_G$ in the i-th row and the j-th column. This means that $\mathbf{A}_{G'}$ is obtained from $\mathbf{A}_G$ by performing π_1 on its rows and π_2 on its columns. But this means

$$\mathbf{P}(\pi_1^{-1})\,\mathbf{A}_G\,\mathbf{P}(\pi_2) = \mathbf{A}_{G'}$$

and thus

$$\mathbf{A}_G\,\mathbf{P}(\pi_2) = \mathbf{P}(\pi_1)\,\mathbf{A}_{G'}\,.$$

Putting $\mathbf{P}(\pi_2) = \mathbf{P}$, $\mathbf{P}(\pi_1) = \mathbf{Q}$ we obtain the required result. The converse assertion can be proved so that we determine π_1, π_2 from $\mathbf{P}, \mathbf{Q}$ and then f_1, f_2.

Corollary 1. *Let G be a digraph with n vertices $u_1, \ldots, u_n$, let $\mathbf{A}_G$ be its adjacency matrix. Let f_1, f_2 be two permutations of the vertex set of G. Let π_1, π_2 be two permutations of the set of numbers $\{1, 2, \ldots, n\}$ such that $f_1(u_i) = u_{\pi_1(i)}$, $f_2(u_i) = u_{\pi_2(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then $\langle f_1, f_2 \rangle$ is an autotopy of G, if and only if*

$$\mathbf{P}(\pi_1)\,\mathbf{A}_G = \mathbf{A}_G\,\mathbf{P}(\pi_2)\,.$$

As mentioned in [1], an isomorphism of a digraph G onto a digraph G' can be considered as a particular case of an isotopy. If $\langle f_1, f_2 \rangle$ is an isotopy of G onto G' and $f_1 \equiv f_2$, then f_1 is an isomorphism of G onto G' and vice versa. Thus we have the following corollaries.

Corollary 2. *Let G and G' be two digraphs with n vertices, let $\mathbf{A}_G$ and $\mathbf{A}_{G'}$ be their adjacency matrices, respectively. The graphs G and G' are isomorphic, if and only if there exists a permutation $n \times n$ matrix $\mathbf{P}$ such that*

$$\mathbf{A}_G \mathbf{P} = \mathbf{P} \mathbf{A}_{G'} .$$

Corollary 3. *Let G be a digraph with n vertices $u_1, \ldots, u_n$, let $\mathbf{A}_G$ be its adjacency matrix. Let f be a permutation of the vertex set of G. Let π be the permutation of the set of numbers $\{1, 2, \ldots, n\}$ such that $f(u_i) = u_{\pi(i)}$ for each $i \in \{1, 2, \ldots, n\}$. Then f is an automorphism of G, if and only if*

$$\mathbf{P}(\pi)\, \mathbf{A}_G = \mathbf{A}_G\, \mathbf{P}(\pi) .$$

Now we shall consider products of digraphs. If G_1 and G_2 are two digraphs with the same vertex set V, then the product $G_1 . G_2$ is the digraph whose vertex set is V and such that for any two vertices u, v of V the number of edges going from u into v is equal to the number of directed paths in the union of G_1 and G_2 of length 2 and with the property that the first edge of such a path belongs to G_1 and the second to G_2. It is well-known that for the adjacency matrix $\mathbf{A}_{G_1 . G_2}$ of the digraph $G_1 . G_2$ the equality $\mathbf{A}_{G_1 . G_2} = \mathbf{A}_{G_1} \mathbf{A}_{G_2}$ holds.

Theorem 2. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2, f_3 be three permutations of the set V such that $\langle f_1, f_2 \rangle$ is an autotopy of G_1 and $\langle f_2, f_3 \rangle$ is an autotopy of G_2. Then $\langle f_1, f_3 \rangle$ is an autotopy of $G_1 . G_2$.*

Proof. Let $V = \{u_1, \ldots, u_n\}$, let π_1, π_2, π_3 be the permutations of $\{1, 2, \ldots, n\}$ such $f_j(u_i) = u_{\pi_j(i)}$ for each $i \in \{1, 2, \ldots, n\}$ and $j = 1, 2, 3$. As $\langle f_1, f_2 \rangle$ is an autotopy of G_1, Corollary 1 yields

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} = \mathbf{A}_{G_1}\, \mathbf{P}(\pi_2) .$$

As $\langle f_2, f_3 \rangle$ is an autotopy of G_2, we have

$$\mathbf{P}(\pi_2)\, \mathbf{A}_{G_2} = \mathbf{A}_{G_2}\, \mathbf{P}(\pi_3) .$$

We multiply the first equation from the right by $\mathbf{A}_{G_2}$; we obtain

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} \mathbf{A}_{G_2} = \mathbf{A}_{G_1}\, \mathbf{P}(\pi_2)\, \mathbf{A}_{G_2} .$$

We substitute for $\mathbf{P}(\pi_2)\, \mathbf{A}_{G_2}$ from the second equation:

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1} \mathbf{A}_{G_2} = \mathbf{A}_{G_1} \mathbf{A}_{G_2}\, \mathbf{P}(\pi_3) .$$

As mentioned above, $\mathbf{A}_{G_1}\mathbf{A}_{G_2} = \mathbf{A}_{G_1 . G_2}$ and thus

$$\mathbf{P}(\pi_1)\, \mathbf{A}_{G_1 . G_2} = \mathbf{A}_{G_1 . G_2}\, \mathbf{P}(\pi_3)\,.$$

Therefore $\langle f_1, f_3 \rangle$ is an autotopy of $G_1 \,.\, G_2$.

Corollary 4. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2 be two permutations of the set V such that f_1 is an automorphism of G_1 and $\langle f_1, f_2 \rangle$ is an autotopy of G_2. Then $\langle f_1, f_2 \rangle$ is an autotopy of $G_1 \,.\, G_2$.*

Corollary 4'. *Let G_1 and G_2 be two digraphs with the same vertex set V. Let f_1, f_2 be two permutations of the set V such that $\langle f_1, f_2 \rangle$ is an autotopy of G_1 and f_2 is an automorphism of G_2. Then $\langle f_1, f_2 \rangle$ is an autotopy of $G_1 \,.\, G_2$.*

The next theorem will concern digraphs with regular adjacency matrices.

Theorem 3. *Let G be a finite digraph whose adjacency matrix $\mathbf{A}_G$ is regular. Let f_1 be a permutation of the vertex set of G. Then there exists at most one permutation f_2 of V such that $\langle f_1, f_2 \rangle$ is an autotopy of G.*

Proof. Let $\langle f_1, f_2 \rangle$ be an autotopy of G, let π_1, π_2 be defined as in the proof of Theorem 1. Then

$$\mathbf{P}(\pi_1)\, \mathbf{A}_G = \mathbf{A}_G\, \mathbf{P}(\pi_2)\,.$$

As $\mathbf{A}_G$ is regular, we have

$$\mathbf{P}(\pi_2) = \mathbf{A}_G^{-1}\, \mathbf{P}(\pi_1)\, \mathbf{A}_G\,.$$

Thus if $\mathbf{A}_G^{-1}\, \mathbf{P}(\pi_1)\, \mathbf{A}_G$ is a permutation matrix, there exists exactly one f_2 to the given f_1. If it is not so, there exists no f_2 with the property that $\langle f_1, f_2 \rangle$ is an autotopy of G.

Theorem 3'. *Let G be a finite digraph whose adjacency matrix $\mathbf{A}_G$ is regular. Let f_2 be a permutation of the vertex set of G. Then there exists at most one permutation f_1 of V such that $\langle f_1, f_2 \rangle$ is an autotopy of G.*

Proof is analogous to that of Theorem 3.

The results of this paper may be used for finding the group of autotopies or automorphisms of a given digraph or for finding the digraphs which have a given autotopy or automorphism.

References

[1] *B. Zelinka:* Isotopy of digraphs. Czech. Math. J. *22* (1972), 353—360.

[2] *B. Zelinka:* The group of autotopies of a digraph. Czech. Math. J. *21* (1971), 619—624.

[3] *B. Zelinka:* Antiisotopy of directed graphs. Sborník věd. prací VŠST Liberec *9* (1970), 15—24.

Author's address: 460 01 Liberec 1, Komenského 2 (katedra matematiky VŠST).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON HADAMARD'S CONCEPTS OF CORRECTNESS

MIROSLAV SOVA, Praha

(Received January 21, 1976)

In the present paper, we first continue in Section 2 the study of well-posedness or correctness of the Duhamel initial value problem in the sense as introduced in [1]. In Section 3, a weakened form of correctness, called here Hadamardian correctness, is newly introduced and studied. It is characterised by the fact that the continuous dependence of solutions on the initial values is omitted, so that the Hadamardian correctness becomes of almost algebraic character. The main results concern the relations between correctness and Hadamardian correctness in Banach spaces. Finally, in Section 5, we obtain the equivalence between these both notions, naturally only under strong restrictions, i.e. for a special system of coefficient operators in Hilbert spaces.

In the text, we use the notation and definitions introduced in [1]. In particular, it is necessary to be acquainted with the points 1.10, 5.1 – 5.3, 7.1, 7.4 and 7.7 of [1]. Moreover, we need some results of [1], which will be quoted when necessary.

1. PRELIMINARIES

1.1 The complex number field will be denoted by C.

1.2 Lemma. *Let $\varphi, \psi, \chi \in R^+ \to R$. If the function φ is continuous on R^+ and bounded on $(0, 1)$, the functions ψ, χ are nondecreasing and*

$$|\varphi(t)| \leqq \psi(t) + \chi(t) \int_0^t |\varphi(\tau)| \, d\tau \quad \textit{for every} \quad t \in R^+ ,$$

then

$$|\varphi(t)| \leqq \psi(t) \, e^{t\chi(t)} \quad \textit{for every} \quad t \in R^+ .$$

Proof. See [3], p. 19.

1.3 By a Fréchet space F we mean a metrizable complete linear topological convex space.

1.4 Lemma. *Let F_1, F_2 be two Fréchet spaces and T a linear transformation from $F_1 \to F_2$. If the transformation T is closed, then it is continuous.*

2. BASIC NOTIONS AND RESULTS

The notions of definiteness, extensiveness and correctness are introduced or recapitulated and some of their properties, needed in the sequel, are discussed. This part should be regarded as a completion and extension of the paper [1].

2.1 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots$ $\ldots, A_n$ will be called definite if every null solution for the operators $A_1, A_2, \ldots, A_n$ is identically zero.

2.2 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ belong to $L(E)$, then the system $A_1, A_2, \ldots, A_n$ is definite.*

Proof. Let u be an arbitrary null solution for the operators $A_1, A_2, \ldots, A_n$. By [1] 5.6

$$(1) \qquad u^{(n-1)}(t) + A_1 \int_0^t u^{(n-1)}(\tau)\,d\tau + \ldots$$
$$\ldots + \frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} u^{(n-1)}(\tau)\,d\tau = 0 \quad \text{for every} \quad t \in R^+ .$$

Let us denote

$$(2) \qquad K = \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|) .$$

It follows from (1) and (2) that

$$(3) \qquad \|u^{(n-1)}(t)\| \leqq K \left(\int_0^t \|u^{(n-1)}(\tau)\|\,d\tau + t \int_0^t \|u^{n-1}(\tau)\|\,d\tau + \ldots \right.$$
$$\left. \ldots + \frac{t^{n-1}}{(n-1)!} \int_0^t \|u^{n-1}(\tau)\|\,d\tau \right) \quad \text{for every} \quad t \in R^+ .$$

We can rewrite (3) in the form

$$(4) \qquad \|u^{(n-1)}(t)\| \leqq K e^t \int_0^t \|u^{(n-1)}(\tau)\|\,d\tau \quad \text{for every} \quad t \in R^+ .$$

Using 1.2, we obtain from (4) that $u^{(n-1)}(t) = 0$ for every $t \in R^+$ which implies according to [1] 2.10 that $u(t) = 0$ for every $t \in R^+$.

The proof is complete.

2.3 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots, A_n$ are closed and if there exists a sequence P_k, $k \in \{1, 2, \ldots\}$, of operators from $\mathsf{L}(E)$ such that*

(α) $P_k^2 = P_k$ *for every* $k \in \{1, 2, \ldots\}$,

(β) $P_k x \to x$ $(k \to \infty)$ *for every* $x \in E$,

(γ) $P_k x \in \mathsf{D}(A_i)$ *for every* $x \in E$, $k \in \{1, 2, \ldots\}$ *and* $i \in \{1, 2, \ldots, n\}$,

(δ) $P_k A_i x = A_i P_k x$ *for every* $k \in \{1, 2, \ldots\}$, $i \in \{1, 2, \ldots, n\}$ *and* $x \in \mathsf{D}(A_i)$,

then the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Proof. Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, we see from (γ) by virtue of [1] 1.11 that

(1) $A_i P_k \in \mathsf{L}(E)$ for every $k \in \{1, 2, \ldots\}$ and $i \in \{1, 2, \ldots, n\}$.

Let now u be an arbitrary null solution for the operators $A_1, A_2, \ldots, A_n$.

Let us denote $u_k(t) = P_k u(t)$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

It follows without difficulty from (α), (γ) and (δ) that

(2) for every $k \in \{1, 2, \ldots\}$, u_k is a null solution for the operators $A_1 P_k, A_2 P_k, \ldots, A_n P_k$.

Using now 2.2 we obtain from (1) and (2) that

(3) $u_k(t) = 0$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

On the other hand, it follows from (β) that

$$u_k(t) \to u(t) \ (k \to \infty) \quad \text{for every} \quad t \in R^+ . \tag{4}$$

It follows from (3) and (4) that $u(t) = 0$ for every $t \in R^+$ which was to be proved.

2.4 Remark. A different criterion of definiteness (of spectral type) was given in [1] 7.3.

2.5 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots, A_n$ are everywhere defined and bounded, then for every $x \in E$ there exists a Duhamel solution u such that $u^{(n-1)}(0_+) = x$ and for every $t \in R^+$*

$$\|u(t)\| \leqq (1 + \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|)) \frac{t^{n-1}}{(n-1)!}$$
$$[\exp(1 + \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|))t] \|x\| .$$

Proof. Let us denote

$$K = \max(\|A_1\|, \|A_2\|, \ldots, \|A_n\|) . \tag{1}$$

Further, let us choose a fixed $x \in E$ and let us put for $t \in R^+$

$$g(t) = A_1 x + t A_2 x + \ldots + \frac{t^{n-1}}{(n-1)!} A_n x . \tag{2}$$

Obviously, by (1) and (2),

$$\|g(t)\| \leqq K\, e^t \|x\| \quad \text{for every} \quad t \in R^+ . \tag{3}$$

Let us now denote by $\mathbf{C}$ the set of all functions $v \in R^+ \to E$ which are continuous on R^+ and bounded on (0, 1).

It is clear from (2) that

$$g \in \mathbf{C} . \tag{4}$$

Further, let us take for $w \in \mathbf{C}$ and $t \in R^+$

$$T\,w(t) = A_1 \int_0^t w(\tau)\, d\tau + A_2 \int_0^t (t - \tau)\, w(\tau)\, d\tau + \ldots \tag{5}$$

$$\ldots + \frac{1}{(n-1)!} A_n \int_0^t (t - \tau)^{n-1}\, w(\tau)\, d\tau .$$

It is clear from (5) that

(6) T transforms $\mathbf{C}$ into itself.

Further, we see without difficulty that

(7) if $w_k \in \mathbf{C}$, $k \in \{1, 2, \ldots\}$, $w \in R^+ \to E$ and $w_k \to w$ $(k \to \infty)$ uniformly on bounded subsets of R^+, then $w \in \mathbf{C}$ and $Tw_k \to Tw$ $(k \to \infty)$ uniformly on bounded subsets of R^+.

On the other hand, if follows from (1) and (5) that

$$\|T\,w(t)\| \leqq Kt \sup_{0<\tau\leqq t} \|w(\tau)\| \quad \text{for every} \quad w \in \mathbf{C} \quad \text{and} \quad t \in R^+ . \tag{8}$$

By induction in virtue of [1] 1.8 and [1] 2.9 we obtain immediately from (8) that

$$\|T^k\, w(t)\| \leqq \frac{K^k t^k}{k!} \sup_{0<\tau\leqq t} \|w(\tau)\| \tag{9}$$

$$\text{for every} \quad w \in \mathbf{C}, \quad t \in R^+ \quad \text{and} \quad k \in \{0, 1, \ldots\} .$$

It follows from (9) that

(10) $\sum_{k=0}^{\infty} (-T)^k\, w$ converges uniformly on bounded subsets of R^+ for every $w \in \mathbf{C}$.

Let us now write

$$v = - \sum_{k=0}^{\infty} (-T)^k\, g . \tag{11}$$

It follows easily from (3), (6), (7) and (9) that

$$v \in \mathbf{C} , \tag{12}$$

(13) $$\|v(t)\| \leqq K\, e^{(K+1)t}\|x\| \quad \text{for every} \quad t \in R^+ ,$$

(14) $$v + Tv = -g .$$

According to (2) and (5) we can write (14) in the form

(15) $$v(t) + A_1 \int_0^t v(\tau)\, d\tau + \ldots + \frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau = $$
$$= -\left[A_1 x + t A_2 x + \ldots + \frac{t^{n-1}}{(n-1)!} A_n x\right]$$

for every $t \in R^+$.

Let us now define for $t \in R^+$

(16) $$u(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau + \frac{t^{n-1}}{(n-1)!} x .$$

It follows from (12), (13) and (16) by means of [1] 1.7 and [1] 2.8 that

(17) the function u is n-times differentiable on R^+,

(18) $$u^{(n)} = v ,$$

(19) $$u(0_+) = u(0_+) = \ldots = u^{(n-2)}(0_+) = 0 , \quad u^{(n-1)}(0_+) = x ,$$

(20) $$u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = v(t) + \left[A_1 \int_0^t v(\tau)\, d\tau + A_1 x\right] + \ldots$$
$$\ldots + \left[\frac{1}{(n-1)!} A_n \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau + \frac{t^{n-1}}{(n-1)!} A_n x\right] \quad \text{for every} \quad t \in R^+ ,$$

(21) $$\|u(t)\| \leqq (K+1) \frac{t^{n-1}}{(n-1)!} e^{(K+1)t}\|x\| \quad \text{for every} \quad t \in R^+ .$$

By (12) and (18) we conclude that

(22) the function $u^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$.

Further, by (15) and (20)

(23) $$u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0 \quad \text{for every} \quad t \in R^+ .$$

Since $x \in E$ was chosen arbitrarily, we see that the statement of our theorem is, with regard to [1] 5.1, contained in (1), (17), (19) and (21)–(23).

2.6 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ will be called extensive if there exists a subset $Z \subseteq E$ dense in $\overline{D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)}$, such that for every $x \in Z$, we can find a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ so that $u^{(n-1)}(0_+) = x$.

2.7 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots, A_n$ belong to $L(E)$, then the system $A_1, A_2, \ldots, A_n$ is extensive.*

Proof. An immediate consequence of 2.5.

2.8 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots, A_n$ are closed and if there exists a set $\mathfrak{P}$ of operators from $L(E)$ such that*

(α) *$P^2 = P$ for every $P \in \mathfrak{P}$,*

(β) *the closure of the set $\{Px : P \in \mathfrak{P}, x \in E\}$ contains $D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$,*

(γ) *$Px \in D(A_i)$ for every $P \in \mathfrak{P}$, $x \in E$ and $i \in \{1, 2, \ldots, n\}$,*

(δ) *$PA_i x = A_i P x$ for every $P \in \mathfrak{P}$, $i \in \{1, 2, \ldots, n\}$ and $x \in D(A_i)$,*

then the system of operators $A_1, A_2, \ldots, A_n$ is extensive.

Proof. Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, we see from (γ) by virtue of [1] 1.11 that

(1) $$A_i P \in L(E) \quad \text{for every} \quad P \in \mathfrak{P} \quad \text{and} \quad i \in \{1, 2, \ldots, n\} .$$

Using 2.5 we obtain from (1) that

(2) for every $x \in E$ and $P \in \mathfrak{P}$, there exists a Duhamel solution v_P for the operators $A_1 P, A_2 P, \ldots, A_n P$ such that $v_P^{(n-1)}(0_+) = x$.

Let us now define for $P \in \mathfrak{P}$

(3) $$u_P = P v_P .$$

It follows easily from (α), (γ) and (8) that

(4) for every $x \in E$ and $P \in \mathfrak{P}$, the function u_P is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u_P^{(n-1)}(0_+) = Px$.

The extensiveness of the system of operators $A_1, A_2, \ldots, A_n$ follows from (β) and (4).

2.9 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ will be called subcorrect of class m if

(A) it is extensive,

(B) there exist two nonnegative constants M, ω such that for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t - \tau)^m A_i u^{(n-i)}(\tau) \, d\tau \right\| \leq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\| .$$

2.10 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ will be called subcorrect if there exists an $m \in \{0, 1, \ldots\}$ so that the system $A_1, A_2, \ldots, A_n$ is subcorrect of class m.

2.11 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$ and $m \in \{0, 1, \ldots\}$. The system of operators $A_1, A_2, \ldots, A_n$ is correct of class m[correct] if and only if it is subcorrect of class m[subcorrect] and the set $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ is dense in E.*

2.12 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect, then it is also definite.*

Proof. Use 2.9 (B).

2.13 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *the set $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ is dense in E,*

(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect of class m,*

then there exists a $\mathscr{W} \in R^+ \times E \to E$ such that

(a) *for every $x \in E$, the function $\mathscr{W}(., x)$ is continuous on R^+ and*

$$\frac{m!}{t^m}\mathscr{W}(t, x) \xrightarrow[t \to 0_+]{} x,$$

(b) *$\int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x)\, d\tau \in \mathsf{D}(A_i)$ for every $x \in E$, $t \in R^+$ and $\tau \in \{1, 2, \ldots, n\}$,*

(c) *for every $x \in E$ and $i \in \{1, 2, \ldots, n\}$, the function $A_i \int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x)\, d\tau$ is continuous on R^+ and bounded on $(0, 1)$,*

(d)
$$\mathscr{W}(t, x) + A_1 \int_0^t \mathscr{W}(\tau, x)\, d\tau + A_2 \int_0^t (t - \tau)\, \mathscr{W}(\tau, x)\, d\tau + \ldots$$
$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1} \mathscr{W}(\tau, x)\, d\tau = \frac{t^m}{m!} x \quad \textit{for every} \quad x \in E \quad \textit{and} \quad t \in R^+,$$

(e) *for every $t \in R^+$, the function $\mathscr{W}(t, \cdot)$ is a linear mapping,*

(f) *there exists two nonnegative constants M, ω so that for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$*

$$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t - \tau)^{i-1} \mathscr{W}(\tau, x)\, d\tau \right\| \leq M e^{\omega t} \|x\|.$$

Proof. It follows immediately from 2.12 that

(1) the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Further, we can choose a dense linear subset $\dot{Z} \subseteq E$ and two nonnegative constants M, ω so that

(2) for every $x \in Z$, there exists a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$,

(3) for every Duhamel solution u for operators $A_1, A_2, \ldots, A_n$, for every $t \in R^+$ and $i \in \{1, 2, \ldots\}$

$$\left\| \frac{1}{m!} \int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau) \right\| \leqq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\| .$$

Now we see easily from the assumptions and from (1)–(3) that the hypotheses of [1] 7.10 and [1] 7.11 are fulfilled. Hence the assertion of our theorem easily follows.

2.14 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ *and* $\mathscr{W} \in R^+ \times E \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the conditions* 2.13 (a)–(d) *are fulfilled,*

then for every $l \in \{0, 1, \ldots\}$

(a) *for every* $x \in E$, *the function* $(d/dt) \int_0^t (t-\tau)^l \mathscr{W}(\tau, x)\, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(b) $\int_0^t (t-\tau)^{i-1+l} \mathscr{W}(\tau, x)\, d\tau \in D(A_i)$ *for every* $x \in E$, $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$,

(c) *for every* $x \in E$ *and* $i \in \{1, 2, \ldots, n\}$, *the function* $A_i \int_0^t (t-\tau)^{i-1+l} \mathscr{W}(\tau, x)\, d\tau$ *is continuous on* R^+ *and bounded on* $(0, 1)$,

(d)
$$\frac{1}{l!} \frac{d}{dt} \int_0^t (t-\tau)^l \mathscr{W}(\tau, x)\, d\tau + A_1 \frac{1}{l!} \int_0^t (t-\tau)^l \mathscr{W}(\tau, x)\, d\tau +$$
$$+ A_2 \frac{1}{(l+1)!} \int_0^t (t-\tau)^{l+1} \mathscr{W}(\tau, x)\, d\tau + \ldots + A_n \frac{1}{(l+n-1)!} \int_0^t (t-\tau)^{l+n-1}$$
$$\mathscr{W}(\tau, x)\, d\tau = \frac{t^{l+m}}{l+m!}\, x \text{ for every } x \in E \text{ and } t \in R^+,$$

(e) *for every* $t \in R^+$, *the function* $(d/dt) \int_0^t (t-\tau)^l \mathscr{W}(\tau, \cdot)\, d\tau$ *is a linear mapping,*

(f) *there exist two nonnegative constants* M, ω *so that for every* $x \in E$, $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$

$$\left\| A_i \frac{1}{(i-1+l)!} \int_0^t (t-\tau)^{i-1+l} \mathscr{W}(\tau, x)\, d\tau \right\| \leqq M e^{\omega t} \frac{t^l}{l!} \|x\| .$$

Proof. An easy consequence of 2.13 by means of [1] 1.8, [1] 2.4, [1] 2.7 and [1] 2.9.

2.15 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$ *and* $\mathscr{W} \in R^+ \times E \to E$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite,*

(γ) *the conditions* 2.13 (a)–(d) *are fulfilled,* *then for every* $x \in D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$ *and for every* $t \in R^+$

$$\mathscr{W}(t, x) + \int_0^t \mathscr{W}(\tau, A_1 x)\, d\tau + \int_0^t (t - \tau)\, \mathscr{W}(\tau, A_2 x)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1}\, \mathscr{W}(\tau, A_n x)\, d\tau = \frac{t^m}{m!}\, x\,.$$

Proof. Let us fix an $x \in D_1(A_1, A_2, \ldots, A_n)$ and let us put for $t \in R^+$

$$w(t) = \mathscr{W}(t, x) + \int_0^t \mathscr{W}(\tau, A_1 x)\, d\tau + \int_0^t (t - \tau)\, \mathscr{W}(\tau, A_2 x)\, d\tau + \ldots$$

$$\ldots + \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1}\, \mathscr{W}(\tau, A_n x)\, d\tau - \frac{t^m}{m!}\, x\,.$$

A simple calculation using conditions 2.13 (a)–(d) and 2.14 (a)–(d) shows that the function w has properties [1] 7.10 (1)–(4). Hence by Lemma [1] 7.10, $w(t) = 0$ for every $t \in R^+$ and this proves our proposition.

2.16 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, and $m \in \{0, 1, \ldots\}$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite,*

(γ) *there exists a function* $\mathscr{W} \in R^+ \times E \to E$ *such that* 2.13 (a)–(f) *hold,*

then

(a) *for every* $x \in D_{m+1}(A_1, A_2, \ldots, A_n)$, *there exists a Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *so that*

$$u^{(n-1)}(0_+) = x\,,$$

(b) *there exists a nonnegative constant* $\varkappa$ *such that for every Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *satisfying* $u^{(n-1)}(0_+) \in D_{m+1}(A_1, A_2, \ldots, A_n)$ *and for every* $i \in \{1, 2, \ldots, n\}$, *the function* $e^{-\varkappa t} A_i\, u^{(n-i)}(t)$ *is bounded on* R^+,

(c) *there exist two nonnegative constants* M, ω *such that for every Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$, *for every* $t \in R^+$ *and every* $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{m!} \int_0^t (t - \tau)^m A_i\, u^{(n-i)}(\tau)\, d\tau \right\| \leqq M e^{\omega t} \left\| u^{(n-1)}(0_+) \right\|.$$

(d) *the set* $D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$ *is dense in* E.

Proof. For the sake of simplicity we shall write

(1) $$\mathfrak{N} = \{1, 2, \ldots, n\}\,.$$

Further we choose, by assumption (γ), a fixed function $\mathscr{W} \in R^+ \times E \to E$ for which

(2) the conditions 2.13 (a)–(f) are fulfilled.

We begin with proving the assertion (a).

To this aim let us fix an arbitrary $x \in D_{m+1}(A_1, A_2, \ldots, A_n)$ and let us write for $t \in R^+$

(3) $$u(t) = \frac{t^{n-1}}{(n-1)!} x - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{n-1+\alpha_1}}{(n-1+\alpha_1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{n-1+\alpha_1+\alpha_2}}{(n-1+\alpha_1+\alpha_2)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{n-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(n-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \int_0^t \frac{(t-\tau)^{n-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(n-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \, . \, \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \, .$$

By means of [1] 1.8 and [1] 2.8 we obtain easily from (3) that

(4) the function u is n-times differentiable on R^+,

(5) $$u^{(n-i)}(t) = \frac{t^{i-1}}{(i-1)!} x - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{n+m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \int_0^t \frac{(t-\tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \, . \, \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\} \, ,$$

(6) $$u^{(n)}(t) = - \sum_{\alpha_1 \in \mathfrak{R}} \frac{t^{\alpha_1 - 1}}{(\alpha_1 - 1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{R}} \frac{t^{\alpha_1+\alpha_2-1}}{(\alpha_1+\alpha_2-1)!} A_{\alpha_1} A_{\alpha_2} x - \ldots \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{R}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_m-1}}{(\alpha_1+\alpha_2+\ldots+\alpha_m-1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{R}} \frac{d}{dt} \int_0^t \frac{(t-\tau)^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \, . \, \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ \, .$$

It follows from (2) and (5) that

(7) $$u^{(n-1)}(0_+) = x \, .$$

With regard to the assumptions of our proposition, we see from (2) that Theorem 2.14 may be applied and therefore

(8) the conditions 2.14 (a)–(f) hold for every $l \in \{0, 1, \ldots\}$.

Using the properties 2.14 (b) and (c) with $l = i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1$, $\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}$, we see easily from (5) and (8) that

(9) $$u^{(n-i)}(t) \in D(A_i) \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\},$$

(10) the functions $A_i u^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

(11) $$A_i u^{(n-i)}(t) = \frac{t^{i-1}}{(i-1)!} A_i x - \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_i A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_i A_{\alpha_1} A_{\alpha_2} x - \ldots + \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x + (-1)^{m+1} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}} A_i \int_0^t \frac{(t-\tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \cdot \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \, d\tau \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Our next objective is to find out that

(12) $$u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0 \quad \text{for every} \quad t \in R^+.$$

To this aim we first consider the terms of the expressions (6) and (11) except the last ones. After a simple calculation we verify that

(13) $$\left[- \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{\alpha_1 - 1}}{(\alpha_1 - 1)!} A_{\alpha_1} x + \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{\alpha_1 + \alpha_2 - 1}}{(\alpha_1 + \alpha_2 - 1)!} A_{\alpha_1} A_{\alpha_2} x - \ldots + (-1)^m \sum_{\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathfrak{N}} \frac{t^{\alpha_1 + \alpha_2 + \ldots + \alpha_m - 1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_m - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x \right] + \left[\sum_{i=1}^n \frac{t^{i-1}}{(i-1)!} A_i x - \sum_{i=1}^n \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!} A_i A_{\alpha_1} x + \sum_{i=1}^n \sum_{\alpha_1, \alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!} A_i A_{\alpha_1} A_{\alpha_2} x - \ldots + \ldots (-1)^{m-1} \sum_{i=1}^n \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m-1}}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m-1})!} \cdot A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m-1}} x + \right.$$

$$+ (-1)^m \sum_{i=1}^{n} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_m)!} \cdot A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} x \Bigg] =$$

$$= (-1)^m \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x$$

$$\text{for every} \quad t \in R^+ .$$

On the other hand, using the properties 2.14 (b)–(d) with $l = \alpha_1 + \alpha_2 + \ldots \ldots + \alpha_{m+1} - m - 1$, $\alpha_1, \alpha_2, \ldots, \alpha_{m+1} \in \mathfrak{N}$, we obtain from (8) that for the last term of (6) and (11) the following identity holds:

(14)
$$\sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \left[\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t - \tau)^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1)!} \cdot \right.$$

$$\left. \cdot \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \,\mathrm{d}\tau + \sum_{i=1}^{n} A_i \int_0^t \frac{(t - \tau)^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(i - 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - m - 1)!} \cdot \right.$$

$$\left. \cdot \mathscr{W}(\tau, A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x) \,\mathrm{d}\tau \right] =$$

$$= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-1}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} - 1)!} A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} x \quad \text{for every} \quad t \in R^+ .$$

Now the identity (12) follows at once from (6), (11), (13) and (14).

The above considerations, namely the points (4), (7), (9), (10) and (12), show that

(15) the function u is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$.

Since $x \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$ has been arbitrary, the property (15) shows that

(16) the statement (a) holds.

Let us now turn to the statement (b).

By (8) [2.14 (f)], we can find fixed nonnegative constants M, ω so that

(17)
$$\left\| A_i \frac{1}{(i - 1 + l)!} \int_0^t (t - \tau)^{i-1+l} \mathscr{W}(\tau, x) \,\mathrm{d}\tau \right\| \leqq M e^{\omega t} \frac{t^l}{l!} \|x\|$$

for every $x \in E$, $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $l \in \{0, 1, \ldots\}$.

Let now u be an arbitrary Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

(18)
$$u^{(n-1)}(0_+) \in \mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n) .$$

Using the definiteness assumption, we obtain from (15) and (18) that

(19) the solution u may be expressed by the formula (3) with $x = u^{(n-1)}(0_+)$.

It follows from (11), (17) and (19) that

$$\|A_i u^{(n-i)}(t)\| \leqq \frac{t^{i-1}}{(i-1)!}\|A_i u^{(n-1)}(0_+)\| + \tag{20}$$

$$+ \sum_{\alpha_1 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1}}{(i-1+\alpha_1)!}\|A_i A_{\alpha_1} u^{(n-1)}(0_+)\| +$$

$$+ \sum_{\alpha_1,\alpha_2 \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2}}{(i-1+\alpha_1+\alpha_2)!}\|A_i A_{\alpha_1} A_{\alpha_2} u^{(n-1)}(0_+)\| + \ldots$$

$$\ldots + \sum_{\alpha_1,\alpha_2,\ldots,\alpha_m \in \mathfrak{N}} \frac{t^{i-1+\alpha_1+\alpha_2+\ldots+\alpha_m}}{(i-1+\alpha_1+\alpha_2+\ldots+\alpha_m)!} \cdot \|A_i A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_m} u^{(n-1)}(0_+)\| +$$

$$+ \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{m+1} \in \mathfrak{N}} M e^{\omega t} \frac{t^{\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1}}{(\alpha_1+\alpha_2+\ldots+\alpha_{m+1}-m-1)!} \cdot$$

$$\cdot \|A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{m+1}} u^{(n-1)}(0_+)\| \quad \text{for every} \quad t \in R^+ \quad \text{and} \quad i \in \{1, 2, \ldots, n\}.$$

Let us now choose

$$\varkappa > \omega . \tag{21}$$

Since ω was chosen nonnegative, we obtain immediately from (20) and (21) that (22) the functions $e^{-\varkappa t} A_i u^{(n-i)}(t)$ are bounded on R^+ for every $i \in \{1, 2, \ldots, n\}$.

Now an immediate consequence of (22) is, if we take into account the assumption on the solution u, that

(23) the assertion (b) holds.

Now we have to prove the assertion (c).

To this aim, let u be an arbitrary Duhamel solution for the operators $A_1, A_2, \ldots, A_n$.

Let us write for $t \in R^+$

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m!}\int_0^t (t-\tau)^m u^{(n-1)}(\tau)\,\mathrm{d}\tau\right). \tag{24}$$

It follows from [1] 2.9, [1] 5.6 and [1] 5.7 that

(25) the function v is continuous on R^+ and bounded on $(0, 1)$,

(26) $\int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau \in \mathsf{D}(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(27) the functions $A_i \int_0^t (t-\tau)^{i-1} v(\tau)\,\mathrm{d}\tau$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

$$v(t) + A_1 \int_0^t v(\tau)\,\mathrm{d}\tau + A_2 \int_0^t (t-\tau)\,v(\tau)\,\mathrm{d}\tau + \ldots \tag{28}$$

$$\ldots + A_n \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} v(\tau)\, d\tau = \frac{t^m}{m!} u^{(n-1)}(0_+)$$

for every $t \in R^+$,

(29) $$\frac{1}{m!} \int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\, d\tau = A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} v(\tau)\, d\tau$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

It follows from (2) [2.13 (a)–(d)] and (25)–(28) by means of [1] 7.10 that

(30) $$v(t) = \mathscr{W}(t, u^{(n-1)}(0_+)) \quad \text{for every} \quad t \in R^+ .$$

Taking $l = 0$ in (17) we can write

(31) $$\left\| A_i \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} \mathscr{W}(\tau, x)\, d\tau \right\| \leqq M\, e^{\omega t} \|x\|$$

for every $x \in E$, $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Now we obtain from (29)–(31) that

(32) $$\left\| \frac{1}{m!} \int_0^t (t-\tau)^m A_i u^{(n-i)}(\tau)\, d\tau \right\| \leqq M\, e^{\omega t} \|u^{(n-1)}(0_+)\|$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$.

Since the Duhamel solution u examined above was arbitrary we obtain from (32) that

(33) the assertion (c) holds.

Finally, by (2) and (8), we can apply 2.13 (a) and 2.14 (b) and we easily obtain that

(34) the assertion (d) holds.

According to (16), (23), (33) and (34), the proof is complete.

2.17 Theorem. *Let* $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$, *and* $m \in \{0, 1, \ldots\}$. *If*

(α) *the operators* $A_1, A_2, \ldots, A_n$ *are closed,*

(β) *the set* $\mathsf{D}_{m+1}(A_1, A_2, \ldots, A_n)$ *is dense in* $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$,

then the following two statements (a) *and* (b) *are equivalent:*

(a) *the system of operators* $A_1, A_2, \ldots, A_n$ *is subcorrect of class* m *and the set* $\mathsf{D}(A_1) \cap \mathsf{D}(A_2) \cap \ldots \cap \mathsf{D}(A_n)$ *is dense in* E,

(b) *the system of operators* $A_1, A_2, \ldots, A_n$ *is definite and there exists a function* $\mathscr{W} \in R^+ \times E \to E$ *such that the properties* 2.13 (a)–(f) *are fulfilled.*

Proof. An immediate consequence of 2.13 and 2.16. •

3. HADAMARDIAN CONCEPTS

In chapter two of book one of his treatise [2], J. HADAMARD introduced different concepts of correctness for partial differential equations which are mostly very general or too weak. An abstract variant of these concepts (but not so general) is defined and studied in the remaining part of this paper.

3.1 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The system of operators $A_1, A_2, \ldots$ $\ldots, A_n$ will be called exponentially Hadamardian if

(A) it is definite

(B) there exists a constant $\varkappa$ such that for every $x \in D_\infty(A_1, A_2, \ldots, A_n)$ we can find a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ for which $u^{(n-1)}(0_+) = x$ and the function $e^{-\varkappa t} A_i u^{(n-i)}(t)$ is bounded on R^+ for every $i \in \{1, 2, \ldots, n\}$.

3.2 Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. In the sequel, we shall consider the linear space $D_\infty(A_1, A_2, \ldots, A_n)$ as a linear topological space determined by the following system of seminorms:

$$|x|_{\alpha_1, \alpha_2, \ldots, \alpha_d} = \|x\| + \|A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x\|$$

for $x \in D_\infty(A_1, A_2, \ldots, A_n)$, $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots n\}$.

3.3 Lemma. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. The linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$ is convex and metrizable.*

3.4 Lemma. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are closed then the linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$ is a Fréchet space.*

Proof. By 3.3 it is only necessary to prove the completeness of $D_\infty(A_1, A_2, \ldots, A_n)$.

Hence, let x_l, $l \in \{1, 2, \ldots\}$, be an arbitrary Cauchy sequence in the linear topological space $D_\infty(A_1, A_2, \ldots, A_n)$.

This implies by 3.2 that

(1) x_l, $l \in \{1, 2, \ldots\}$ is a Cauchy sequence in E,

(2) for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x_l$, $l \in$ $\in \{1, 2, \ldots\}$, is a Cauchy sequence in E.

It follows from (1) that there exists an $x \in E$ such that

(3) $x_l \to x$ $(l \to \infty)$.

It is clear that it suffices to prove that

(4) $x \in D_d(A_1, A_2, \ldots, A_n)$ for every $d \in \{1, 2, \ldots\}$,

(5) for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x_l \to$ $\to A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} x$ $(l \to \infty)$.

To prove this we proceed by induction on d.

First, it follows immediately from the closedness of operators $A_1, A_2, \ldots, A_n$ that

(6) $x \in D_1(A_1, A_2, \ldots, A_n)$,

(7) for every $\alpha_1 \in \{1, 2, \ldots, n\}$, $A_{\alpha_1} x_l \to A_{\alpha_1} x$ $(l \to \infty)$.

Now we suppose that (4) and (5) are true for some fixed $d \in \{1, 2, \ldots\}$. Using this assumption and the closedness of operators $A_1, A_2, \ldots, A_n$, we obtain easily that

(8) $x \in D_{d+1}(A_1, A_2, \ldots, A_n)$,

(9) for every $\alpha_1, \alpha_2, \ldots, \alpha_{d+1}$, $A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{d+1}} x_l \to A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_{d+1}} \times$ $(l \to \infty)$.

This argument implies that the assertions (4) and (5) hold for every $d \in \{1, 2, \ldots\}$ and this completes the proof.

3.5 Proposition. *Let* $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. *If the operators* $A_1, A_2, \ldots, A_n$ *are closed, then the system of operators* $A_1, A_2, \ldots, A_n$ *is exponentially Hadamardian if and only if*

(A) *there exists a set* $Z \subseteq D_\infty(A_1, A_2, \ldots, A_n)$ *dense in the linear topological space* $D_\infty(A_1, A_2, \ldots, A_n)$ *such that for every* $x \in Z$ *we can find a Duhamel solution* u *for the operators* $A_1, A_2, \ldots, A_n$ *fulfilling* $u^{(n-1)}(0_+) = x$,

(B) *there exist two nonnegative constants* N, $\varkappa$ *and a finite sequence* $q_1, q_2, \ldots, q_r \in \{1, 2, \ldots, n\}$, $r \in \{1, 2, \ldots\}$, *so that for every Duhamel solution* u *fulfilling* $u^{(k-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, *for every* $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$

$$\|A_i u^{(n-i)}(t)\| \leqq N e^{\varkappa t}[\|u^{(n-1)}(0_+)\| + \|A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+)\|] .$$

Proof. "Only if" part.

Let us assume that the system $A_1, A_2, \ldots, A_n$ is exponentially Hadamardian and let us try to verify the properties 3.5 (A) and 3.5 (B).

The property 3.5 (A) being evident we should only prove 3.5 (B).

To this aim, let us introduce some notation.

First we choose a fixed constant $\varkappa$ such that the condition 3.1 (B) holds.

We denote by $\mathbf{Q}$ the linear space of all functions $f \in R^+ \to E$ such that

(1) f is n-times differentiable on R^+,

(2) $f^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$,

(3) $f^{(n-i)}(t) \in D(A_i)$ for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$,

(4) the functions $A_i f^{(n-i)}$ are continuous on R^+ and bounded on $(0, 1)$ for every $i \in \{1, 2, \ldots, n\}$,

(5) the functions $e^{-\varkappa t} A_i f^{(n-i)}(t)$ are bounded on R^+ for every $i \in \{1, 2, \ldots, n\}$.

The space $\mathbf{Q}$ will be equipped with the following system of seminorms:

$$|f|_0 = \sup_{t \in R^+} e^{-\varkappa t} \|A_i f^{(n-i)}(t)\| , \tag{6}$$

$$|f|_T = \sup_{0 < t < T} \{\|f(t)\| + \|f'(t)\| + \ldots + \|f^{(n)}(t)\| + \tag{7}$$

$$+ \|A_1 f^{(n-1)}(t)\| + \|A_2 f^{(n-2)}(t)\| + \ldots + \|A_n f(t)\|\} \quad \text{for} \quad T > 0 .$$

Clearly

(8) **Q** is a linear topological space.

Moreover, it is almost evident that

(9) the linear topological space **Q** is convex and metrizable.

Now, utilizing the assumed closedness of the operators $A_1, A_2, \ldots, A_n$ we obtain easily that

(10) the linear topological space **Q** is complete.

Hence, by (8)–(10), we can state that

(11) the space **Q** is a Fréchet space.

After these preparatory constructions, we can define, in virtue of the properties 3.1 (A), (B), a linear transformation $U \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n) \to \mathbf{Q}$ in the following way:

(12) for $x \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$, we denote by Ux the unique Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ fulfilling

$$u^{(n-1)}(0_+) = x\,.$$

Using the assumed closedness of the operators $A_1, A_2, \ldots, A_n$ we deduce easily from the properties defining the spaces $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ and **Q** that

(13) the operator U is closed as a transformation of the linear topological space $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ into the linear topological space **Q**.

Applying now the closed graph theorem 1.4 we get from 3.4 and from (11) and (13) that

(14) the operator U is continuous as a transformation of the linear topological space $\mathsf{D}_\infty(A_1, A_2, \ldots, A_n)$ into the linear topological space **Q**.

The required property (B) is an immediate consequence of (14).

The proof of the "only if" part is complete.

The "if" part.

Now we suppose that the conditions 3.5 (A), 3.5 (B) hold and we try to prove 3.1 (A), (B).

Since the property 3.1 (A) is an immediate consequence of 3.5 (B), it remains in fact to prove only 3.1 (B).

To this aim let us choose

(15) $$x \in \mathsf{D}_\infty(A_1, A_2, \ldots, A_n)\,.$$

Further, we choose fixed nonnegative constants $N, \varkappa$, a number $r \in \{1, 2, \ldots\}$ and a finite sequence $q_1, q_2, \ldots, q_r \in \{1, 2, \ldots, n\}$ so that 3.5 (B) holds.

Now it is easy to conclude from 3.5 (A), 3.5 (B) that there exists a sequence $u_l \in R^+ \to E$, $l \in \{1, 2, \ldots\}$ so that

(16) for every $l \in \{1, 2, \ldots\}$, the function u_l is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$,

$$u_l^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n) \quad \text{for every} \quad l \in \{1, 2, \ldots\}\,, \tag{17}$$

$$u_l^{(n-1)}(0_+) \to x \ (l \to \infty)\,, \tag{18}$$

$$A_{\alpha_1}A_{\alpha_2} \ldots A_{\alpha_d} u_l^{(n-1)}(0_+) \to A_{\alpha_1}A_{\alpha_2} \ldots A_{\alpha_d} x \ (l \to \infty) \tag{19}$$

for every $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$,

$$\|A_i u_{l_1}^{(n-i)}(t) - A_i u_{l_2}^{(n-i)}(t)\| \leqq \tag{20}$$

$$\leqq Ne^{\varkappa t}[\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| + \|A_{q_1}A_{q_2} \ldots A_{q_r}(u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|]$$

for every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$ and $l_1, l_2 \in \{1, 2, \ldots\}$.

It follows from (20) that

$$\|u_{l_1}^{(n)}(t) - u_{l_2}^{(n)}(t)\| \leqq nNe^{\varkappa t}[\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| + \tag{21}$$

$$+ \|A_{q_1}A_{q_2} \ldots A_{q_r}(u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|] \quad \text{for every} \quad t \in R^+ \text{ and } l_1, l_2 \in \{1, 2, \ldots\}.$$

Now using [1] 2.10, we obtain from (21) that

$$\|u_{l_1}^{(j)}(t) - u_{l_2}^{(j)}(t)\| \leqq \tag{22}$$

$$\leqq \left[nNe^{\varkappa t}\frac{t^{n-j}}{(n-j)!} + 1\right][\|u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+)\| +$$

$$+ \|A_{q_1}A_{q_2} \ldots A_{q_r}(u_{l_1}^{(n-1)}(0_+) - u_{l_2}^{(n-1)}(0_+))\|]$$

for every $t \in R^+$, $j \in \{0, 1, \ldots, n\}$ and $l_1, l_2 \in \{1, 2, \ldots\}$.

It follows from (18), (19) and (22) that there exists a function $u \in R^+ \to E$ such that

$$u_l(t) \to u(t) \ (l \to \infty) \quad \text{for every} \quad t \in R^+\,. \tag{23}$$

Since the operators $A_1, A_2, \ldots, A_n$ are assumed to be closed, it is easy to obtain from (18), (19), (22) and (23) by means of [1] 2.6 that u is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$ and this was to prove.

The proof of "if" part is complete.

3.6 Remark. The exponential Hadamardian property is related with the Hadamard notion of "correctly set" problem (cf. [2], p. 4). Since it does not involve the class of correctness, it is interesting to study its relations with the notion of correctness.

In the sequel, we prove that correctness always implies exponential Hadamardian property, but as to the converse we are able to get it only under strong a priori restrictions on operators $A_1, A_2, \ldots, A_n$ as shown in the section 5.

It should be said that a more general property would be adequate to the (roughly described) Hadamardian notion of correctly set problem, i.e. it would be necessary to replace the property 3.1 (B) by

(B′) for every $x \in D_\infty(A_1, A_2, \ldots, A_n)$, there exists a Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) = x$.

Such systems will be called Hadamardian.

It is easy to see that every exponentially Hadamardian system is also Hadamardian.

3.7 Theorem. *Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$. If*

(α) *the operators $A_1, A_2, \ldots, A_n$ are closed,*

(β) *the set $D(A_1) \cap D(A_2) \cap \ldots \cap D(A_n)$ is dense in E,*

(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is subcorrect,*

then this system is also exponentially Hadamardian.

Proof. An immediate consequence of 2.9, 2.10, 2.12, 2.13 and 2.16.

3.8 Example. *There exist a Banach space E and an operator $A \in L^+(E)$ so that*

(a) *the operator A is closed,*

(b) *the system of operators $0, -A$ is subcorrect of class zero,*

(c) *the system of operators $0, A$ is definite,*

(d) *the system of operators $0, A$ is extensive,*

(e) *the system of operators $0, A$ is not exponentially Hadamardian and consequently also not subcorrect.*

Proof. Let

(1) $$E = L_2(0, \pi)$$

and assume that the operator A is defined as follows:

(2) $x \in D(A)$ if and only if $x \in E$, $x(0_+) = x(1_-) = 0$, x is differentiable on $(0, \pi)$ and there exists a $y \in E$ so that for every $0 < \xi_1 < \xi_2 < \pi$, there is $x'(\xi_2) - x'(\xi_1) = \int_{\xi_1}^{\xi_2} y(\eta)\, d\eta$; then $Ax = y$.

It is easy to prove by elementary means that the assertion (a) holds.

Let us now denote

(3) $e_k(\xi) = (2/\pi)^{1/2} \sin k\xi$ for every $0 < \xi < \pi$ and $k \in \{1, 2, \ldots\}$,

(4) $Z = \{\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_k e_k;\ \alpha_1, \alpha_2, \ldots, \alpha_k \in C,\ k \in \{1, 2, \ldots\}\}$.

It is easy to prove that

(5) $e_k \in D(A)$ and $Ae_k = k^2 e_k$ for every $k \in \{1, 2, \ldots\}$,

(6) the sequence e_k, $k \in \{1, 2, \ldots\}$, is orthonormal,

(7) the set Z is dense in E.

Now the assertion (b) can be derived easily from (5)–(7) by means of Fourier series developments.

The assertions (c) and (d) follow from (5)–(7) by means of 2.3 and 2.8 or simply by direct verification.

It remains to prove (e).

Since $D_\infty(0, A) = \bigcap_{r=1}^{\infty} D(A^r)$, we prove easily that

(8) Z is a dense subset of the Fréchet space $D_\infty(0, A)$.

Further, let us denote

(9) $u_k(t) = \sinh kt\, e_k$ for every $t \in R^+$ and $k \in \{1, 2, \ldots\}$.

It is obvious that

(10) for every $k \in \{1, 2, \ldots\}$, the function u_k is a Duhamel solution for the operators 0, A such that $u_k^{(n-1)}(0_+) = e_k$.

We see from (4) and (8)–(10) that the condition 3.5 (A) is fulfilled. But it is an easy matter to show by means of the sequence u_k, $k \in \{1, 2, \ldots\}$, that 3.5 (B) cannot be fulfilled due to the exponential growth of hyperbolic sinus.

Hence the system 0, A cannot be exponentially Hadamardian and, by 3.8, not even subcorrect.

But this says that (e) holds.

The proof is complete.

3.9 Remark. The above example 3.8 is a somewhat elaborated version of the famous example of a non-correctly set problem, given for the first time by Hadamard in 1917 (cf. [2], pp. 33 and 37).

4. SOME AUXILIARY RESULTS

This section collects some mostly known results on polynomials, on solutions of ordinary differential equations with constant coefficients and on normal operators in Hilbert spaces which will be necessary in Section 5.

4.1 Let $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, and $\varphi \in R^+ \to C$. The fuuction φ will be called a standard solution for the numbers $a_1, a_2, \ldots, a_n$ if

(1) the function φ is n-times differentiable on R^+,

(2) the function $\varphi^{(n)}$ is continuous on R^+ and bounded on $(0, 1)$,

(3) $\varphi^{(n)}(t) + a_1\, \varphi^{(n-1)}(t) + \ldots + a_n\, \varphi(t) = 0$ for every $t \in R^+$,

(4) $\varphi(0_+) = \varphi'(0_+) = \ldots = \varphi^{(n-2)}(0_+) = 0$, $\varphi^{(n-1)}(0_+) = 1$.

4.2 Lemma. *For every $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, there exists a unique standard solution φ for the numbers $a_1, a_2, \ldots, a_n$.*

Proof. Well-known result which is also an immediate consequence of 2.2 and 2.5.

4.3 Lemma. *Let $a_1, a_2, \ldots, a_n \in C$, $z_1, z_2, \ldots, z_n \in C$, $n \in \{1, 2, \ldots\}$, ω a real constant and $\varphi \in R^+ \to C$. If*

(α) $z^n + a_1 z^{n-1} + \ldots + a_n = (z - z_1)(z - z_2)\ldots(z - z_n)$ *for every* $z \in C$,

(β) $\operatorname{Re} z_i \leqq \omega$ *for every* $i \in \{1, 2, \ldots, n\}$,

(γ) *the function* φ *is a standard solution for the numbers* $a_1, a_2, \ldots, a_n$,

then

(a) $|\varphi(t)| \leqq 3^n(1 + t)^n e^{\omega t}$ *for every* $t \in R^+$,

(b) $\left|\frac{a_i}{(i-1)!}\int_0^t (t - \tau)^{i-1} \varphi(\tau)\, d\tau\right| \leqq 3^n(1 + t)^n e^{\omega t}$ *for every* $t \in R^+$ *and* $i \in \{1, 2, \ldots, n\}$.

Proof. We proceed by induction on n.

The case $n = 1$ is verified by a simple calculation.

Now let us assume the estimates (a), (b) take place for $n - 1$, $n > 1$ and try to prove them for n.

To this aim, we need some preparatory considerations.

By Fundamental Theorem of Algebra, we can find numbers $\alpha \in C$ and $b_1, b_2, \ldots, b_{n-1}$ so that

(1) $$z^n + a_1 z^{n-1} + \ldots + a_n = (z - \alpha)(z^{n-1} + b_1 z^{n-2} + \ldots + b_{n-1}) \quad \text{for every} \quad z \in C\,.$$

For the sake of simplicity we shall write

(2) $$b_0 = 1\,.$$

It is easy to see from (1) and (2) that

(3) $$a_1 = b_1 - \alpha b_0,\ a_2 = b_2 - \alpha b_1, \ldots, a_{n-1} = b_{n-1} - \alpha b_{n-2},\ a_n = -\alpha b_{n-1}\,.$$

Let now

(4) ψ be a standard solution for the numbers $b_1, b_2, \ldots, b_{n-1}$.

It is an easy matter to prove using (1) and (4) that

(5) $$\varphi(t) = \int_0^t e^{\alpha(t-\tau)} \psi(\tau)\, d\tau \quad \text{for every} \quad t \in R^+\,.$$

Using (5), we obtain easily the following identities:

(6) $$\frac{1}{p!}\int_0^t (t - \tau)^p \varphi(\tau)\, d\tau = \int_0^t e^{\alpha(t-\tau)} \frac{1}{p!}\int_0^\tau (\tau - \sigma)^p \psi(\sigma)\, d\sigma\, d\tau$$
$$\text{for every} \quad t \in R^+ \quad \text{and} \quad p \in \{0, 1, \ldots\}\,,$$

(7) $$\alpha \int_0^t \varphi(\tau)\, d\tau = \int_0^t (e^{\alpha(t-\tau)} - 1)\, \psi(\tau)\, d\tau \quad \text{for every} \quad t \in R^+\,,$$

$$(8) \qquad \alpha \frac{1}{(p+1)!} \int_0^t (t-\tau)^{p+1} \varphi(\tau)\, d\tau = \int_0^t (e^{\alpha(t-\tau)} - 1) \frac{1}{p!} \int_0^t (\tau - \sigma)^p \psi(\sigma)\, d\sigma\, d\tau$$

for every $t \in R^+$ and $p \in \{0, 1, \ldots\}$.

On the other hand, we have by induction hypothesis that

$$(9) \qquad |\psi(t)| \leqq 3^{n-1}(1+t)^{n-1} e^{\omega t} \quad \text{for every} \quad t \in R^+,$$

$$(10) \qquad \left| b_j \frac{1}{(j-1)!} \int_0^t (t-\tau)^{j-1} \psi(\tau)\, d\tau \right| \leqq 3^{n-1}(1+t)^{n-1} e^{\omega t}$$

for every $t \in R^+$ and $j \in \{1, 2, \ldots, n-1\}$.

The desired estimates are now simple consequences of (2), (3) and (5)–(10).

4.4 Lemma. *Let* $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, *and* $\varphi \in R^+ \to C$. *If the function* φ *is a standard solution for the numbers* $a_1, a_2, \ldots, a_n$, *then*
(a) $|\varphi^{(j)}(t)| \leqq e^{[1+\max(|a_1|,|a_2|,\ldots,|a_n|)]t}$ *for every* $t \in R^+$ *and* $j \in \{0, 1, \ldots, n-1\}$,
(b) $|\varphi^{(n)}(t)| \leqq |a_1|\, e^{[1+\max(|a_1|,|a_2|,\ldots,|a_n|)]t}$ *for every* $t \in R^+$.

Proof. Using the properties 4.1 (1)–(4) we obtain easily the following two identities:

$$(1) \qquad \varphi^{(n-1)}(t) = 1 - a_1 \int_0^t \varphi^{(n-1)}(\tau)\, d\tau - a_2 \int_0^t \varphi^{(n-2)}(\tau)\, d\tau - \ldots - a_n \int_0^t \varphi(\tau)\, d\tau,$$

$$(2) \qquad \varphi^{(n)}(t) = -a_1 - a_1 \int_0^t \varphi^{(n)}(\tau)\, d\tau - a_2 \int_0^t \varphi^{(n-1)}(\tau)\, d\tau - \ldots - a_n \int_0^t \varphi'(\tau)\, d\tau$$

for every $t \in R^+$.

The identities (1) and (2) give the estimates

$$(3) \quad |\varphi^{(n-1)}(t)| \leqq 1 + \max(|a_1|, a_2|, \ldots, |a_n|) \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)|)\, d\tau,$$

$$(4) \quad |\varphi^{(n)}(t)| \leqq |a_1| + \max(|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n)}(\tau)|)\, d\tau$$

for every $t \in R^+$.

Using the inequalities (3) and (4) we see easily that

$$(5) \qquad |\varphi(t)| + |\varphi'(t)| + \ldots + |\varphi^{(n-2)}(t)| + |\varphi^{(n-1)}(t)| =$$

$$= \left| \int_0^t \varphi'(\tau)\, d\tau \right| + \left| \int_0^t \varphi''(\tau)\, d\tau \right| + \ldots + \left| \int_0^t \varphi^{(n-1)}(\tau)\, d\tau \right| + |\varphi^{(n-1)}(t)| \leqq$$

$$\leqq \int_0^t |\varphi'(\tau)| + \int_0^t |\varphi''(\tau)| \, d\tau + \ldots + \int_0^t |\varphi^{(n-2)}(\tau)| \, d\tau + \int_0^t |\varphi^{(n-1)}(\tau)| \, d\tau +$$

$$+ 1 + \max(|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)|) \, d\tau \leqq$$

$$\leqq 1 + [1 + \max(|a_1|, |a_2|, \ldots, |a_n|)] \int_0^t (|\varphi(\tau)| + |\varphi'(\tau)| + \ldots + |\varphi^{(n-2)}(\tau)| +$$

$$+ |\varphi^{(n-1)}(\tau)|) \, d\tau ,$$

(6)
$$|\varphi'(t)| + |\varphi''(t)| + \ldots + |\varphi^{(n-1)}(t)| + |\varphi^{(n)}(t)| =$$

$$= \left|\int_0^t \varphi''(\tau) \, d\tau\right| + \left|\int_0^t \varphi'''(\tau) \, d\tau\right| + \ldots + \left|\int_0^t \varphi^{(n)}(\tau) \, d\tau\right| + |\varphi^{n}(t)| \leqq$$

$$\leqq \int_0^t |\varphi''(\tau)| \, d\tau + \int_0^t |\varphi'''(\tau)| \, d\tau + \ldots + \int_0^t |\varphi^{(n)}(\tau)| \, d\tau +$$

$$+ |a_1| + \max(|a_1|, |a_2|, \ldots, |a_n|) \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n)}(\tau)|) \, d\tau \leqq$$

$$\leqq |a_1| + [1 + \max(|a_1|, |a_2|, \ldots, |a_n|)] \int_0^t (|\varphi'(\tau)| + |\varphi''(\tau)| + \ldots + |\varphi^{(n-1)}(\tau)| +$$

$$+ |\varphi^{(n)}(\tau)| \, d\tau \quad \text{for every} \quad t \in R^+ .$$

The inequalities (a), (b) follow immediately from (5), (6) by means of 1.2.

4.5 Lemma. *Let* $a_1, a_2, \ldots, a_n \in C$, $n \in \{1, 2, \ldots\}$, $z \in C$ *and* $\varphi \in R^+ \to C$. *If*

(α) $z^n + a_1 z^{n-1} + \ldots + a_n = 0$,

(β) *the function* φ *is a standard solution for the numbers* $a_1, a_2, \ldots, a_n$, *then for every* $t \in R^+$

$$e^{zt} = [\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-1} \varphi(t)] +$$

$$+ z[\varphi^{(n-2)}(t) + a_1 \varphi^{(n-3)}(t) + \ldots + a_{n-2} \varphi(t)] + \ldots +$$

$$+ z^{n-2}[\varphi'(t) + a_1 \varphi(t)] + z^{n-1} \varphi(t) .$$

Proof. Let us denote the right hand side of the identity to be proved by $\psi(t)$. Now we have

(1)
$$\psi'(t) - z \psi(t) = [\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t)] -$$

$$- z[\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-1} \varphi(t)] +$$

$$+ z[\varphi^{(n-1)}(t) + a_1 \varphi^{(n-2)}(t) + \ldots + a_{n-2} \varphi'(t)] -$$

$$- z^2[\varphi^{(n-2)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-2} \varphi(t)] + \ldots +$$
$$+ z^{n-2}[\varphi''(t) + a_1 \varphi'(t)] - z^{n-1}[\varphi'(t) + a_1 \varphi(t)] +$$
$$+ z^{n-1} \varphi'(t) - z^n \varphi(t) =$$
$$= [\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t)] -$$
$$- [z^n + a_1 z^{n-1} + \ldots + a_{n-1} z] \varphi(t) \quad \text{for every} \quad t \in R^+ .$$

Since by assumptions (α) and (β)

$$\varphi^{(n)}(t) + a_1 \varphi^{(n-1)}(t) + \ldots + a_{n-1} \varphi'(t) = -a_n \varphi(t),$$
$$z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

we see immediately from (1) that

$$\psi'(t) - z \psi(t) = 0 \quad \text{for every} \quad t \in R^+ . \tag{2}$$

On the other hand, it is easily verified that

$$\psi(0_+) = 1 . \tag{3}$$

Now we prove without difficulty from (2) and (3) that for $t \in R^+$

$$e^{zt} - \psi(t) - z \int_0^t (e^{z\tau} - \psi(\tau)) \, d\tau = 0$$

and consequently

$$|e^{zt} - \psi(t)| \leqq |z| \int_0^t |e^{z\tau} - \psi(\tau)| \, d\tau \quad \text{for every} \quad t \in R^+ . \tag{4}$$

Now it suffices to apply 1.2 and it follows from (4) that $e^{zt} - \psi(t) = 0$ for every $t \in R^+$ which was to prove.

4.6 Lemma. *Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in C$ and $\varphi, \psi \in R^+ \to C$. If φ is a standard solution for the numbers $a_1, a_2, \ldots, a_n$ and ψ for the numbers $b_1, b_2, \ldots \ldots, b_n$, then, writing*

$$K = \max(|a_1|, |a_2|, \ldots, |a_n|),$$
$$L = \max(|b_1|, |b_2|, \ldots, |b_n|),$$
$$\delta = \max(|a_1 - b_1|, |a_2 - b_2|, \ldots, |a_n - b_n|),$$

we have for every $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$

$$|\varphi^{(j)}(t) - \psi^{(j)}(t)| \leqq \delta(K + 1) \, e^{(3 + K + Le^t)t} .$$

Proof. By [1] 2.10 we can write for every $t \in R^+$

$$(1) \qquad \varphi^{(n)}(t) + a_1 \int_0^t \varphi^{(n)}(\tau)\,d\tau + \ldots + \frac{a_n}{(n-1)!} \int_0^t (t-\tau)^{n-1} \varphi^{(n)}(\tau)\,d\tau = -a_1\,,$$

$$(2) \qquad \psi^{(n)}(t) + b_1 \int_0^t \psi^{(n)}(\tau)\,d\tau + \ldots + \frac{b_n}{(n-1)!} \int_0^t (t-\tau)^{n-1} \psi^{(n)}(\tau)\,d\tau = -b_1\,.$$

It follows from (1) and (2) that for every $t \in R^+$

$$(3) \qquad \varphi^{(n)}(t) - \psi^{(n)}(t) = -(a_1 - b_1) -$$

$$- \left[(a_1 - b_1) \int_0^t \varphi^{(n)}(\tau)\,d\tau + \ldots + \frac{a_n - b_n}{(n-1)!} \int_0^t (t-\tau)^{n-1} \varphi^{(n)}(\tau)\,d\tau\right] -$$

$$- \left[b_1 \int_0^t (\varphi^{(n)}(\tau) - \psi^{(n)}(\tau))\,d\tau + \ldots + \frac{b_n}{(n-1)!} \int_0^t (t-\tau)^{n-1} (\varphi^{(n)}(\tau) - \psi^{(n)}(\tau))\,d\tau\right].$$

Moreover, we have by 4.4 (b) for every $t \in R^+$

$$(4) \qquad |\varphi^{(n)}(t)| \leqq K e^{(1+K)t}\,.$$

It follows from (3) and (4) that for every $t \in R^+$

$$(5) \qquad |\varphi^{(n)}(t) - \psi^{(n)}(t)| \leqq$$

$$\leqq \delta + \delta\left[\int_0^t |\varphi^{(n)}(\tau)|\,d\tau + \ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} |\varphi^{(n)}(\tau)|\,d\tau\right] +$$

$$+ L\left[\int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau + \ldots + \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\right] \leqq$$

$$\leqq \delta + \delta\left[t \max_{0<\tau<t} |\varphi^{(n)}(\tau)| + \ldots + \frac{t^n}{n!} \max_{0<\tau<t} |\varphi^{(n)}(\tau)|\right] +$$

$$+ L\left[\int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau + \ldots + \frac{t^{n-1}}{(n-1)!} \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\right] \leqq$$

$$\leqq \delta + \delta e^t \max_{0<\tau<t} (|\varphi^{(n)}(\tau)|) + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau \leqq$$

$$\leqq \delta + \delta e^t K e^{(1+K)t} + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau \leqq$$

$$\leqq \delta(K+1)\, e^{(2+K)t} + L e^t \int_0^t |\varphi^{(n)}(\tau) - \psi^{(n)}(\tau)|\,d\tau\,.$$

Applying now 1.2 to the inequality (5) we obtain immediately for every $t \in R^+$

$$(6) \qquad |\varphi^{(n)}(t) - \psi^{(n)}(t)| \leqq \delta(K+1)\, e^{(2+K+Le^t)t}\,.$$

Now the desired inequality follows easily from (6).

4.7 The system of all Borel subsets of C is denoted by $\mathscr{B}(C)$.

4.8 Lemma. *Let* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in C \to C$, $n \in \{1, 2, \ldots\}$, *and* $\boldsymbol{m} \in R^+ \times C \to C$. *If*
(α) *the functions* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ *are Borel measurable,*
(β) *for every* $s \in C$, *the function* $\boldsymbol{m}(\cdot, s)$ *is a standard solution for the numbers* $\boldsymbol{a}_1(s), \boldsymbol{a}_2(s), \ldots, \boldsymbol{a}_n(s)$,
then for every $t \in R^+$ *and* $j \in \{0, 1, \ldots, n\}$, *the functions* $\boldsymbol{m}_t^{(j)}(t, \cdot)$ *are Borel measurable.*

Proof. It follows from (α) that there exist a sequence X_k, $k \in \{1, 2, \ldots\}$, of Borel subsets and a sequence K_k, $k \in \{1, 2, \ldots\}$, of nonnegative constants such that

(1) $$\bigcup_{k=1}^{\infty} X_k = C\,,$$

(2) $$|a_i(s)| \leqq K_k \quad \text{for every} \quad s \in X_k \quad \text{and} \quad i \in \{1, 2, \ldots, n\}\,.$$

Let us now fix $t \in R^+$ and $\varepsilon > 0$.
We take for $k \in \{1, 2, \ldots\}$

(3) $$\delta_k = \frac{\varepsilon}{(K_k + 1)\, e^{(3 + K_k + K_k e^t)t}}\,.$$

By (α), there exists for every $k \in \{1, 2, \ldots\}$ a subset $\Delta_k \subseteq \mathscr{B}(C)$ such that

(4) $$\bigcup \Delta_k = X_k\,,$$

(5) $$\text{for every} \quad i \in \{1, 2, \ldots, n\}\,, \quad X \in \Delta_k \quad \text{and} \quad s_1, s_2 \in X\,,$$
$$\text{we have} \quad |\boldsymbol{a}_i(s_1) - \boldsymbol{a}_i(s_2)| \leqq \delta_k\,.$$

Now we use 4.6 with $\delta = \delta_k$, $K = L = K_k$ for every $k \in \{1, 2, \ldots\}$ and we obtain from (β), (2), (3) and (5) that
(6) for every $j \in \{0, 1, \ldots, n\}$, $X \in \Delta_k$ and $s_1, s_2 \in X$, we have
$$|\boldsymbol{m}_t^{(j)}(t, s_1) - \boldsymbol{m}_t^{(j)}(t, s_2)| \leqq \delta_k (K_k + 1)\, e^{(3 + K_k + K_k e^t)t} = \varepsilon\,.$$

Let us now denote $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$.
Then by (1), (4) and (6)

(7) $$\bigcup \Delta = C\,,$$

(8) $$\text{for every} \quad j \in \{0, 1, \ldots, n\}\,, \quad X \in \Delta \quad \text{and} \quad s_1, s_2 \in X\,, \quad \text{we have}$$
$$|\boldsymbol{m}_t^{(j)}(t, s_1) - \boldsymbol{m}_t^{(j)}(t, s_2)| \leqq \varepsilon\,.$$

Since $t \in R^+$ and $\varepsilon > 0$ have been arbitrary, the assertion of our lemma follows immediately from (7) and (8).

4.9 A Banach space E will be called Hilbert space if $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for every $x, y \in E$. In a Hilbert space E we introduce the so-called scalar product $\langle x, y\rangle$ for every $x, y \in E$ in the following way: $\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2]$ in the real case, $\langle x, y\rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2]$ in the complex case. This scalar product has the usual well-known properties. The notion of the adjoint operator A^* to an operator $A \in L^+(E)$ is introduced in the usual way.

4.10. In the sequel we always suppose that E is a complex Hilbert space.

4.11. An operator $A \in L^+(E)$ is called normal if $AA^* = A^*A$.

4.12. Let $\mathscr{E} \in \mathscr{B}(C) \to L(E)$. The function $\mathscr{E}$ is called a spectral measure if $\mathscr{E}(C) = I$, $\mathscr{E}(X)$ is an orthogonal (symmetric) projector for every $X \in \mathscr{B}(C)$, $\mathscr{E}(X \cup Y) = \mathscr{E}(X) + \mathscr{E}(Y) - \mathscr{E}(X \cap Y)$ for every $X, Y \in \mathscr{B}(C)$ and $\mathscr{E}(X_k)\, x \underset{k\to\infty}{\to} 0$ for every $x \in E$ and every nondecreasing sequence $X_k \in \mathscr{B}(C)$, $k \in \{1, 2, \ldots\}$, such that $\bigcap_{k=1}^{\infty} X_k = \emptyset$.

4.13. Lemma. *For every spectral measure $\mathscr{E}$ in E, an integral calculus can be developed* (see [4, Chap. VII] and [5, Chap. XVIII]). *The elementary rules of this calculus will be frequently applied in Section 5 and we refer to them by quoting this point.*

The following facts are particularly important

(a) $\|\mathscr{E}(\cdot)\, x\|^2$ *is a nonnegative measure on* $\mathscr{B}(C)$ *for every* $x \in E$,

(b) *if f is a Borel measurable function from* $C \to C$, *then for some* $x \in E$ *and* $X \in \mathscr{B}(C)$:

$$\int_X f(s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \textit{exists if and only if} \quad \int_X |f(s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2 \quad \textit{and}$$

$$\left\| \int_X f(s)\, \mathscr{E}(\mathrm{d}s)\, x \right\|^2 = \int_X |f(s)|^2\, \|\mathscr{E}(\mathrm{d}s)\, x\|^2 .$$

4.14. Lemma. *Let $A \in L^+(E)$. If the operator A is normal, then there is a unique spectral measure $\mathscr{E}$ such that*

(I) $x \in D(A)$ *if and only if* $\int_C s\mathscr{E}(\mathrm{d}s)\, x$ *exists,*

(II) $Ax = \int_C s\, \mathscr{E}(\mathrm{d}s)\, x$ *for every* $x \in D(A)$.

Proof. See [4, Chap. VIII].

4.15. Let $A_1, A_2, \ldots, A_n \in L^+(E)$, $n \in \{1, 2, \ldots\}$, be normal operators. This system is called abelian if the corresponding spectral measures $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ (cf. 4.14) are commutative, i.e. $\mathscr{E}_i(X_i)\,\mathscr{E}_j(X_j) = \mathscr{E}_j(X_j)\,\mathscr{E}_i(X_i)$ for every $X_1, X_2 \in \mathscr{B}(C)$, $i, j \in \{1, 2, \ldots, n\}$.

4.16. Lemma. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the operators $A_1, A_2, \ldots$ $\ldots, A_n$ are normal and this system is abelian, then there exists a spectral measure $\mathscr{E} \in \mathscr{B}(C) \to \mathsf{L}(E)$ and Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in C \to C$ so that for every $i \in \{1, 2, \ldots, n\}$*

(I) *$x \in \mathsf{D}(A_i)$ if and only if $\int_C \boldsymbol{a}_i(s)\,\mathscr{E}(\mathrm{d}s)\,x$ exists,*

(II) *$A_i(x) = \int_C \boldsymbol{a}_i(s)\,\mathscr{E}(\mathrm{d}s)\,x$ for every $x \in \mathsf{D}(A_i)$.*

Proof. See [4, Chap. X, especially § 3].

5. ABELIAN SYSTEMS OF NORMAL OPERATORS IN HILBERT SPACES

In this section, we shall study linear differential equations in a Hilbert space over C whose coefficients form an abelian system of normal operators. In particular, we show that in this class of operators, the exponentially Hadamardian systems are correct.

5.1 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If*

(α) *E is a Hilbert space over C,*

(β) *the operators $A_1, A_2, \ldots, A_n$ are normal,*

(γ) *the system of operators. $A_1, A_2, \ldots, A_n$ is abelian,*

then the system of operators $A_1, A_2, \ldots, A_n$ is definite.

Proof. Let us choose by 4.16 Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and a spectral measure $\mathscr{E}$ so that 4.16 (I), (II) hold.

Let us now define for $k \in \{1, 2, \ldots\}$

(1) $$S_k = \{s : |\boldsymbol{a}_i(s)| \leqq k \text{ for every } i \in \{1, 2, \ldots, n\}\}\,.$$

It is clear that the sets S_k are Borel measurable for every $k \in \{1, 2, \ldots\}$ and hence we can take

(2) $$P_k = \mathscr{E}(S_k) \quad \text{for} \quad k \in \{1, 2, \ldots\}\,.$$

It follows from 4.13 that the assumptions of 2.3 are fulfilled and hence the statement is true.

5.2 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If the assumptions 5.1 (α)–(γ) are fulfilled, then the system of operators $A_1, A_2, \ldots, A_n$ is extensive.*

Proof. Let us choose by 4.16 Borel measurable functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ and a spectral measure $\mathscr{E}$ so that 4.16 (I), (II) hold.

Let us now define

(1) $\mathscr{S} = \{S : S \in \mathscr{B}(C),$ the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are bounded on $S\}$.

Now we take

(2) $$\mathfrak{P} = \{\mathscr{E}(S) : S \in \mathscr{S}\} .$$

It follows from 4.13 that the assumptions of 2.8 are fulfilled and consequently the statement is true.

5.3 Remark. We see immediately that the preceding Theorem 5.2 gives more, namely, under the assumptions of 5.2, the Duhamel solutions exist in fact for initial data from a dense subset of E.

5.4 Theorem. *Let $A_1, A_2, \ldots, A_n \in \mathsf{L}^+(E)$, $n \in \{1, 2, \ldots\}$. If*
(α) *E is a Hilbert space over C,*
(β) *the operators $A_1, A_2, \ldots, A_n$ are normal,*
(γ) *the system of operators $A_1, A_2, \ldots, A_n$ is abelian,*
(δ) *the system of operators $A_1, A_2, \ldots, A_n$ is exponentially Hadamardian,*
then this system is correct (of class $n - 1$).

Proof. It follows from 5.1 that

(1) the system of operators $A_1, A_2, \ldots, A_n$ is definite.
Further, by 5.2

(2) the system of operators $A_1, A_2, \ldots, A_n$ is extensive.
With regard to (2), it suffices to prove that

(3) the condition 2.9 (B) is satisfied.
To prove (3), we need a series of preparatory considerations.
First, using 4.16, we obtain from (α)–(γ) that there exist functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in C \to C$ and a function $\mathscr{E} \in \mathscr{B}(C) \to \mathsf{L}(E)$ so that

(4) the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are Borel measurable,

(5) the function $\mathscr{E}$ is a spectral measure,

(6) for every $i \in \{1, 2, \ldots, n\}$, $x \in \mathsf{D}(A_i)$ if and only if $\int_C \boldsymbol{a}_i(s)\, \mathscr{E}(\mathrm{d}s)\, x$ exists ,

(7) $$A_i x = \int_C \boldsymbol{a}_i(s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every} \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad x \in \mathsf{D}(A_i) .$$

By 4.13, we obtain from (4)–(7) that

(8) $$\mathscr{E}(X)\, A_i \leqq A_i\, \mathscr{E}(X) \quad \text{for every} \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad X \in \mathscr{B}(C) .$$

Let us now denote

(9) $\mathscr{S} = \{X : X \in \mathscr{B}(C)$, the functions $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ are bounded on $X\}$.

By 4.13, we obtain from (4)–(7) and (9) that

(10) $\mathscr{E}(X)\, x \in \mathrm{D}_\infty(A_1, A_2, \ldots, A_n)$ for every $x \in E$ and $X \in \mathscr{S}$,

(11) $$\|A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_d} \mathscr{E}(X)\, x\| = \left[\int_X |a_{\alpha_1}(s)\, a_{\alpha_2}(s) \ldots a_{\alpha_d}(s)|^2 \, \|\mathscr{E}(\mathrm{d}s)\, x\|^2\right]^{1/2}$$

for every $x \in E$, $X \in \mathscr{S}$, $d \in \{1, 2, \ldots\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_d \in \{1, 2, \ldots, n\}$,

(12) there exists a sequence $X_\nu \in \mathscr{S}$, $\nu = \{1, 2, \ldots\}$ such that $X_\nu \subseteq X_{\nu+1}$ for every $\nu \in \{1, 2, \ldots\}$ and $\bigcup\{X_\nu : \nu \in \{1, 2, \ldots\}\} = C$.

On the other hand, by 4.2 there exists a unique function $m \in R^+ \times C \to C$ such that

(13) for every $s \in C$, the function $m(\cdot, s)$ is a standard solution for the numbers $a_1(s), a_2(s), \ldots, a_n(s)$.

Using 4.8, we obtain from (4) and (13) that

(14) the functions $m_t^j(t, \cdot)$ are Borel measurable for every $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$.

Further, using 4.4 we obtain from (9) and (13) that

(15) for every $X \in \mathscr{S}$, there exists a constant K so that for every $t \in R^+$, $s \in X$ and $j \in \{0, 1, \ldots, n\}$

$$|m_t^{(j)}(t, s)| \leqq K e^{Kt}.$$

By 4.13, we obtain from (13)–(15) that

(16) for every $x \in E$ and $X \in \mathscr{S}$, the function $\int_X m(\cdot, s)\, \mathscr{E}(\mathrm{d}s)\, x$ is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

$$\left(\int_X m(t, s)\, \mathscr{E}(\mathrm{d}s)\, x\right) \xrightarrow[t \to 0_+]{} \mathscr{E}(X)\, x,$$

(17) $$\frac{\mathrm{d}^j}{\mathrm{d}t^j} \int_X m(t, s)\, \mathscr{E}(\mathrm{d}s)\, x = \int_X m_t^{(j)}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every}$$

$t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$,

(18) $$\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i \frac{\mathrm{d}^{n-i}}{\mathrm{d}\tau^{n-i}} \left(\int_X m(\tau, s)\, \mathscr{E}(\mathrm{d}s)\, x\right) \mathrm{d}\tau =$$
$$= \int_X a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s)\, \mathrm{d}\tau\, \mathscr{E}(\mathrm{d}s)\, x \quad \text{for every}$$

$t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $i \in \{1, 2, \ldots, n\}$,

(19) $$\left\|\int_X m_t^{(j)}(t, s)\, \mathscr{E}(\mathrm{d}s)\, x\right\| = \left[\int_X |m_t^{(j)}(t, s)|^2 \, \|\mathscr{E}(\mathrm{d}s)\, x\|^2\right]^{1/2}$$

for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$,

(20)
$$\left\| \int_X a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \, \mathscr{E}(ds) \, x \right\| =$$
$$= \left[\int_X \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \, d\tau \right|^2 \| \mathscr{E}(ds) \, x \|^2 \right]^{1/2}$$
for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $i \in \{1, 2, \ldots, n\}$.

Our next purpose is to establish some estimates of growth of the function m.

It follows from Theorem 3.5 that we can fix two nonnegative constants $N, \varkappa$, a number $r \in \{1, 2, \ldots\}$ and a finite sequence $q_1, q_2, \ldots, q_r$ so that

(21) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, for every $t \in R^+$ and every $i \in \{1, 2, \ldots, n\}$
$$\| A_i u^{(n-i)}(t) \| \leqq N e^{\varkappa t} [\| u^{(n-1)}(0_+) \| + \| A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+) \|] .$$

Since for every $t \in R^+$
$$u^{(n)}(t) = -[A_1 u^{(n-1)}(t) + A_2 u^{(n-2)}(t) + \ldots + A_n u(t)] ,$$
for every $t \in R^+$ and $k \in \{0, 1, \ldots, n-1\}$
$$u^{(k)}(t) = \frac{1}{(n-1-k)!} \int_0^t (t-\tau)^{n-1-k} u^{(n)}(\tau) \, d\tau + \frac{t^{n-1-k}}{(n-1-k)!} u^{(n-1)}(0_+)$$
and for every $t \in R^+$, $\delta > 0$ and $l \in \{0, 1, \ldots\}$
$$\frac{t^l}{l!} \leqq \frac{1}{\delta^l} e^{\delta t} ,$$
we deduce from (21) after a simple calculation that

(22) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$ such that $u^{(n-1)}(0_+) \in D_\infty(A_1, A_2, \ldots, A_n)$, for every $t \in R^+$, $j \in \{0, 1, \ldots, n\}$ and $\delta > 0$
$$\| u^{(j)}(t) \| \leqq \frac{1}{\delta^{n-j}} (nN + \delta) \, e^{(\varkappa + \delta)t} [\| u^{(n-1)}(0_+) \| + \| A_{q_1} A_{q_2} \ldots A_{q_r} u^{(n-1)}(0_+) \|] .$$

It follows from (1), (10), (16), (17) and (22) that

(23)
$$\left\| \int_X m_t^{(j)}(t, s) \, \mathscr{E}(ds) \, x \right\| \leqq$$
$$\leqq \frac{1}{\delta^{n-j}} (nN + \delta) \, e^{(\varkappa + \delta)t} [\| \mathscr{E}(X) \, x \| + \| A_{q_1} A_{q_2} \ldots A_{q_r} \mathscr{E}(X) \, x \|]$$
for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$.

Now (11), (19) and (23) give, with regard to the inequality $(a^{1/2} + b^{1/2})^2 \leqq \leqq 2(a + b)$ for $a \geqq 0$, $b \geqq 0$

$$(24)\quad \int_X |m_t^{(j)}(t,s)|^2 \, \|\mathscr{E}(\mathrm{d}s)\,x\|^2 \leqq \frac{1}{\delta^{2(n-j)}}(nN+\delta)^2\, e^{2(\varkappa+\delta)t}\Big[\Big(\Big(\int_X \|\mathscr{E}(\mathrm{d}s)\,x\|^2\Big)^{1/2} +$$

$$+\Big(\int_X |a_{q_1}(s)\,a_{q_2}(s)\ldots a_{q_r}(s)|^2\,\|\mathscr{E}(\mathrm{d}s)\,x\|^2\Big)^{1/2}\Big]^2 \leqq$$

$$\leqq \frac{2}{\delta^{2(n-j)}}(nN+\delta)^2\,e^{2(\varkappa+\delta)t}\int_X (1+|a_{q_1}(s)\,a_{q_2}(s)\ldots a_{q_r}(s)|^2)\,\|\mathscr{E}(\mathrm{d}s)\,x\|^2 \leqq$$

$$\leqq \frac{2}{\delta^{2(n-j)}}(nN+\delta)^2\,e^{2(\varkappa+\delta)t}\int_X (1+|a_{q_1}(s)\,a_{q_2}(s)\ldots a_{q_r}(s)|)^2\,\|\mathscr{E}(\mathrm{d}s)\,x\|^2$$

for every $t \in R^+$, $x \in E$, $X \in \mathscr{S}$ and $j \in \{0, 1, \ldots, n\}$.

Let us now define for $\delta > 0$, $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$

$$(25)\quad N_{\delta,t,j} = \Big\{s : s \in C,\ |m_t^{(j)}(t,s)| > \frac{\sqrt{2}}{\delta^{n-j}}(nN+\delta)\,e^{(\varkappa+\delta)t}(1+|a_{q_1}(s)\,a_{q_2}(s)\ldots a_{q_r}(s)|)\Big\}.$$

It is clear from (14) and (25) that

(26) the set $N_{\delta,t,j}$ is Borel measurable for every $\delta > 0$, $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$.

Let us now put for $\delta > 0$

$$(27)\quad N_\delta = \bigcup\{N_{\delta,t,j} : t \in R^+,\ t \text{ rational},\ j \in \{0, 1, \ldots, n\}\}.$$

We see from (26) and (27) that

(28) the set N_δ is Borel measurable for every $\delta > 0$.

It follows from (13) (the continuity of $m_t^{(j)}(\cdot, s)$ follows by 4.1), (25) and (27) that

$$(29)\quad C \setminus N_\delta = \Big\{s : |m_t^{(j)}(t,s)| \leqq \frac{\sqrt{2}}{\delta^{(n-j)}}(nN+\delta)\,e^{(\varkappa+\delta)t}(1+|a_{q_1}(s)\,a_{q_2}(s)\ldots a_{q_r}(s)|) \text{ for every } t \in R^+ \text{ and } j \in \{0, 1, \ldots, n\}\Big\} \text{ for every } \delta > 0.$$

Now we need to prove that

$$(30)\quad \mathscr{E}(N_\delta) = 0 \quad \text{for every} \quad \delta > 0.$$

It is seen from (12) that it is sufficient for the validity of (30) to prove that $\mathscr{E}(N_\delta)\,\mathscr{E}(X)\,x = 0$ for every $\delta > 0$, $x \in E$ and $X \in \mathscr{S}$, i.e. that

$$(31)\quad \mathscr{E}(N_\delta \cap X)\,x = 0 \quad \text{for every} \quad \delta > 0,\quad x \in E \quad \text{and} \quad X \in \mathscr{S}.$$

On the contrary, suppose that (31) is not true. Then there exist $\delta > 0$, $x \in E$ and $X \in \mathcal{S}$ so that $\mathcal{E}(N_\delta \cap X)\, x \neq 0$. Consequently, by (27) we can find $t \in R^+$ and $j \in \{0, 1, \ldots, n\}$ so that

$$\mathcal{E}(N_{\delta,t,j} \cap X)\, x \neq 0 \,.$$

Hence by (25)

$$\int_{N_{\delta,t,j} \cap X} |m_t^{(j)}(t, s)|^2 \, \|\mathcal{E}(\mathrm{d}s)\, x\|^2 >$$

$$> \frac{2}{\delta^{2(n-j)}} (nN + \delta)^2 \, e^{(\varkappa+\delta)t} \int_{N_{\delta,t,j} \cap X} (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)^2 \, \|\mathcal{E}(\mathrm{d}s)\, x\|^2 \,.$$

Since $N_{\delta,t,j} \cap X \in \mathcal{S}$ by (9) and (26), the last inequality obviously contradicts (24) and this proves (31).

The statements (29) and (30) represent the needed growth properties of the function m and will now be used to estimate the roots of the characteristic polynomial.

By Fundamental Theorem of Algebra, there exist functions $z_1, z_2, \ldots, z_n \in C \to C$ such that

(32)
$$z^n + a_1(s)\, z^{n-1} + \ldots + a_n(s) =$$
$$= (z - z_1(s))\,(z - z_2(s)) \ldots (z - z_n(s)) \quad \text{for every} \quad s, z \in C \,.$$

Applying 4.5 to (32) we obtain easily

(33)
$$|e^{z_i(s)t}| \leqq (1 + |z_i(s)|)^{n-1} (1 + |a_1(s)| + |a_2(s)| + \ldots$$
$$+ |a_{n-1}(s)|) (|m(t, s)| + |m'(t, s)| + \ldots + |m_t^{(n-1)}(t, s)|)$$
$$\text{for every} \quad t \in R^+ \,, \quad s \in C \quad \text{and} \quad i \in \{1, 2, \ldots, n\} \,.$$

We get from (29) and (33)

(34)
$$e^{\operatorname{Re} z_i(s)t} = |e^{z_i(s)t}| \leqq (1 + |z_i(s)|)^{n-1} (1 + |a_1(s)| +$$
$$+ |a_2(s)| + \ldots + |a_{n-1}(s)|) \sqrt{2} \left(\frac{1}{\delta^n} + \frac{1}{\delta^{n-1}} + \ldots + \frac{1}{\delta}\right) (nN + \delta) \,.$$
$$. \, e^{(\varkappa+\delta)t} (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|) =$$
$$= e^{(\varkappa+\delta)t} \left[\sqrt{2} \left(\frac{1}{\delta} + \frac{1}{\delta^2} + \ldots + \frac{1}{\delta^n}\right) (nN + \delta) (1 + |z_i(s)|)^{n-1} \,.\right.$$
$$\left. . \, (1 + |a_1(s)| + |a_2(s)| + \ldots + |a_n(s)|) (1 + |a_{q_1}(s)\, a_{q_2}(s) \ldots a_{q_r}(s)|)\right]$$
$$\text{for every} \quad t \in R^+ \,, \quad \delta > 0 \quad \text{and} \quad s \in C \setminus N_\delta \,.$$

Since the member in the last brackets does not depend on t, it follows immediately from (34) that

(35) $$\operatorname{Re} z_i(s) \leqq \varkappa + \delta \quad \text{for every} \quad \delta > 0 \quad \text{and} \quad s \in C \setminus N_\delta .$$

Let us now put

(36) $$N = \bigcup\{N_{1/k} : k \in \{1, 2, \ldots\}\} .$$

It follows from (30) that

(37) $$\mathscr{E}(N) = 0 .$$

On the other hand, by (35) and (36)

(38) $$\operatorname{Re} z_i(s) \leqq \varkappa \quad \text{for every} \quad s \in C \setminus N .$$

The last results (37) and (38) allow us to estimate the growth of a general Duhamel solution which is our task from (3).

However, to this aim we need still an auxiliary result, namely

(39) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u^{(n-i)}(\tau) \,\mathrm{d}\tau \right\| \leqq$$

$$\leqq \left[\int_C \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \,\mathrm{d}\tau \right|^2 \|\mathscr{E}(\mathrm{d}s)\, u^{(n-1)}(0_+)\|^2 \right]^{1/2} .$$

To prove (39) let u be an arbitrary Duhamel solution for the operators $A_1, A_2, \ldots, A_n$.

By (12), we can choose a sequence X_ν, $\nu \in \{1, 2, \ldots\}$ such that

(40) $X_\nu \in \mathscr{S}$ for every $\nu \in \{1, 2, \ldots\}$, $X_\nu \subseteq X_{\nu+1}$ for every $\nu \in \{1, 2, \ldots\}$ and $\bigcup\{X_\nu : \nu \in \{1, 2, \ldots\}\} = C$,

(41) $$\mathscr{E}(X_\nu)\, x \to x \ (\nu \to \infty) \quad \text{for every} \quad x \in E .$$

By (16), (18), (20), (40) and (41), there exists a sequence u_ν, $\nu \in \{1, 2, \ldots\}$, such that

(42) for every $\nu \in \{1, 2, \ldots\}$, the function u_ν is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that

$$u_\nu^{(n-1)}(0_+) = \mathscr{E}(X_\nu)\, u^{(n-1)}(0_+) .$$

(43) $$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i u(\tau) \,\mathrm{d}\tau \right\| =$$

$$= \left[\int_{X_\nu} \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1} m(\tau, s) \,\mathrm{d}\tau \right|^2 \|\mathscr{E}(\mathrm{d}s)\, u^{(n-1)}(0_+)\|^2 \right]^{1/2}$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

On the other hand, we establish easily by means of (8) that

(44) for every $\nu \in \{1, 2, \ldots\}$, the function $\mathscr{E}(X_\nu)\, u$ is a Duhamel solution for the operators $A_1, A_2, \ldots, A_n$ such that $(\mathscr{E}(X_\nu)\, u)^{(n-1)}(0_+) = \mathscr{E}(X_\nu)\, u^{(n-1)}(0_+)$.

Now we get from (1), (42) and (44) that

$$u_\nu = \mathscr{E}(X_\nu)\, u \quad \text{for every} \quad \nu \in \{1, 2, \ldots\}\,. \tag{45}$$

It follows from (43) and (45) that

$$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i\, \mathscr{E}(X_\nu)\, u^{(n-i)}(\tau)\, d\tau \right\| = \tag{46}$$

$$= \left[\int_{X_\nu} \left| a_i(s) \frac{1}{(i-1)!} \int_0^t (t-\tau)^{i-1}\, m(\tau, s) \right|^2 \left\| \mathscr{E}(ds)\, u^{(n-1)}(0_+) \right\|^2 \right]^{1/2}$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

By (8)

$$\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i\, \mathscr{E}(X_\nu)\, u^{(n-i)}(\tau)\, d\tau = \tag{47}$$

$$= \mathscr{E}(X_\nu) \left[\frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i\, u^{(n-i)}(\tau)\, d\tau \right]$$

for every $t \in R^+$, $i \in \{1, 2, \ldots, n\}$ and $\nu \in \{1, 2, \ldots\}$.

Letting $\nu \to \infty$, we see easily from (40), (41), (46) and (47) that (39) is valid.

Using Lemma 4.3 we see from (13), (32), (38) and (39) that (48) for every Duhamel solution u for the operators $A_1, A_2, \ldots, A_n$, every $t \in R^+$ and $i \in \{1, 2, \ldots, n\}$

$$\left\| \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} A_i\, u^{(n-i)}(\tau)\, d\tau \right\| \leqq 3^n (1+t)^n e^{\varkappa t} \left\| u^{(n-1)}(0_+) \right\|.$$

But (48) clearly yields (3) if we take $M = 3^n$, $\omega = \varkappa + 1$.

The proof is complete.

5.5 Remark. The preceding theorem shows that *the system of operators $A_1, A_2, \ldots$ $\ldots, A_n$ with the properties* 5.4 (α), (β), (γ) *is correct if and only if it is exponentially Hadamardian.*

Moreover, in the course of the proof, we have shown that *the system of operators $A_1, A_2, \ldots, A_n$ with the properties* 5.4 (α), (β), (γ) *is correct if and only if it is correct of class $n - 1$.*

For Hadamardian systems, Theorem 5.4 does not hold and certain additional restrictive assumptions on the operators $A_1, A_2, \ldots, A_n$ must be introduced.

One of such conditions is known from Gårding's theory of hyperbolic equations and says, roughly speaking, that the operators $A_1, A_2, \ldots, A_n$ are polynomials of certain n fixed operators.

We intend to return to these problems in another paper.

References

[1] *Sova, M.*: Linear differential equations in Banach spaces, Rozpravy Československé Akademie věd, Řada matematických a přírodních věd, *85* (1975), No 6, 1—82.

[2] *Hadamard, J.*: Lectures on Cauchy's problem in linear partial differential equations, 1923.

[3] *Hille, E.*: Lectures on ordinary differential equations, 1969.

[4] *Sz.-Nagy, B.*: Spektraldarstellung linearer Transformationen des Hilbertschen Raumes, 1967.

[5] *Dunford, N., Schwartz, J. T.*: Linear operators, Part III, 1971.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

GENERALIZED LC-IDENTITY ON GD-GROUPOIDS

V. SATHYABHAMA, Waterloo

(Received March 17, 1976)

Introduction. A generalized LC-identity [1, 2] is given by

$$A_1(A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z))) . \tag{1}$$

This functional equation, when all the functions (operations) A_i $(i = 1, 2, \ldots, 6)$ are quasigroups defined on the same non-empty set G is investigated in [2] and its general solution is obtained by reducing it to a simpler equation. If this equation is satisfied on non-empty sets G_i $(i = 1, 2, \ldots, 7)$, then each of the operations A_i $(i = 1, 2, \ldots, 6)$ can be regarded as a GD-groupoid in a natural way.

In this paper we find the general solution of equation (1) defined on GD-groupoids, in terms of a loop operation $(+)$ and an arbitrary mapping ψ such that ψ is a mapping into the left nucleus of the loop $(+)$.

Basic definitions and notaitons. *A loop* $G(\cdot)$ is a quasigroup with an identity. If the loop $G(\cdot)$ satisfies the identity

$$(x . (x . y)) . z = x . (x . (y . z)) , \quad \text{for all} \quad x, y, z \in G ,$$

it is called an *LC-loop* [1]. When the operation $(\cdot)$ is replaced by quasigroups A_i $(i = 1, 2, \ldots, 6)$ defined on G, we get the functional equation (1).

A *GD-groupoid* is an ordered quadruple $(G_1, G_2, G; A)$ involving three non-empty sets G_1, G_2, G and the mapping A: $G_1 \times G_2 \to G$ such that the equations $A(a, y) = c$ and $A(x, b) = c$ always have solutions in $y \in G_2$ and $x \in G_1$ respectively, for every $a \in G_1$, $b \in G_2$ and $c \in G$. When these solutions are unique, the GD-groupoid is called a *G-quasigroup*. Throughout this paper we denote the GD-groupoids simply by the operation involved in it.

A GD-groupoid $(G_1, G_2, G; A_1)$ is homotopic to another GD-groupoid $(H_1, H_2, H; A_2)$ if there exist three surjections α: $G_1 \to H_1$, β: $G_2 \to H_2$ and γ: $G \to H$ such that $\gamma\, A_1(x, y) = A_2(\alpha x, \beta y)$, for every $x \in G_1$, $y \in G_2$ in this case the triple $[\alpha, \beta, \gamma]$ is called *a homotopy*.

The following notations are used.

$$L_i(a)\, y = A_i(a, y) , \quad R_i(b)\, x = A_i(x, b) , \quad (i = 1, 2, \ldots, 6) .$$

We consider the functional equation (1), namely $A_1(A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z)))$, for all $x \in G_1$, $y \in G_2$ and $z \in G_3$ where the operations ($i = 1, 2, \ldots, 6$) are the GD-groupoids $(G_5, G_3, G; A_1)$, $(G_1, G_4, G_5; A_2)$, $(G_1, G_2, G_4; A_3)$, $(G_1, G_7, G; A_4)$, $(G_1, G_6, G_7; A_5)$ and $(G_2, G_3, G_6; A_6)$. Further, we assume that A_6 is a G-quasigroup and $R_1(c)$: $G_5 \to G$, $L_5(a)$: $G_6 \to G_7$ and $L_4(a)$: $G_7 \to G$ are bijections for fixed $c \in G_3$ and $a \in G_1$ respectively.

Putting $x = a$ in equation (1), we get

$$(2) \qquad A_1(L_2(a)\, L_3(a)\, y, z) = L_4(a)\, L_5(a)\, A_6(y, z)\,.$$

Also, with $x = a$ and $z = c$ simultaneously in (1), we have

$$(3) \qquad R_1(c)\, L_2(a)\, L_3(a)\, y = L_4(a)\, L_5(a)\, R_6(c)\, y\,.$$

Since $L_4(a)$, $L_5(a)$, $R_6(c)$ and $R_1(c)$ are bijections, from (3), we see that $L_2(a)\, L_3(a)$ is a bijection for $x = a \in G_1$. Hence, from (2) and (1) we obtain

$$(4) \qquad L_4(a)\, L_5(a)\, A_6((L_2(a)\, L_3(a))^{-1}\, A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z)))\,.$$

Putting $z = c \in G_3$, it follows from (4) that

$$(5) \qquad L_4(a)\, L_5(a)\, R_6(c)\, ((L_2(a)\, L_3(a))^{-1}\, A_2(x, A_3(x, y))) = A_4(x, A_5(x, R_6(c)\, y))\,.$$

From (4) and (5) we have

$$(6) \qquad L_4(a)\, L_5(a)\, A_6(R_6(c)^{-1}\, L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, R_6(c)\, y), z) = A_4(x, A_5(x, A_6(y, z)))\,.$$

Equation (6) could be rewritten as:

$$(7) \qquad L_4(a)\, L_5(a)\, A_6(R_6(c)^{-1}\, L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, u)), z) = A_4(x, A_5(x, A_6(R_6(c)^{-1}\, u, z)))\,,$$

where $R_6(c)\, y = u \in G_6$.

Now let

$$(8) \qquad L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, u)) = K(x, u)\,, \quad x \in G_1\,, \quad u \in G_6\,.$$

Then K is the mapping $G_1 \times G_6 \to G_6$. By means of (8), (7) becomes,

$$(9) \qquad A_6(R_6(c)^{-1}\, K(x, u), z) = K(x, A_6(R_6(c)^{-1}\, u, z))\,, \quad x \in G_1\,, \quad u \in G_6\,, \quad z \in G_3\,.$$

On G_6 define an operation $(+)$ as follows:

$$s + t = A_6(R_6(c)^{-1}\, s\,, \quad L_6(b)^{-1}\, t)\,, \quad \text{for every} \quad s, t \in G_6\,.$$

That is

$$A_6(y, z) = R_6(c)\, y + L_6(b)\, z\,. \tag{10}$$

For the element $s \in G_6$, there is only one element $y \in G_2$ such that $s = R_6(c)\, y$, because $R_6(c)$ is a bijection. A similar argument holds for $L_6(b)\, z$ also. Thus, the operation $(+)$ is well-defined on G_6. Further, we note from (10) that $G_6(+)$ is the homotopic image of the GD-groupoid A_6 and is itself a GD-groupoid [3]. Besides, since the equations $A_6(y, c) = d$ and $A_6(b, z) = d$ have unique solutions for $y \in G_2$ and $z \in G_3$ (since A_6 is a G-quasigroup) $G_6(+)$ is a quasigroup.

Next, we will show that $G_6(+)$ is a loop. That is, $G_6(+)$ has an identity. Putting $y = b$ in (10), we have $L_6(b)\, z = R_6(c)\, b + L_6(b)\, z$, which implies that $R_6(c)\, b$ is the left identity in $G_6(+)$, since for every $u \in G_6$, there is a unique $z \in G_3$ such that $L_6(b)\, z = u$. Similarly, by putting $z = c$ in (10), we get $R_6(c)\, y = R_6(c)\, y + L_6(b)\, c$, showing thereby that $L_6(b)\, c$ is the right identity in $G_6(+)$. Thus, $G_6(+)$, having a left and a right identity, has an identity namely $A_6(b, c)$ denoted by 0, and therefore $G_6(+)$ is a loop.

From (9) and (10) we have

$$K(x, u) + v = K(x, u + v)\,, \quad x \in G_1\,, \quad u \in G_6\,, \quad L_6(b)\, z = v \in G_6\,. \tag{11}$$

Put $u = 0$, the identity element in $G_6(+)$. Then from (11), we get

$$K(x, 0) + v = K(x, v)\,. \tag{12}$$

Let

$$K(x, 0) = \psi x\,, \quad \text{where } \psi \text{ is the mapping } \quad G_1 \to G_6\,. \tag{13}$$

Then, (11), (12) and (13) yield,

$$(\psi x + u) + v = \psi x + (u + v)\,, \tag{14}$$

where ψ is a map $G_1 \to G_6$ and $(+)$ is a loop operation defined on G_6 and hence ψx belongs to the left nucleus of $(+)$.

Equations (2) and (10) yield,

$$A_1(w, z) = L_4(a)\, L_5(a)\, (R_6(c)\, (L_2(a)\, L_3(a))^{-1}\, w + L_6(b)\, z)\,, \qquad w \in G_5\,, \quad z \in G_3\,. \tag{15}$$

From (5) and (8), using (3) and (12), we get

$$\begin{aligned} A_2(x, A_3(x, y)) &= L_2(a)\, L_3(a)\, R_6(c)^{-1}\, K(x, R_6(c)\, y)\,, \\ &= L_2(a)\, L_3(a)\, R_6(c)^{-1}\, (\psi x + R_6(c)\, y)\,. \end{aligned} \tag{16}$$

From (8) and (12), we have

$$A_4(x, A_5(x, u)) = L_4(a) L_5(a) (\psi x + u) . \tag{17}$$

Putting $L_2(a) L_3(a) = \alpha$, $L_4(a) L_5(a) = \beta$, $R_6(c) = \gamma$, $L_6(b) = \delta$, equations (15), (16), (17) and (10) yield

$$\begin{aligned} A_1(w, z) &= \beta(\gamma\alpha^{-1}w + \delta z) , \quad w \in G_5 , \quad z \in G_3 , \\ A_2(x, A_3(x, y)) &= \alpha\gamma^{-1}(\psi x + \gamma y) , \quad x \in G_1 , \quad y \in G_2 , \\ A_4(x, A_5(x, u)) &= \beta(\psi x + u) , \quad x \in G_1 , \quad u \in G_6 , \\ A_6(y, z) &= \gamma y + \delta z , \quad y \in G_2 , \quad z \in G_3 . \end{aligned} \tag{18}$$

Thus, we have proved part of the following theorem.

Theorem. *Let $(G_5, G_3, G; A_1)$, $(G_1, G_4, G_5; A_2)$, $(G_1, G_2, G_4; A_3)$, $(G_1, G_7, G; A_4)$, $(G_1, G_6, G_7; A_5)$ and $(G_2, G_3, G_6; A_6)$ be GD-groupoids satisfying the functional equation (1) and let $R_1(c)$: $G_5 \to G$, $L_5(a)$: $G_6 \to G_7$, $L_4(a)$: $G_7 \to G$ be bijections for fixed $c \in G_3$, $a \in G_1$. Further, let A_6 be a G-quasigroup. Then there exists a loop $(+)$ defined on the set G_6 and a mapping ψ: $G_1 \to G_6$ such that ψ is a mapping into the left nucleus of the loop $(+)$ and the general solution of equation (1) is given by (18) and conversely.*

The converse part of this theorem can esily be established by simply substituting (18) into (1) and taking into account that ψx belongs to the left nucleus of the loop $G_6, (+)$.

Now we will deduce the result proved in [2] from Theorem 1, that is let us consider the case when all the GD-groupoids A_i $(i = 1, 2, \ldots, 6)$ are quasigroups defined on the same set G. If we represent the quasigroups A_5 and A_4 as

$$A_5(x, y) = C(x, y) , \tag{19}$$

and

$$A_4(x, y) = \beta K(x, y) , \tag{20}$$

then C and K are quasigroups. From (5) we have

$$A_2(x, A_3(x, y)) = \alpha\gamma^{-1} K(x, C(x, \gamma y)) , \tag{21}$$

and, from (18),

$$A_1(x, y) = \beta(\gamma\alpha^{-1}x + \delta y) , \quad A_6(x, y) = \gamma x + \delta y . \tag{22}$$

Substituting (19), (20), (21) and (22) into (1) and in the resulting equation replacing γy by y and δz by z, we get

$$K(x, C(x, y)) + z = K(x, C(x, y + z)) , \tag{23}$$

which is precisely the reduced equation (7) in [2]. Also, with $y = 0$, the identity of the loop $(+)$, and writing $K(x, C(x, 0)) = \psi(x)$, from (23) we obtain

$$\psi(x) + z = K(x, C(x, z)) . \tag{24}$$

From (23) and (24) we see that

$$(\psi(x) + y) + z = \psi(x) + (y + z) ,$$

which shows that $\psi(x)$ belongs to the left nucleus of the loop $G, (+)$, [2].

I sincerely thank Professor PL. KANNAPPAN for taking pains to go through this work and for his valuable suggestions.

References

[1] *Fenyves, F.:* Extra Loops II, Publ. Math. (Debrecen) *16*, 187—192 (1969).

[2] *D. A. Robinson:* Concerning Functional Equations of the Generalized Bol-Moufang Type (to appear).

[3] *S. Milic:* On GD-groupoids with Applications to *n*-ary Quasigroups, Pulb. Inst. Math. *T 13* (*27*), 65—76, 1972.

Author's address: University of Waterloo, Waterloo, Ontario, Canada. N2L 3G1.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

NOTE ON VOLTERRA-STIELTJES INTEGRAL EQUATIONS

ŠTEFAN SCHWABIK, Praha

(Received May 6, 1976)

This note is a supplement to the paper [2] which is devoted to the Volterra-Stieltjes integral equation in the space $BV_n[0, 1]$ of n-vector functions of bounded variation on the interval $[0, 1]$.

Assume that $\mathbf{K}(t, s)$ is an $n \times n$-matrix valued function defined on the square $[0, 1] \times [0, 1] = J$ such that

(1) $$v(\mathbf{K}) < \infty$$

and

(2) $$\operatorname{var}_0^1 \mathbf{K}(0, \cdot) < \infty$$

where $v(\mathbf{K})$ denotes the twodimensional Vitali variation of $\mathbf{K}$ on the square J and $\operatorname{var}_0^1 \mathbf{K}(0, \cdot)$ is the variation of $\mathbf{K}(0, s)$ in the second variable on the interval $[0, 1]$. The notions of variation are defined in the usual way by the norm in the space $L(R_n)$ of all $n \times n$-matrices which is the operator norm for linear operators on R_n (see [1], [2], [3]).

In [2], Theorem 3.1 asserts the following:

If $\mathbf{K} : J \to L(R_n)$ satisfies (1), (2) and for any $t \in (0, 1]$ the inverse matrix $[I - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))]^{-1}$ exists then the homogeneous Volterra-Stieltjes integral equation

(3) $$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{0}$$

possesses only the trivial solution $\mathbf{x} = \mathbf{0}$ in $BV_n[0, 1]$.

This states that the condition

(4) $$\mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-)) \quad \text{is a regular matrix for all} \quad t \in (0, 1]$$

is sufficient for the equation (3) to have only the trivial solution $\mathbf{x} = \mathbf{0} \in BV_n$. Our aim is to prove that (4) is also a necessary condition for the equation (3) to have this property.

Note that the limit $\lim_{\tau \to t-} \mathbf{K}(t, \tau) = \mathbf{K}(t, t-)$ exists since (1) and (2) hold (see [1]).

1. Theorem. *If $\boldsymbol{K}: J \to L(R_n)$ satisfies* (1) *and* (2) *then the homogeneous Volterra-Stieltjes integral equation* (3) *has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ in BV_n if and only if the condition* (4) *is satisfied.*

Proof. The sufficiency of (4) is stated in the above quoted theorem from [2]. It remains to prove the necessity. We show in the sequel that if (4) is not satisfied then (3) has a nonzero solution in the space BV_n.

It was shown in [2] that for the operator

$$\boldsymbol{x} \in BV_n \to \int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) \in BV_n$$

we have

$$\int_0^t \mathrm{d}_s[\boldsymbol{K}(t, s)]\, \boldsymbol{x}(s) = \int_0^1 \mathrm{d}_s[\boldsymbol{K}^{\Delta}(t, s)]\, \boldsymbol{x}(s) \tag{5}$$

where

$$\begin{aligned} \boldsymbol{K}^{\Delta}(t, s) &= \boldsymbol{K}(t, s) - \boldsymbol{K}(t, 0) \quad \text{if} \quad 0 \leqq s \leqq t \leqq 1\,, \\ \boldsymbol{K}^{\Delta}(t, s) &= \boldsymbol{K}(t, t) - \boldsymbol{K}(t, 0) = \boldsymbol{K}^{\Delta}(t, t) \quad \text{if} \quad 0 \leqq t < s \leqq 1\,. \end{aligned} \tag{6}$$

For the new "triangular" kernel $\boldsymbol{K}^{\Delta}$ we have $\operatorname{var}_0^1 \boldsymbol{K}^{\Delta}(0, \cdot) < \infty$, $v(\boldsymbol{K}^{\Delta}) < \infty$, $\boldsymbol{K}^{\Delta}(t, 0) = 0$ for $t \in [0, 1]$ if (1) and (2) is satisfied for the kernel $\boldsymbol{K}$. Hence the equation (3) can be written in the Fredholm-Stieltjes form

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\boldsymbol{K}^{\Delta}(t, s)]\, \boldsymbol{x}(s) = \boldsymbol{0}\,.$$

Since (1) and (2) hold we have $\operatorname{var}_0^1 \boldsymbol{H} < \infty$ for the matrix valued function $\boldsymbol{H}: [0, 1] \to L(R_n)$ defined by the relations

$$\boldsymbol{H}(t) = \boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-) \quad \text{for} \quad t \in (0, 1]\,, \quad \boldsymbol{H}(0) = \boldsymbol{0}$$

and there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \in (0, 1]$ such that $\boldsymbol{H}(t) = \boldsymbol{0}$ for $t \in [0, 1]$, $t \neq t_i$, $i = 1, 2, \ldots$ (see Lemma 3.1 in [2]). Hence $\sum_{i=1}^{\infty} \|\boldsymbol{H}(t_i)\| < \infty$ because $\operatorname{var}_0^1 \boldsymbol{H} = 2 \sum_{t_i \in (0,1)} \|\boldsymbol{H}(t_i)\| + \|\boldsymbol{H}(1)\|$. This implies that $\|\boldsymbol{H}(t)\| < \frac{1}{2}$ for $t \in [0, 1]$ except for a finite set of points in (0, 1). Hence the matrix $\boldsymbol{I} - \boldsymbol{H}(t)$ can be singular only at a finite set of points T_i, $i = 1, \ldots, k$, $0 < T_1 < T_2 < \ldots < T_k \leqq 1$.

Let us assume that the condition (4) is not satisfied. Then by the facts shown above there is a point $T_1 \in (0, 1]$ such that $\boldsymbol{I} - \boldsymbol{H}(t) = \boldsymbol{I} - (\boldsymbol{K}(t, t) - \boldsymbol{K}(t, t-))$ is a regular matrix for $t \in [0, T_1)$ but $\boldsymbol{I} - \boldsymbol{H}(T_1) = \boldsymbol{I} - (\boldsymbol{K}(T_1, T_1) - \boldsymbol{K}(T_1, T_1-))$ is not regular. Hence there exists $\boldsymbol{z} \in R_n$ such that the linear algebraic equation

$$[\boldsymbol{I} - (\boldsymbol{K}(T_1, T_1) - \boldsymbol{K}(T_1, T_1-))]\, \boldsymbol{x} = \boldsymbol{z} \tag{7}$$

has no solution in R_n. If we define the function $\mathbf{y}^\wedge : [0, 1] \to R_n$ by the relations $\mathbf{y}^\wedge(t) = \mathbf{0}$ for $t \in [0, 1]$, $t \neq T_1$ and $\mathbf{y}^\wedge(T_1) = \mathbf{z}$ then $\mathbf{y}^\wedge \in BV_n$. Let us now consider the Volterra-Stieltjes integral equation

$$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{y}^\wedge(t)\,. \tag{8}$$

Since $\mathbf{I} - (\mathbf{K}(t, t) - \mathbf{K}(t, t-))$ is regular for $t \in [0, T_1)$, every solution $\mathbf{x}$ of (8) vanishes on the interval $[0, T_1)$ by the first part of the theorem and for $t = T_1$ we have

$$\mathbf{x}(T_1) - \int_0^{T_1} \mathrm{d}_s[\mathbf{K}(T_1, s)]\, \mathbf{x}(s) = \mathbf{z}\,.$$

Using the relation

$$\int_0^{T_1} \mathrm{d}_s[\mathbf{K}(T_1, s)]\, \mathbf{x}(s) = (\mathbf{K}(T_1, T_1) - \mathbf{K}(T_1, T_1-))\, \mathbf{x}(T_1)$$

(see [1]) we get

$$\mathbf{x}(T_1) - (\mathbf{K}(T_1, T_1) - \mathbf{K}(T_1, T_1-))\, \mathbf{x}(T_1) = \mathbf{z}$$

but the value $\mathbf{x}(T_1)$ cannot be determined since the linear algebraic equation (7) has no solution. Hence there is no $\mathbf{x} \in BV_n[0, 1]$ satisfying the equation (8), i.e. the range of the operator

$$\mathbf{x} \in BV_n \to \mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) \in BV_n$$

is a proper subspace in $BV_n[0, 1]$.

Since the Volterra-Stieltjes integral equation is a special case of the Fredholm-Stieltjes integral equation we obtain by the Fredholm Theorem (see Theorem 6 in [3]) that there exists in BV_n a nonzero solution of the homogeneous equation (3) and our theorem is completely proved.

2. Corollary. *Let $\mathbf{K} : J \to L(R_n)$ satisfy (1) and (2). Then the nonhomogeneous Volterra-Stieltjes integral equation*

$$\mathbf{x}(t) - \int_0^t \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{y}(t) \tag{9}$$

has a unique solution $\mathbf{x} \in BV_n[0, 1]$ for any $\mathbf{y} \in BV_n[0, 1]$ if and only if the condition (4) is satisfied.

Proof. Since (5) holds the equation (9) can be written in the Fredholm-Stieltjes form

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}^\Delta(t, s)]\, \mathbf{x}(s) = \mathbf{y}(t)$$

where $\mathbf{K}^\Delta : J \to L(R_n)$ is given by (6). By Theorem 1 the corresponding homogeneous equation has only the trivial solution $\mathbf{x} = \mathbf{0}$ in BV_n and consequently by the Fredholm Theorem (see Theorem 6. in [3]) we obtain the statement of the corollary.

3. Theorem. *Let $\mathbf{K} : J \to L(R_n)$ satisfy (1) and (2). If the condition (4) is satisfied then for every $\mathbf{y} \in BV_n[0, 1]$ the unique solution of the equation (9) is given by the formula*

$$\mathbf{x}(t) = \mathbf{y}(t) + \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{y}(s) \qquad t \in [0, 1] \tag{10}$$

where $\boldsymbol{\Gamma}(t, s)$, $0 \leqq s \leqq t \leqq 1$ is a uniquely determined $n \times n$ – matrix valued function such that

$$\boldsymbol{\Gamma}(t, s) = \mathbf{K}(t, s) - \mathbf{K}(t, 0) + \int_0^t \mathrm{d}_r[\mathbf{K}(t, r)]\, \boldsymbol{\Gamma}(r, s) \tag{11}$$

if $0 \leqq s \leqq t \leqq 1$. If we define $\boldsymbol{\Gamma}(t, s) = \boldsymbol{\Gamma}(t, t)$ for $0 \leqq t < s \leqq 1$ then $v(\boldsymbol{\Gamma}) < \infty$ and $\mathrm{var}_0^1\, \boldsymbol{\Gamma}(t, \cdot) < \infty$ for every $t \in [0, 1]$.

Proof. Since the equation (9) can be rewritten in the form of a Fredholm-Stieltjes integral equation

$$\mathbf{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}^\Delta(t, s)]\, \mathbf{x}(s) = \mathbf{y}(t)$$

we obtain by Theorem 8. from [3] that the unique solution of this equation can be given by the formula

$$\mathbf{x}(t) = \mathbf{y}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{y}(s) \tag{12}$$

where $\boldsymbol{\Gamma} : J \to L(R_n)$ satisfies the equality

$$\boldsymbol{\Gamma}(t, s) = \mathbf{K}^\Delta(t, s) - \mathbf{K}^\Delta(t, 0) + \int_0^1 \mathrm{d}_r[\mathbf{K}^\Delta(t, r)]\, \boldsymbol{\Gamma}(r, s)$$

for all $t, s \in [0, 1]$, $\mathrm{var}_0^1\, \boldsymbol{\Gamma}(0, \cdot) < \infty$, $\boldsymbol{\Gamma}(t, 0) = 0$ for all $t \in [0, 1]$, and $v(\boldsymbol{\Gamma}) < \infty$. Using the definition (6) of the "triangular" kernel $\mathbf{K}^\Delta$ and the relation (5) we obtain

$$\int_0^1 \mathrm{d}_r[\mathbf{K}^\Delta(t, r)]\, \boldsymbol{\Gamma}(r, s) = \int_0^t \mathrm{d}_r[\mathbf{K}(t, r)]\, \boldsymbol{\Gamma}(r, s)$$

and this yields the relation (11) for $0 \leqq s \leqq t \leqq 1$. Further, evidently $\boldsymbol{\Gamma}(t, s) = \boldsymbol{\Gamma}(t, t)$ for $0 \leqq t < s \leqq 1$ and also

$$\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{y}(s) = \int_0^t \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{y}(s)$$

for every $\mathbf{y} \in BV_n$. Hence by (12) we obtain the representation (10) for the solution of the equation (7). Let us finally mention that by Theorem 8. in [3] the matrix valued function $\boldsymbol{\Gamma}(t, s)$ is uniquely determined on the square J.

References

[1] *Schwabik Š.*: On an integral operator in the space of functions with bounded variation. Časopis pěst. mat. *97* (1972), 297—330.
[2] *Schwabik Š.*: On Volterra-Stieltjes integral equations. Časopis pěst. mat. *99* (1974), 255—278.
[3] *Schwabik Š.*: On an integral operator in the space of functions with bounded variation, II. Časopis pěst. mat. *102* (1977), 189—202.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

IDEALS OF BINARY RELATIONAL SYSTEMS

JAROMÍR DUDA, Brno and IVAN CHAJDA, Přerov

(Received May 20, 1976)

The concept of an ideal of a partially ordered set was introduced for the purpose of investigating systems with a partial ordering. This concept is a generalization of the lattice ideal (see [1], [7]). However, in [6] another definition of an ideal of a partially ordered set is given which is more general than the classical one and makes it possible to obtain deeper results for some partially ordered systems, especially for l-groups. The aim of this paper is to generalize this definition to the case of general binary relation and to show its applicability to some problems in binary relational systems.

1. ELEMENTARY PROPERTIES OF ϱ-IDEALS

Let ϱ be a binary relation on a set A. The pair $\langle A, \varrho \rangle$ is called a *binary relational system*. We introduce $U(a, b) = \{x \in A;\ a \varrho x,\ b \varrho x\}$ and $L(a, b) = \{x \in A;\ x \varrho a,\ x \varrho b\}$ for arbitrary $a, b \in A$. The system $\langle A, \varrho \rangle$ is said to be *ϱu-directed* (*ϱl-directed*) if $U(a, b) \neq \emptyset$ ($L(a, b) \neq \emptyset$, respectively) for each $a, b \in A$. If $\langle A, \varrho \rangle$ is both ϱu-directed and ϱl-directed, it will be called *ϱ-directed*. The set B is called a *ϱu-directed subset* of A if $\langle A, \varrho \rangle$ is a binary relational system, $B \subseteq A$ and $U(a, b) \cap B \neq \emptyset$ for each $a, b \in B$. Analogously we introduce *ϱl-directed* and *ϱ-directed subsets*.

Definition 1. Let $\langle A, \varrho \rangle$ be a binary relational system and I a non-void subset of A. If the conditions

(I_1) $a \in A,\ i \in I,\ a \varrho i$ imply $a \in I$,

(I_2) $i, j \in I$ implies $U(i, j) \cap I \neq \emptyset$

are satisfied, then I is called a *ϱ-ideal of* $\langle A, \varrho \rangle$.

An arbitrary subset I of A fulfilling the condition (I_1) is called a *semi ϱ-ideal of* $\langle A, \varrho \rangle$.

A non-void subset D of A is called a *dual ϱ-ideal of* $\langle A, \varrho \rangle$ if the following conditions (dual to (I_1), (I_2)) are satisfied:

(D_1) $b \in A,\ d \in D,\ d \varrho b$ imply $b \in D$,

(D_2) $d, g \in D$ implies $L(d, g) \cap D \neq \emptyset$.

The set of all ϱ-ideals of $\langle A, \varrho \rangle$ will be denoted by $\mathcal{I}(A)$. It is clear that $\langle \mathcal{I}(A), \subseteq \rangle$ is a partially ordered set.

Definition 2. A ϱ-ideal I of $\langle A, \varrho \rangle$ is called *maximal*, if the conditions $I \subseteq J$, $I \neq J$ are fulfilled by no ϱ-ideal J of $\langle A, \varrho \rangle$. A ϱ-ideal I of $\langle A, \varrho \rangle$ is called *prime*, if

$$\text{(P)} \quad a, b \in A, \quad \emptyset \neq L(a, b) \subseteq I \quad \text{imply} \quad a \in I \quad \text{or} \quad b \in I.$$

Dually we obtain the concept of a *dual prime ϱ-ideal*.

An arbitrary subset C of A is called a *ϱ-convex subset of* $\langle A, \varrho \rangle$, if $a, b \in C$, $x \in A$, $a \varrho x$, $x \varrho b$ imply $x \in C$.

Notation. Let ϱ be a binary relation on the set A. The transitive hull of ϱ is denoted by the symbol $t(\varrho)$; i.e. for $a, b \in A$ we have $a \, t(\varrho) \, b$ if and only if there exist $a_0, \ldots, a_n \in A$ with $a_0 = a$, $a_n = b$, $a_{i-1} \varrho a_i$ for $i = 1, \ldots, n$.

Example 1. If ϱ is a partial ordering on A, Definition 1 introduces the concept of an o-ideal from [6]. Moreover, if $\langle A, \varrho \rangle$ is a lattice, the concept of a ϱ-ideal coincides with that of a lattice ideal. If ϱ is an equivalence relation on A, then $\mathcal{I}(A) = A/\varrho$.

Proposition 1. *Let ϱ be a binary relation on a set A. Then*

(a) *Each ϱ-ideal of $\langle A, \varrho \rangle$ is a ϱ-convex and ϱu-directed subset of A.*

(b) *If $\langle A, \varrho \rangle$ is ϱl-directed, then each ϱ-ideal of $\langle A, \varrho \rangle$ is a ϱ-directed subset of A.*

(c) *$\langle A, \varrho \rangle$ is ϱu-directed if and only if $A \in \mathcal{I}(A)$.*

Proof. Let I be a ϱ-ideal of $\langle A, \varrho \rangle$. By (I_1), I is ϱ-convex and, by (I_2), I is ϱu-directed. If $\langle A, \varrho \rangle$ is ϱl-directed, then $L(a, b) \neq \emptyset$ for each $a, b \in I$. Let $t \in L(a, b)$. Then $t \varrho a$, hence by (I_1) it is $t \in I$. Thus $\emptyset \neq L(a, b) \subseteq I$, i.e. I is also ϱl-directed; (a) and (b) are proved. If A is a ϱ-ideal of $\langle A, \varrho \rangle$, then $\emptyset \neq U(a, b) \cap A = U(a, b)$ for ach $a, b \in A$, thus $\langle A, \varrho \rangle$ is ϱu-directed. Conversely, if $\langle A, \varrho \rangle$ is ϱu-directed, then $\emptyset \neq U(a, b) = U(a, b) \cap A$. As (I_1) is satisfied automatically, we obtain $A \in \mathcal{I}(A)$.

Proposition 2. *Let $\{I_\gamma; \gamma \in \Gamma\}$ be a chain of ϱ-ideals of $\langle A, \varrho \rangle$ (i.e. $I_\gamma \subseteq I_\delta$ or $I_\delta \subseteq I_\gamma$ for each $\gamma, \delta \in \Gamma$). Then $I = \bigcup_{\gamma \in \Gamma} I_\gamma$ is also a ϱ-ideal of $\langle A, \varrho \rangle$.*

Proof. Let $a \in A$, $i \in I$ and $a \varrho i$. Then $i \in I_\gamma$ for some $\gamma \in \Gamma$ and, by (I_1), $a \in I_\gamma$. Hence $a \in I$. If $i, j \in I$, then $i \in I_\gamma$, $j \in I_\delta$ for some $\gamma, \delta \in \Gamma$. Without loss of generality, suppose $I_\gamma \subseteq I_\delta$. Then $i, j \in I_\delta$, thus $U(i, j) \cap I_\delta \neq \emptyset$. As $I_\delta \subseteq I$, also $U(i, j) \cap I \neq \emptyset$, which completes the proof.

Corollary. *Each ϱ-ideal of $\langle A, \varrho \rangle$ is contained in a maximal ϱ-ideal of $\langle A, \varrho \rangle$.*

This follows directly from Proposition 2 by Kuratowski-Zorn theorem.

Proposition 3. *Let $\langle A, \varrho \rangle$ be a ϱl-directed binary relational system and I a prime ϱ-ideal of $\langle A, \varrho \rangle$. If $A - I \neq \emptyset$, then $D = A - I$ is a dual prime ϱ-ideal of $\langle A, \varrho \rangle$.*

Proof. Let $D = A - I \neq \emptyset$. Let $b \in A$, $d \in D$ and $d \varrho b$. If $b \notin D$, then $b \in I$ and, by (I_1), $d \in I$, a contradiction. Thus (D_1) is satisfied.

Let $c, d \in D$ and $L(c, d) \cap D = \emptyset$. As $\langle A, \varrho \rangle$ is ϱl-directed, we have $\emptyset \neq L(c, d) \subseteq \subseteq I$. By the assumptions, I is a prime ϱ-ideal of $\langle A, \varrho \rangle$, thus $c \in I$ or $d \in I$, also a contradiction. Thus also (D_2) is satisfied and D is a dual ϱ-ideal of $\langle A, \varrho \rangle$.

Suppose $a, b \in A$ and $\emptyset \neq U(a, b) \subseteq D$. If $a \in I$ and $b \in I$, by (I_2) we have $\emptyset \neq \neq U(a, b) \cap I$, which is a contradiction to $U(a, b) \subseteq D$. Thus either $a \in D$ or $b \in D$, i.e. D is a prime dual ϱ-ideal of $\langle A, \varrho \rangle$.

Proposition 4. *Let $\langle A, \varrho \rangle$ be a ϱl-directed binary relational system and I a prime ϱ-ideal of $\langle A, \varrho \rangle$. Then $I_1 \cap I_2 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ for each two ϱ-ideals I_1, I_2 of $\langle A, \varrho \rangle$.*

Proof. The assertion is evident for $I = A$. Let $I \neq A$. By Proposition 4, $D = = A - I$ is a dual prime ϱ-ideal of $\langle A, \varrho \rangle$. If $x_1 \in I_1 - I$, $x_2 \in I_2 - I$, then $x_1, x_2 \in D$ and, by (D_2), $L(x_1, x_2) \cap D \neq \emptyset$. If $t \in L(x_1, x_2) \cap D$, then $t \varrho x_1$, $t \varrho x_2$ and by (I_1) we have $t \in I_1 \cap I_2 \subseteq I$, which is a contradiction. Thus $I_1 - I = \emptyset$ or $I_2 - I = \emptyset$, which implies the assertion.

2. PRINCIPAL ϱ-IDEALS AND SUPREMAL RELATIONS

Definition 3. Let $\langle A, \varrho \rangle$ be a binary relational system and $\emptyset \neq M \subseteq A$. If the intersection of all ϱ-ideals of $\langle A, \varrho \rangle$ containing M is also a ϱ-ideal of $\langle A, \varrho \rangle$, we denote it by $I(M)$ and call it a *ϱ-ideal generated by M*. If $M = \{a_1, \ldots, a_n\}$ is a finite set, $I(M)$ is denoted briefly by $I(a_1, \ldots, a_n)$ and called *a finitely generated ϱ-ideal*. For $M = \{a\}$, $I(a)$ is called a *principal ϱ-ideal generated by a*. If $I(a)$ exists for each $a \in A$, $\langle A, \varrho \rangle$ is called *principal*.

Notation. If $\langle A, \varrho \rangle$ is principal, $\mathscr{I}_0(A)$ denotes the set of all principal ϱ-ideals of $\langle A, \varrho \rangle$.

Lemma 1. *Let ϱ be a binary relation on A, $a, b \in A$ and let $I(a)$, $I(b)$ exist. If $a\, t(\varrho)\, b$, then $I(a) \subseteq I(b)$.*

Proof. By Definition 3, $b \in I(b)$. If $a\, t(\varrho)\, b$, then there exist $a_0, \ldots, a_n \in A$, $a_0 = = a$, $a_n = b$ and $a_{i-1} \varrho a_i$ for $i = 1, \ldots, n$; thus by (I_1) also $a_{n-1} \in I(b)$ and inductively $a = a_0 \in I(b)$. Hence $I(a) \subseteq I(b)$.

Definition 4. A binary relation ϱ is called *supremal* on A, if for each $a, b \in A$ there exists at least one element $s(a, b) \in U(a, b)$ such that $x \in U(a, b)$ implies $s(a, b) =$

$= x$ or $s(a, b)\ \varrho\ x$. Each element $s(a, b)$ with this property is called a *ϱ-supremum of a, b.*

It is clear that the ϱ-supremum of a, b need not be determined uniquely. If for example $A = \{a, b\}$ and $a\ \varrho\ a$, $a\ \varrho\ b$, $b\ \varrho\ a$, $b\ \varrho\ b$, then a is a ϱ-supremum of a, b as well as b is. However, if $s(a, b) \neq s'(a, b)$ are two ϱ-suprema of a, b, then $s(a, b)\ \varrho\ s'(a, b)$ and $s'(a, b)\ \varrho\ s(a, b)$.

If ϱ is supremal on A and each $a, b \in A$ has just one ϱ-supremum, ϱ is called *uniquely supremal* on A. Clearly, each antisymmetrical supremal relation on A is uniquely supremal on A. The dual concepts are *infimal* and *uniquely infimal relation* on A.

The following examples show that for a uniquely supremal binary relation ϱ the system $\langle A, \varrho \rangle$ need not be a semilattice.

Example 2. Let A be the set of all integers and $a\ \varrho\ b$ if and only if $b - a \geqq 1$. Then ϱ is uniquely supremal on A and $s(a, b) = \max\{a, b\} + 1$. However, $s(a, a) \neq a$, thus $\langle A, \varrho \rangle$ is not a semilattice.

Example 3. Let $\precsim$ be a reflexive, uniquely supremal and uniquely infimal relation on A. Then $\langle A, \precsim \rangle$ is a *weakly associative lattice* (see [3]). However, $\langle A, \precsim \rangle$ is not generally a semilattice, since it is not necessarily transitive (see [2]).

Lemma 2. *Let ϱ be a supremal relation on A and J a ϱ-ideal of $\langle A, \varrho \rangle$. Then $s(a, b) \in J$ for each $a, b \in J$ and for an arbitrary ϱ-supremum $s(a, b)$ of a, b.*

Proof. Let $a, b \in J$, $s(a, b)$ be a ϱ-supremum of a, b and $s(a, b) \notin J$. As J is a ϱ-ideal of $\langle A, \varrho \rangle$, there exists $x \in U(a, b) \cap J$. Thus $x \neq s(a, b)$. By Definition 4 we have $s(a, b)\ \varrho\ x$, thus $x \in J$ implies $s(a, b) \in J$, a contradiction.

Proposition 5. *If ϱ is a supremal relation on A, then every set $\{I_\gamma; \gamma \in \Gamma\}$ of ϱ-ideals of $\langle A, \varrho \rangle$ has an infimum $I = \bigcap_{\gamma \in \Gamma} I_\gamma$ in $\langle \mathscr{I}(A), \subseteq \rangle$ provided $I \neq \emptyset$. Moreover, if $\langle A, \varrho \rangle$ is also ϱl-directed, then $\langle \mathscr{I}(A), \subseteq \rangle$ is a conditionally complete and join complete lattice.*

Proof. If ϱ is supremal on A, then $\langle A, \varrho \rangle$ is ϱu-directed and, by Proposition 1(c), A is the greatest element of $\langle \mathscr{I}(A), \subseteq \rangle$. Let $\{I\gamma; \gamma \in \Gamma\} \subseteq \mathscr{I}(A)$ and $\emptyset \neq I = \bigcap_{\gamma \in \Gamma} I_\gamma$. If $a \in A$, $i \in I$, $a\ \varrho\ i$, then $i \in I_\gamma$ for each $\gamma \in \Gamma$ and, by (I_1), also $a \in I_\gamma$ for each $\gamma \in \Gamma$. Hence $a \in I$. If $i, j \in I$, then, by Lemma 2, $s(i, j) \in U(i, j) \cap I_\gamma$ for each $\gamma \in \Gamma$ and an arbitrary ϱ-supremum $s(i, j)$ of i, j. Hence $s(i, j) \in U(i, j) \cap I$. Accordingly, I is a ϱ-ideal of $\langle A, \varrho \rangle$. It is evident that I is the infimum of $\{I_\gamma; \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq \rangle$.

Let $\langle A, \varrho \rangle$ be ϱl-directed and $I_1, I_2 \in \mathscr{I}(A)$. Then $I_1 \cap I_2 \neq \emptyset$, since the relations $a \in I_1$, $b \in I_2$ imply $x \in I_1 \cap I_2$ for each $x \in L(a, b) \neq \emptyset$. By the former result, $I_1 \cap I_2$ is the infimum of $\{I_1, I_2\}$ in $\langle \mathscr{I}(A), \subseteq \rangle$. Let $\{I_\gamma; \gamma \in \Gamma\} \subseteq \mathscr{I}(A)$. Denote by $\mathscr{S}$ the set of all ϱ-ideals of $\langle A, \varrho\rangle$ containing $\bigcap_{\gamma\in\Gamma} I_\gamma$. By the first result, $A \in \mathscr{S}$, thus $\mathscr{S} \neq \emptyset$. Then $J = \bigcap \mathscr{S}$ is a ϱ-ideal of $\langle A, \varrho\rangle$. Clearly J is the supremum of $\{I_\gamma; \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq\rangle$. The proof is complete.

Corollary. *Let ϱ be a supremal relation on A. Then $\langle A, \varrho\rangle$ is principal and, moreover, there exists $I(M)$ for each $\emptyset \neq M \subseteq A$.*

Proposition 6. *Let ϱ be a supremal relation on A. If $\langle \mathscr{I}(A), \subseteq\rangle$ contains the least element, then it is an algebraic lattice and the finitely generated ϱ-ideals are its compact elements.*

Proof. If $\langle \mathscr{I}(A), \subseteq\rangle$ contains the least element, then by Proposition 5 it is a complete lattice. By Corollary of Proposition 5, $\langle A, \varrho\rangle$ is principal and $I(M)$ exists for each $\emptyset \neq M \subseteq A$.

Let $I \in \mathscr{I}(A)$. Then clearly $I(x) \subseteq I$ for each $x \in I$. Hence $\bigcup_{x\in I} I(x) \subseteq I$. As $x \in I(x)$, also $I \subseteq \bigcup_{x\in I} I(x)$, thus $I = \bigcup_{x\in I} I(x)$. Now $\bigcup_{x\in I} I(x)$ is a ϱ-ideal of $\langle A, \varrho\rangle$, hence $I = \bigcup_{x\in I} I(x) = \bigvee_{x\in I} I(x)$ (where $\bigvee$ stands for the supremum in the lattice $\langle \mathscr{I}(A), \subseteq\rangle$).

Let $a \in A$ and $I(a) \subseteq \bigvee_{\gamma\in\Gamma} I_\gamma$ for some $I_\gamma \in \mathscr{I}(A)$, $\gamma \in \Gamma$. By the proof of Proposition 5, $\bigvee_{\gamma\in\Gamma} I_\gamma = I(\bigcup_{\gamma\in\Gamma} I_\gamma)$, i.e. $a \in I(a) \subseteq \bigvee_{\gamma\in\Gamma} I_\gamma = I(\bigcup_{\gamma\in\Gamma} I_\gamma)$. By Proposition 2 and Proposition 5, $\mathscr{I}(A)$ is the algebraic closure system with $M \to I(M)$ as an algebraic closure operator on A (see [4], Theorem 1.2). This means that there exists a finite subset M of $\bigcup_{\gamma\in\Gamma} I_\gamma$, such that $a \in I(M)$. Now there exists a finite subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ with $M \subseteq \bigcup_{i=1}^{n} I_{\gamma_i}$. This yields $a \in I(M) \subseteq I(\bigcup_{i=1}^{n} I_{\gamma_i}) = \bigvee_{i=1}^{n} I_{\gamma_i}$, i.e. $I(a) \subseteq \bigvee_{i=1}^{n} I_{\gamma_i}$. Thus $I(a)$ is a compact element in $\langle \mathscr{I}(A), \subseteq\rangle$ for each $a \in A$. As ϱ is supremal, each finitely generated ϱ-ideal is principal, which completes the proof.

Notation. Let ϱ be a binary relation on A. We introduce operators

$$\mathscr{L}, L : 2^A - \{\emptyset\} \to 2^A$$

by the rules

$$\mathscr{L}(X) = \{a \in A;\ a \,\varrho\, x \text{ for some } x \in X\},$$

$$L(X) = \mathscr{L}(X) \cup X .$$

If ϱ is supremal on A, we introduce operators $\mathscr{S}, S : 2^A - \{\emptyset\} \to 2^A$ by

$$\mathscr{S}(X) = \{a \in A;\ a = s(x, y) \text{ for some } x \in X,\ y \in X \text{ and } \varrho\text{-supremum } s(x, y)\},$$

$$S(X) = \mathscr{S}(X) \cup X .$$

Lemma 3. *Let ϱ be a binary relation on A and $\emptyset \neq X \subseteq Y \subseteq A$. Then*

$$X \subseteq L(X) \subseteq L(Y).$$

If ϱ is also supremal on A, then

$$X \subseteq S(X) \subseteq S(Y).$$

The proof is clear.

Notation. Let ϱ be supremal on A and $\emptyset \neq X \subseteq A$. Define $(SL)^1(X) = (SL)(X) = S(L(X))$ and for any integer n recursively

$$(SL)^{n+1}(X) = (SL)((SL)^n(X)).$$

Analogously, for the operators $\mathscr{S}$ and $\mathscr{L}$ let us write $(\mathscr{S}\mathscr{L})^1(X) = (\mathscr{S}\mathscr{L})(X) = \mathscr{S}(\mathscr{L}(X))$ if $\mathscr{L}(X) \neq \emptyset$ and $(\mathscr{S}\mathscr{L})^{n+1}(X) = (\mathscr{S}\mathscr{L})((\mathscr{S}\mathscr{L})^n(X))$ if $\mathscr{L}((\mathscr{S}\mathscr{L})^n(X)) \neq \emptyset$.

Proposition 7. *Let ϱ be a supremal relation on A. Then $I(M) = \bigcup_{n=1}^{\infty} (SL)^n(M)$ for each $\emptyset \neq M \subseteq A$.*

Proof. Let M be a non-void subset of A. First we prove that $I_M = \bigcup_{n=1}^{\infty} (SL)^n(M)$ is a ϱ-ideal $\langle A, \varrho \rangle$.

Let $a \in A$, $x \in I_M$ and $a \varrho x$. Then $x \in (SL)^n(M)$ for an integer n, thus $a \in L((SL)^n(M))$ and, by Lemma 3, $a \in (SL)((SL)^n(M)) = (SL)^{n+1}(M)$. Hence $a \in I_M$. If $i, j \in I_M$, then there exist integers n, m with $i \in (SL)^n(M)$, $j \in (SL)^m(M)$. By Lemma 3, for $k = \max\{n, m\}$ we have $i, j \in (SL)^k(M)$, thus $s(i, j) \in (SL)(SL)^k(M)) = (SL)^{k+1}(M) \subseteq I_M$ for each ϱ-upremum $s(i, j)$ of i, j. Hence $U(i, j) \cap I_M \neq \emptyset$, thus I_M is a ϱ-ideal of $\langle A, \varrho \rangle$. Clearly $M \subseteq I_M$.

It remains to prove $I_M = I(M)$. Let I be a ϱ-ideal of $\langle A, \varrho \rangle$ with $M \subseteq I$. From (I_1) and Lemma 2 we obtain $(SL)(M) \subseteq I$. By induction we can easily extend it to $(SL)^k(M) \subseteq I$ for each integer k, thus $I_M \subseteq I$, i.e. $I_M \subseteq I(M)$. The converse inclusion is evident, thus $I_M = I(M)$.

Corollary. *Let ϱ be a reflexive and supremal binary relation on A. Then $I(M) = \bigcup_{n=1}^{\infty} (\mathscr{S}\mathscr{L})^n(M)$ for each non-void subset M of A.*

Remark. From Proposition 7 we can derive an explicite description of the suprema of $\{I_\gamma; \gamma \in \Gamma\}$ in $\langle \mathscr{I}(A), \subseteq \rangle$ in the case ϱ is supremal on A. Indeed,

$$\bigvee_{\gamma \in \Gamma} I_\gamma = \bigcup_{n=1}^{\infty} (SL)^n \left(\bigcup_{\gamma \in \Gamma} I_\gamma \right).$$

For some special binary relations frequently used in mathematical investigations the set of ϱ-ideals can be characterized more precisely.

A binary relation ϱ on the set A is called *complete*, if either $a \varrho b$ or $b \varrho a$ is satisfied for each $a, b \in A$. Clearly, ϱ is complete if and only if its symmetrical hull is a universal relation on A.

Proposition 8. *If ϱ is a complete binary relation on a set A, then*
(a) *$\langle A, \varrho \rangle$ is principal and $I(a) = \{x \in A;\ x\, t(\varrho)\, a\}$ for each $a \in A$.*
(b) *Every finitely generated ϱ-ideal of $\langle A, \varrho \rangle$ is principal.*
(c) *Each ϱ-ideal of $\langle A, \varrho \rangle$ is prime.*
(d) *$\langle \mathscr{I}(A), \subseteq \rangle$ is a chain.*

Proof. (a) Le ϱ be a complete relation on A. Then $a \varrho a$ for each $a \in A$, i.e. ϱ is reflexive. If $a, b \in A$, then $a \varrho b$ or $b \varrho a$. As $a \varrho a$, $b \varrho b$, it implies $a \in U(a, b)$ or $b \in U(a, b)$. Suppose $a \in U(a, b)$. If $c \in U(a, b)$, then $a \varrho c$, $b \varrho c$, thus $a = s(a, b)$. For $b \in U(a, b)$ clearly $b = s(a, b)$. Thus ϱ is also supremal and, by Corollary of Proposition 5, $\langle A, \varrho \rangle$ is principal. For $a \in A$ fix denote $M = \{x \in A;\ x\, t(\varrho)\, a\}$. Clearly $a \in M$.

If $b \in A$, $x \in M$, $b \varrho x$, then there exist $a_0, \ldots, a_n \in A$ with $a_0 = x$, $a_n = a$ and $a_{i-1} \varrho a_i$ for $i = 1, \ldots, n$. Thus $b \varrho x$ implies $b\, t(\varrho)\, a$, i.e. $b \in M$. If $i, j \in M$, then either $i \in U(i, j)$ or $j \in U(i, j)$. Hence $U(i, j) \cap M \neq \emptyset$ and M is a ϱ-ideal of $\langle A, \varrho \rangle$ containing a.

Conversely, let I be a ϱ-ideal of $\langle A, \varrho \rangle$ containing a. If $t \in M$, then $t \varrho a_1, \ldots$ $\ldots, a_{n-1} \varrho a_n = a$ for some $a_1, \ldots, a_n \in A$. As $a \in I$, it is also $a_{n-1} \in I$ and, inductively by (I_1), $t \in I$. Hence $M \subseteq I$, i.e. $M = I(a)$. As $a \in A$ was chosen arbitrary, the statement (a) is proved.

(b) By Corollary of Proposition 5, there exists finitely generated ϱ-ideal $I(a_1, \ldots, a_n)$ for every finite subset $\{a_1, \ldots, a_n\}$ of A. Without loss of generality, suppose $a_1 \varrho a_2$. Then clearly $I(a_1, \ldots, a_n) = I(a_2, \ldots, a_n)$. With respect to the finiteness of $\{a_1, \ldots, a_n\}$, by $n - 1$ steps we obtain $I(a_1, \ldots, a_n) = I(a_i)$ for some $i \in \{1, \ldots, n\}$.

(c) Let I be a ϱ-ideal of $\langle A, \varrho \rangle$ and $i, j \in A$. As ϱ is complete, $i \in L(i, j)$ or $j \in$ $\in L(i, j)$ is fulfilled. Then $\emptyset \neq L(i, j) \subseteq I$ implies $i \in I$ or $j \in I$, thus I is prime.

(d) Let I, J be ϱ-ideals of $\langle A, \varrho \rangle$. By Proposition 5, $I \cap J$ is also a ϱ-ideal of $\langle A, \varrho \rangle$ and by (c) $I \cap J$ is prime. As $I \cap J \subseteq I \cap J$, by Proposition 4 we obtain $I \subseteq I \cap J \subseteq J$ or $J \subseteq I \cap J \subseteq I$, thus $\langle \mathscr{I}(A), \subseteq \rangle$ is a chain.

Remark. If ϱ is complete on A, clearly $S(X) = X$ for each $\emptyset \neq X \subseteq A$. As ϱ is also reflexive, we have $L = \mathscr{L}$. Then by Proposition 7 we have $I(M) = \bigcup_{n=1}^{\infty} \mathscr{L}^n(M)$ and by Proposition 8, $\{x \in A;\ x\, t(\varrho)\, a\} = \bigcup_{n=1}^{\infty} \mathscr{L}^n(\{a\})$.

Definition 5. Let ϱ be a binary relation on a set A, $c \in B \subseteq A$. We call c the *ϱ-greatest element of B*, if $b \varrho c$ is true for all $b \in B$.

An element $d \in B$ is called *ϱ-maximal of B*, if $d \varrho b$ is true for none of the elements $b \in B$, $b \neq d$.

We say that $\langle A, \varrho \rangle$ satisfies the *ϱ-maximal condition* if each non-void subset of A has a ϱ-maximal element.

Lemma 4. *Let B be a semi ϱ-ideal of $\langle A, \varrho \rangle$ with the ϱ-greatest element $b \in B$. Then B is the principal ϱ-ideal and $B = I(b)$.*

Proof. If $x, y \in B$, then $x \varrho b$, $y \varrho b$ and it means $b \in U(x, y) \cap B$. As B is a semi ϱ-ideal, B is a ϱ-ideal of $\langle A, \varrho \rangle$. Further, if I is a ϱ-ideal of $\langle A, \varrho \rangle$ containing b, then $t \varrho b$ implies $t \in I$ for each $t \in A$. However, $t \varrho b$ is true for each $t \in B$, thus $B \subseteq I$. Hence $B = I(b)$.

Lemma 5. *Every ϱu-directed subset B in a binary relational system (A, ϱ) has at most one ϱ-maximal element. If such an element exists in B, it is at the same time the ϱ-greatest element of B.*

Proof. If B is a ϱu-directed subset of A and $a, b \in B$ are ϱ-maximal elements of B, then $a \varrho t$, $b \varrho t$ for each $t \in U(a, b) \cap B \neq \emptyset$, thus it remains only $a = t = b$. Let B have a ϱ-maximal element m. If $x \in B$, then there exists $s \in U(x, m) \cap B$ since B is ϱu-directed, i.e. $x \varrho s$ and $m \varrho s$. As m is ϱ-maximal, we have $m = s$, thus $x \varrho m$. As x was chosen arbitrary, m is the ϱ-greatest element of B.

Proposition 9. *Let $\langle A, \varrho \rangle$ satisfy the ϱ-maximal condition. Then each ϱ-ideal of $\langle A, \varrho \rangle$ is principal and has a ϱ-greatest element.*

Proof. By Proposition 1, each ϱ-ideal I of $\langle A, \varrho \rangle$ is ϱu-directed and, by Lemma 5, I has the ϱ-greatest element because $\langle A, \varrho \rangle$ satisfies the ϱ-maximal condition. By Lemma 4, I is principal.

Definition 6. Let $\langle A, \varrho \rangle$, $\langle B, \sigma \rangle$ be binary relational systems. A *homomorphism of $\langle A, \varrho \rangle$ into $\langle B, \sigma \rangle$* is a mapping h of A into B such that $a \varrho b$ implies $h(a) \sigma h(b)$. If h is a surjective and injective homomorphism of $\langle A, \varrho \rangle$ onto $\langle B, \sigma \rangle$ and h^{-1} is also a homomorphism of $\langle B, \sigma \rangle$ onto $\langle A, \varrho \rangle$ we call h an *isomorphism of $\langle A, \varrho \rangle$ onto $\langle B, \sigma \rangle$* and wirte $\langle A, \varrho \rangle \cong \langle B, \sigma \rangle$. For this definition see e.g. to [5].

Notation. If $\langle A, \varrho \rangle$ is principal, then by Lemma 1 the mapping $J_0 : a \to I(a)$ is a homomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathcal{I}_0(A), \subseteq \rangle$. Denote by Θ_0 the equivalence relation induced by J_0 on A. By the notation introduced in [5], $\langle A, \varrho \rangle / \Theta_0$ means the binary relational system $\langle A', \varrho' \rangle$, the support A' of which is the factor set A/Θ_0 and the relation ϱ' on A/Θ_0 is defined by $X, Y \in A/\Theta_0$, $X \varrho' Y$ if and only if $x \varrho y$ for some $x \in X$, $y \in Y$.

Denote by $[a]$ the class of A/Θ_0 containing the element a.

Proposition 10. *Let $\langle A, \varrho \rangle$ be principal. If each principal ϱ-ideal of $\langle A, \varrho \rangle$ has the ϱ-greatest element, then $\langle \mathscr{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle / \Theta_0$.*

Proof. Clearly the mapping $[a] \to I(a)$ is a bijection of A/Θ_0 onto $\mathscr{I}_0(A)$. Suppose $a, b \in A$, $[a]\ \varrho'\ [b]$. Then there exist $a' \in [a]$, $b' \in [b]$ with $a'\ \varrho\ b'$. By Lemma 1, $I(a') \subseteq I(b')$, hence $I(a) \subseteq I(b)$ and the mapping $[a] \to I(a)$ is a homomorphism. Let $I(a) \subseteq I(b)$. Denote by c the ϱ-greatest element of $I(b)$. Then $a\ \varrho\ c$, $b\ \varrho\ c$ and $c \in I(b)$, i.e. $I(b) \subseteq I(c)$. Clearly $I(c) \subseteq I(b)$, thus $I(b) = I(c)$. From $a\ \varrho\ c$ we have $[a]\ \varrho'\ [c]$ and from $I(b) = I(c)$ it follows that $[b] = [c]$, thus also $[a]\ \varrho'\ [b]$. Accordingly, also the converse mapping of $[a] \to I(a)$ is a homomorphism of $\langle A, \varrho \rangle / \Theta_0$ onto $\langle \mathscr{I}_0(A), \subseteq \rangle$, thus $\langle \mathscr{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle / \Theta_0$.

Corollary. *Let $\langle A, \varrho \rangle$ be a principal binary relational system satisfying the ϱ-maximal condition. Then $\langle \mathscr{I}(A), \subseteq \rangle$ is a lattice if and only if $\langle A, \varrho \rangle / \Theta_0$ is a lattice.*

This follows directly from Proposition 10, since by Proposition 9 each ϱ-ideal of $\langle A, \varrho \rangle$ is principal and has the ϱ-greatest element.

It is well-known (see e.g. [1]) that for a partial order $\leqq$ the mapping $a \to I(a)$ is an isomorphism of $\langle A, \leqq \rangle$ onto $\langle \mathscr{I}_0(A), \subseteq \rangle$. It can be proved that also the converse proposition is true. These facts show that partially ordered sets can be fully characterized by their sets of principal $\leqq$-ideals. This characterization is given by the following

Proposition 11. *Let $\langle A, \varrho \rangle$ be a binary relational system. The following conditions are equivalent:*

(a) *$\langle A, \varrho \rangle$ is principal and a is the ϱ-maximal element of $I(a)$ for each $a \in A$.*
(b) *J_0 is an isomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathscr{I}_0(A), \subseteq \rangle$.*
(c) *J_0 is an injective mapping of A onto $\mathscr{I}_0(A)$.*
(d) *ϱ is a partial ordering on A.*

Proof. Clearly (b) $\Rightarrow$ (c) and $(d) \Rightarrow (b)$. Prove (c) $\Rightarrow$ (a). The existence of J_0 implies that $\langle A, \varrho \rangle$ is principal. Let $a \in A$. Suppose the existence of $b \in I(a)$ with $a\ \varrho\ b$. By Lemma 1, $a\ \varrho\ b$ implies $I(a) \subseteq I(b)$, from $b \in I(a)$ we have $I(b) \subseteq I(a)$, thus $I(a) = I(b)$. From the injectivity of J_0 we have $a = b$. Thus a is the ϱ-maximal element of $I(a)$ for each $a \in A$.

It remains to prove (a) $\Rightarrow$ (d). Let $a \in A$ be the ϱ-maximal element of $I(a)$. As $I(a)$ is a ϱu-directed subset of A, by Lemma 5 a is the ϱ-greatest element of $I(a)$. Thus $a\ \varrho\ a$, i.e. ϱ is reflexive on A. Let $a, b \in A$ and $a\ \varrho\ b$, $b\ \varrho\ a$. By Lemma 1 we have $I(a) = I(b)$ and, by Lemma 5, $a = b$, since $I(a) = I(b)$ has just one ϱ-maximal element. Thus ϱ is also antisymmetrical. Suppose $a\ \varrho\ b$, $b\ \varrho\ c$ for $a, b, c \in A$. Then $I(a) \subseteq I(b) \subseteq I(c)$, i.e. $a \in I(c)$. As c is the ϱ-greatest element in $I(c)$ (by Lemma 5), we have $a\ \varrho\ c$. Accordingly, ϱ is also transitive, i.e. ϱ is a partial order on A.

Lemma 6. *Let ϱ be a transitive binary relation on A. If $a \in A$ and $a \varrho a$, then $I(a)$ exists and $I(a) = \{x \in A;\ x \varrho a\}$.*

Proof. Suppose $a \in A$ and $a \varrho a$. Denote $M = \{x \in A;\ x \varrho a\}$. Then $a \in M$ and $x, y \in M$ implies $x \varrho a$, $y \varrho a$, thus $a \in U(x, y) \cap M$. If $b \in M$, $x \in A$, $x \varrho b$, then $b \varrho a$ and the transitivity of ϱ implies $x \varrho a$ and hence $x \in M$. Accordingly, M is the ϱ-ideal of $\langle A, \varrho \rangle$ containing a. If I is also a ϱ-ideal of $\langle A, \varrho \rangle$ containing a, then $x \in M$ implies $x \varrho a$, thus, by (I_1), $x \in I$, i.e. $M \subseteq I$. Hence $I(a) = M$.

Proposition 12. *For an arbitrary binary relational system $\langle A, \varrho \rangle$ the following conditions are equivalent:*

(a) *$\langle A, \varrho \rangle$ is principal and $I(a) \subseteq I(b)$ if and only if $a \varrho b$;*
(b) *$\langle A, \varrho \rangle$ is principal and $I(a) = \{x \in A;\ x \varrho a\}$;*
(c) *ϱ is a quasiorder on A.*

Proof. If ϱ is a quasiorder, by Lemma 6 we obtain the implication (c) $\Rightarrow$ (b). Suppose (b). Then $I(a) \subseteq I(b)$ implies $a \varrho b$, the converse implication is given by Lemma 1, thus (b) $\Rightarrow$ (a). Suppose (a). $I(a) \subseteq I(a)$ for each $a \in A$, ϱ is reflexive. Let $a, b, c \in A$ and $a \varrho b$, $b \varrho c$. By Lemma 1 we obtain $I(a) \subseteq I(c)$ and the assumption (a) implies $a \varrho c$, thus ϱ is also transitive. Thus also (a) $\Rightarrow$ (c), which completes the proof.

Proposition 13. *Let ϱ be a quasiorder on A. If ϱ is uniquely supremal on A, then $\langle \mathscr{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle$.*

Proof. Let ϱ be uniquely supremal on A. As ϱ is reflexive and transitive, from unique supremality we have also the antisymmetry, thus ϱ is a partial ordering on A and, by Proposition 11, $\langle \mathscr{I}_0(A), \subseteq \rangle \cong \langle A, \varrho \rangle$.

Remark. Proposition 13 can be clearly dualized for ϱ uniquely infimal on A.

4. EMBEDDING OF RELATIONAL SYSTEMS INTO POSETS

The concept of a replica for the general case of algebraic structures is introduced in [5]. Its modification for the case of binary relational systems is given by

Definition 7. Let $\mathscr{C}$ be class of binary relational systems and let $\langle A, \varrho \rangle$ be an arbitrary system not necessarily from $\mathscr{C}$. A homomorphism h of $\langle A, \varrho \rangle$ onto a system $\langle D, \delta \rangle \in \mathscr{C}$ is called an *embedding of $\langle A, \varrho \rangle$ into $\mathscr{C}$* and $\langle D, \delta \rangle$ is called a *$\mathscr{C}$-replica*, if for each system $\langle B, \beta \rangle \in \mathscr{C}$ and an arbitrary homomorphism g of $\langle A, \varrho \rangle$ onto $\langle B, \beta \rangle$ there exists a homomorphism f of $\langle D, \delta \rangle$ onto $\langle B, \beta \rangle$ with $g = f \cdot h$.

Denote by $\mathscr{P}$ the class of all partially ordered sets. It is known (see e.g. [5], § 11.3) that $\mathscr{P}$ forms a quasivariety of algebraic systems. Thus, by Theorem 5 from § 11.3

in [5], for an arbitrary binary relational system $\langle A, \varrho \rangle$ there exists an embedding into $\mathscr{P}$ and a $\mathscr{P}$-replica. In this section we shall give a condition for $\langle A, \varrho \rangle$ to have a $\mathscr{P}$-replica $\langle \mathscr{I}_0(A), \subseteq \rangle$.

Definition 8. A binary relational system $\langle A, \varrho \rangle$ is called *strictly principal*, if it is principal and $I(a) \subseteq I(b)$ implies $a\ t(\varrho)\ b$.

Example 4. If ϱ is a complete relation on A, then, by Proposition 8 (a), $\langle A, \varrho \rangle$ is strictly principal.

If ϱ is a quasiorder on A, then $\langle A, \varrho \rangle$ is strictly principal by Proposition 12 (a).

If $\langle A, \varrho \rangle$ is a *finite cycle*, i.e. $A = \{a_1, \ldots, a_n\}$ and $a_1\ \varrho\ a_2, \ldots, a_{n-1}\ \varrho\ a_n$, $a_n\ \varrho\ a_1$ (ϱ need not be transitive or reflexive), then $I(a) = A$ for each $a \in A$ and $a\ t(\varrho)\ b$ is also true for each $a, b \in A$, thus $\langle A, \varrho \rangle$ is strictly principal.

Proposition 14. *Let $\langle A, \varrho \rangle$ be a strictly principal binary relational system. Then $\langle \mathscr{I}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica and J_0 is an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$.*

Proof. By Lemma 1, J_0 is a homomorphism of $\langle A, \varrho \rangle$ onto $\langle \mathscr{I}_0(A), \subseteq \rangle \in \mathscr{P}$. Let $\langle P, \leqq \rangle \in \mathscr{P}$ and let g be a homomorphism of $\langle A, \varrho \rangle$ onto $\langle P, \leqq \rangle$. Introduce the relation $\mathscr{I}_0(A) \to P$ by the rule $I(a) \to g(a)$ for each $a \in A$.

1°. If $I(a) = I(b)$, then $a\ t(\varrho)\ b$, $b\ t(\varrho)\ a$, i.e. there exist $a_0, \ldots, a_n, b_0, \ldots, b_m \in A$ such that $a_0 = a = b_m$, $b_0 = b = a_n$ and $a_{i-1}\ \varrho\ a_i$ $(i = 1, \ldots, n)$, $b_{j-1}\ \varrho\ b_j$ $(j = 1, \ldots, m)$. As g is a homomorphism, it follows that $g(a) \leqq g(b)$ and $g(b) \leqq g(a)$. As $\leqq$ is a partial order, $g(a) = g(b)$. Accordingly, the relation $\to$ is a mapping of $\mathscr{I}_0(A)$ onto P. Denote this mapping by f.

2°. If $I(a) \subseteq I(b)$, then $a\ t(\varrho)\ b$ because $\langle A, \varrho \rangle$ is strictly principal, i.e. there exist $a_0, \ldots, a_n \in A$ with $a_0 = a$, $a_n = b$, $a_{i-1}\ \varrho\ a_i$ for $i = 1, \ldots, n$. As g is a homomorphism, we have $g(a) \leqq g(b)$. Thus the mapping f is a homomorphism of $\langle \mathscr{I}_0(A), \subseteq \rangle$ onto $\langle P, \leqq \rangle$.

3°. Evidently, $f(J_0(a)) = f(I(a)) = g(a)$ for each $a \in A$, thus $\langle \mathscr{I}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica and J_0 an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$.

Corollary 1. *Let ϱ be a reflexive binary relation on a set A and let $\langle A, \varrho \rangle$ be principal. If a principal ϱ-ideal generated by $a \in A$ in $\langle A, \varrho \rangle$ is equal to the principal $t(\varrho)$-ideal generated by $a \in A$ in $\langle A, t(\varrho) \rangle$ for each $a \in A$, then J_0 is an embedding of $\langle A, \varrho \rangle$ into $\mathscr{P}$ and $\langle \mathscr{I}_0(A), \subseteq \rangle$ is a $\mathscr{P}$-replica.*

Proof. If ϱ is reflexive, then $\sigma = t(\varrho)$ is a quasiorder on A, and by Proposition 12, $\langle A, \varrho \rangle$ is principal and $I(a) \subseteq I(b) \Rightarrow a\ \sigma\ b$, i.e. $a\ t(\varrho)\ b$. As $I(a)$ is the same in $\langle A, \varrho \rangle$ as in $\langle A, \sigma \rangle$, it follows that $\langle A, \varrho \rangle$ is strictly principal and, by Proposition 14, we obtain the result.

Corollary 2. *Let ϱ be a complete relation on A. Then the $\mathcal{P}$-replica $\langle \mathcal{I}_0(A), \subseteq \rangle$ of $\langle A, \varrho \rangle$ is a chain.*

It follows directly from Proposition 14 and Proposition 8.

Corollary 3. *Let ϱ be an equivalence relation on a set A. Then the $\mathcal{P}$-replica of $\langle A, \varrho \rangle$ is the antichain (i.e. a complete unordered set) $\langle A/\varrho, \subseteq \rangle$.*

Proof. By example 1, $\mathcal{I}(A) = A/\varrho$ for an equivalence relation ϱ on A. Then clearly $I(a) = [a]$ for each $a \in A$, where $[a]$ denotes the class of the partition A/ϱ, $I(a) \subseteq \subseteq I(b)$ is equivalent to $[a] \subseteq [b]$, which is equivalent to $[a] = [b]$, i.e. $a \varrho b$. Hence $\langle A, \varrho \rangle$ is also strictly principal and, by Proposition 14, the assertion is obtained, because $\mathcal{I}_0(A) = \mathcal{I}(A) = A/\varrho$.

References

[1] *Birkhoff G.:* Lattice Theory, New York 1940.

[2] *Fried E.:* Tournaments and non-associative lattices, Annales Univ. Sci. Budapest., Sectio Math., *13* (1970), 151—164.

[3] *Fried E., Grätzer G.:* Some examples of weakly associative lattices, Colloq. Math., *27* (1973), 215—221.

[4] *Cohn P. M.:* Universal Algebra, Harper and Row, New York 1965.

[5] *Мальцев А. И.:* Алгебраические системы, Москва 1970.

[6] *Rachůnek J.:* o-idéaux des ensembles ordonnés, Acta Univ. Palack. Olomouc., fac. rer. natur., tom. *45* (1974), 77—81.

[7] *Szász G.:* Instroduction to lattice theory, Akadémiai Kiadó, Budapest 1963.

Authors' addresses: J. Duda, 616 00 Brno, Kroftova 21, I. Chajda, 750 00 Přerov, tř. Lidových milicí 290.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

UNIVERSAL SIMULTANEOUS APPROXIMATIONS OF THE COEFFICIENT FUNCTIONALS

PETR PŘIKRYL, Praha

(Received June 14, 1976)

Universal approximations, a concept which appeared in numerical mathematics some years ago [1], [2], had resulted from the attempt to avoid problems connected with the choice of the space over which the given functional should be (best) approximated. BABUŠKA and SOBOLEV [3] pointed out that the dependence of the best approximation on the space can have unpleasant numerical consequences. The information at our disposal is usually not sufficient to determine a unique space over which the given functional should be approximated optimally and the conclusions on the advantage of optimum methods are thus "unstable" in practice. This implies the importance of finding approximations the error of which does not differ "too much" from those of the best approximations in a wide class of spaces. Such approximations are then called universal.

In an earlier paper of the author the universal approximations of Fourier coefficients in a particular class of Hilbert spaces were studied [5]. Later, the author announced some results valid for general classes of Hilbert spaces [6]. This paper treats the approximations of the coefficient functionals associated with a basis of Banach spaces and the conclusions of [6] are here contained as a special case. Since the fundamental ideas here are similar to those of [5] we proceed rather briefly in this paper, referring to [5] whenever convenient*).

1. BEST APPROXIMATIONS: SOME LOWER BOUNDS

We shall deal with classes $\mathfrak{B}$ of Banach spaces (B-spaces) E over the field of complex numbers, generated by a common Schauder basis $\{x_j\}$. Let $E \in \mathfrak{B}$ and assume that we want to compute the values $F_j(x)$, $j = 1, 2, \ldots, r$ where

$$x = \sum_{j=1}^{\infty} F_j(x)\, x_j$$

and $r > 1$.

*) The results of this paper were presented at the Third Conference on Basic Problems of Numerical Mathematics (Prague 1973) (cf. [7]).

We shall approximate the vector $[F_j]$ of functionals from E^* by another vector $[G_j]$, $G_j \in E^*$. The approximating functionals are assumed to be of the form

$$G_j(x) = \sum_{k=1}^{n} a_k(x)\, g_k(j)\,, \tag{1.1}$$

where $1 \leqq n < r$, $a_k \in E^*$ and g_k are complex-valued functions of an integer argument j. Thus instead of calculating r values $F_j(x)$ we compute n values $a_k(x)$, $n < r$. The main question we ask in this paper is how to choose the matrix $[g_k(j)]$ ($k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, r$) properly.

For a given n, denote by M_n the set of all the approximations $[G_j]$ where G_j is of the form (1.1). We define the *error of the approximation* as

$$\omega_E([G_j]) = \max_{j=1,2,\ldots,r} \|F_j - G_j\|_{E^*}\,. \tag{1.2}$$

Let $M \subset M_n$ for some n. Then the *best* (or *optimal*) *approximation* from the set M (if it exists) has the error

$$\Omega_E(M) = \inf_{[G_j]\in M} \omega_E([G_j])\,. \tag{1.3}$$

Obviously $\Omega_E(M) \geqq \Omega_E(M_n)$ for any $M \subset M_n$.

A positive lower bound can be derived for $\Omega_E(M_n)$, which is of decisive importance for further considerations.

Theorem 1.1. *Let $E \in \mathfrak{B}$ and choose $n + 1$ integers $j_1, j_2, \ldots, j_{n+1}$ in such a way that $1 \leqq j_s \leqq r$ and $j_s \neq j_t$ whenever $s \neq t$. Then*

$$\Omega_E(M_n) \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\|\right)^{-1}. \tag{1.4}$$

Proof. We shall make use of Lemma 4.1 of [5], which is obviously valid also in B-spaces. We reformulate it for the reader's convenience, but without proof.

Lemma 1.1. *Let $E \in \mathfrak{B}$ and denote by θ the zero element of E. If for every $x \in E$ and all approximations $[G_j] \in M_n$*

$$\inf_{\substack{a_k \\ k=1,2,\ldots,n}} \max_{j=1,2,\ldots,r} |F_j(x) - G_j(x)| \geqq C_E(x, g_1, g_2, \ldots, g_n) \tag{1.5}$$

is valid, then

$$\Omega_E(M_n) \geqq \inf_{\substack{g_k \\ k=1,2,\ldots,n}} \sup_{\substack{x\in E \\ x\neq\theta}} \left(\|x\|^{-1} \cdot C_E(x, g_1, g_2, \ldots, g_n)\right). \tag{1.6}$$

Therefore, we need a lower bound for

$$\inf_{\substack{a_k\\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} |F_j(x) - G_j(x)| =$$

$$= \inf_{\substack{a_k\\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} |F_j(x) - \sum_{k=1}^{n} a_k(x)\, g_k(j)| .$$

This can be found in the same manner as in [5]. We choose $n + 1$ integers j_s, $s = 1, 2, \dots, n + 1$ satisfying the hypotheses of the theorem. Then we compose $n + 1$ n-dimensional vectors

$$[g_1(j_s), g_2(j_s), \dots, g_n(j_s)] , \quad s = 1, 2, \dots, n + 1 .$$

These vectors are linearly dependent and we can find numbers $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ such that

$$\sum_{s=1}^{n+1} \lambda_s\, g_k(j_s) = 0 , \quad k = 1, 2, \dots, n , \tag{1.7}$$

and

$$\sum_{s=1}^{n+1} |\lambda_s| = 1 . \tag{1.8}$$

Every vector $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots, \lambda_{n+1}]$ satisfying (1.7) and (1.8) will be called *determined by the matrix* $[g_k(j_s)]$. For any such $\boldsymbol{\lambda}$ and any $x \in E$ we have

$$\sum_{s=1}^{n+1} \lambda_s (F_{j_s}(x) - G_{j_s}(x)) = \sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x) .$$

In virtue of (1.8) we obtain

$$\max_{s=1,2,\dots,n+1} |F_{j_s}(x) - G_{j_s}(x)| \geqq |\sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x)|$$

and further

$$\inf_{\substack{a_k\\ k=1,2,\dots,n}} \max_{j=1,2,\dots,r} |F_j(x) - G_j(x)| \geqq |\sum_{s=1}^{n+1} \lambda_s\, F_{j_s}(x)| \tag{1.9}$$

for every $x \in E$ and $[G_j] \in M_n$. The right-hand side of (1.9) is independent of a_k's and this bound thus satisfies the hypothesis of Lemma 1.1. In fact, for every $[G_j] \in M_n$ (1.9) generally represents a family of bounds (we can obtain different bounds with different solutions of the problem (1.7), (1.8)). This ambiguity is, however, insignificant and we can avoid it by assigning each $[g_k(j)]$ some fixed vector determined by $[g_k(j_s)]$.

Let $N \subset E$. Using Lemma 1.1 we now get easily

$$(1.10) \qquad \Omega_E(M_n) \geqq \inf_{\substack{g_k \\ k=1,2,\ldots,n}} \sup_{\substack{x\in N \\ x\neq\theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|} \geqq \inf_{\lambda} \sup_{\substack{x\in N \\ x\neq\theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|}$$

where the second infimum is taken over all λ's satisfying (1.8)*). We shall take for N the linear subspace of E spanned by $x_{j_1}, x_{j_2}, \ldots, x_{j_{n+1}}$. Put

$$\hat{x}(\lambda) = \sum_{s=1}^{n+1} \frac{\bar{\lambda}_s}{\|x_{j_s}\|} x_{j_s} ;$$

obviously $\hat{x}(\lambda) \in N$. For every λ satisfying (1.8) we have

$$(1.11) \qquad \sup_{\substack{x\in N \\ x\neq\theta}} \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(x)\right|}{\|x\|} \geqq \frac{\left|\sum_{s=1}^{n+1} \lambda_s F_{j_s}(\hat{x})\right|}{\|\hat{x}\|} = \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|},$$

as $F_{j_s}(\hat{x}) = \bar{\lambda}_s \,.\, \|x_{j_s}\|^{-1}$ and $\|\hat{x}\| \leqq 1$. In view of (1.10) we have thus obtained

$$(1.12) \qquad \Omega_E(M_n) \geqq \inf_{\lambda} \left(\sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|} \right)$$

where the infimum has the same meaning as above.

The proof can now be readily finished. Firstly,

$$(1.13) \qquad \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|} \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\|\right)^{-1},$$

which can be easily verified by the Cauchy inequality, and this lower bound is actually the least one since for

$$\lambda_s = \frac{\|x_{j_s}\|}{\sum_{t=1}^{n+1} \|x_{j_t}\|}$$

(1.13) becomes equality. The theorem is proved.

Proving Theorem 1.1 we have obtained also the following result regarding the set $M_n^g \subset M_n$ of the approximations with a fixed matrix $[g_k(j)]$ (cf. (1.11)).

Theorem 1.2. *Let $E \in \mathfrak{B}$. Given a matrix $[g_k(j)]$, choose integers $j_1, j_2, \ldots, j_{n+1}$ in such a way that $1 \leqq j_s \leqq r$ and $j_s \neq j_t$ whenever $s \neq t$.*

---

*) It can be shown (by assigning each λ a fixed $[g_k(j)]$ such that (1.7) holds) that even the equality sign could be written in the second part of (1.10).

Then

$$(1.14) \qquad \Omega_E(M_n^g) \geqq \sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|}$$

for any $\boldsymbol{\lambda}$ *determined by* $[g_k(j_s)]$.

The above bounds on Ω are generally improvable since e.g. for a special class of Hilbert spaces we obtained in [5]

$$\Omega_H(M_n) \geqq \left(\sum_{s=1}^{n+1} \|x_{j_s}\|^2\right)^{-1/2}$$

and

$$\Omega_H(M_n^g) \geqq \left(\sum_{s=1}^{n+1} \frac{|\lambda_s|^2}{\|x_{j_s}\|^2}\right)^{1/2}.$$

Nevertheless, for our further qualitative considerations the bounds of Theorems 1.1 and 1.2 are sufficient.

We can see from Theorem 1.1 that $\Omega_E(M_n) > 0$. Hence, we can form the ratio

$$\varrho_E(M, [G_j]) = \frac{\omega_E([G_j])}{\Omega_E(M)}$$

and use this ratio to measure the quality of a given approximation $[G_j] \in M \subset M_n$ with respect to the set M.

2. UNIVERSAL APPROXIMATIONS

In [5], we constructed the approximation $[K_j^*]$ which was optimal in a given Hilbert space H_0 from a class $\mathfrak{H}$ and we showed that this approximation can be very "bad" in other spaces from the class $\mathfrak{H}$ considered. Namely, we proved that for any D positive a space $H_D \in \mathfrak{H}$ existed such that $\varrho_{H_D}(M_n, [K_j^*]) > D$ – even if $\varrho_{H_0}(M_n, [K_j^*]) = 1$. This effect led us to the introduction of the concept of a universal approximation.

Definition 2.1. An approximation $[G_j] \in M_n$ is said to be *universal* for a given class $\mathfrak{B}$ of B-spaces if there exists a constant D such that

$$(2.1) \qquad \varrho_E(M_n, [G_j]) \leqq D$$

for any $E \in \mathfrak{B}$.

So, in contrast with the optimality, universality is related to some *class* of spaces.

It is clear that for a sufficiently small class $\mathfrak{B}$ (e.g. consisting of only a finite number of spaces E) every approximation from M_n would be universal. On the other hand, for wider classes of spaces a universal approximation need not exist [5]. It is therefore reasonable to search for some (as general as possible) conditions on $\mathfrak{B}$ that would guarantee the existence of a universal approximation. The concept of a conservative class of spaces will play an important role in such conditions.

Definition 2.2. We shall call the class $\mathfrak{B}$ of B-spaces E *conservative*, if the elements of the common basis can be assigned subscripts in such a way that

$$\|x_1\| \leqq \|x_2\| \leqq \ldots \leqq \|x_r\| \tag{2.2}$$

in every $E \in \mathfrak{B}$.

In the remainder of the paper we shall be concerned with conservative classes of B-spaces only and we shall assume that the basis $\{x_j\}$ has been ordered in such a way that (2.2) holds.

We now formulate some conditions on $\mathfrak{B}$ that are sufficient for a universal approximation to exist. We denote by S_n the (continuous) linear operator given by

$$S_n(x) = \sum_{j=1}^{n} F_j(x)\, x_j\,,$$

($x \in E,\ n = 1, 2, \ldots$). Further denote

$$v_r(E) = \sup_{1 \leqq n \leqq r} \|S_n\|$$

and

$$v(E) = \sup_{1 \leqq n < \infty} \|S_n\|$$

(the *norm* of the basis $\{x_j\}$).

Theorem 2.1. *Let $\mathfrak{B}$ be a conservative class of B-spaces. If*

$$v_r(E) \leqq K \tag{2.3}$$

for every $E \in \mathfrak{B}$ and K is independent of E, then for each n, $1 \leqq n \leqq r$, there exists an approximation $[B_j] \in M_n$ universal with respect to $\mathfrak{B}$. This approximation is defined by (1.1), *where*

$$a_k = F_k\,, \quad k = 1, 2, \ldots, n\,, \tag{2.4}$$

$$g_k(j) = \delta_{kj}\,, \quad k = 1, 2, \ldots, n\,, \quad j = 1, 2, \ldots, r\,.\text{*)}$$

Moreover,

$$Q_E(M_n, [B_j]) \leqq 2K(n + 1) \tag{2.5}$$

in every $E \in \mathfrak{B}$.

---

*) $\delta_{kj} = 0$ unless $k = j$, in which case $\delta_{kj} = 1$.

Proof. The error of $[B_j]$ is

$$\omega_E([B_j]) = \max_{j=n+1,\dots,r} \|F_j\|_{E^*} \equiv \|F_q\|_{E^*} \tag{2.6}$$

where $n + 1 \leqq q \leqq r$. Using Theorem 1.1 with $j_s = s$, $s = 1, 2, \dots, n + 1$ we obtain

$$Q_E(M_n, [B_j]) \leqq \|F_q\|_{E^*} \cdot \sum_{s=1}^{n+1} \|x_s\| . \tag{2.7}$$

We need to estimate $\|F_q\|_{E^*}$ by means of $\|x_q\|^{-1}$. It will prove sufficient to proceed in a very simple manner. We write

$$|F_q(x)| = \frac{\|F_q(x)\, x_q\|}{\|x_q\|} .$$

This yields

$$|F_q(x)| \leqq \frac{\|\sum_{s=1}^{q} F_s(x)\, x_s\| + \|\sum_{s=1}^{q-1} F_s(x)\, x_s\|}{\|x_q\|} \leqq \frac{2\, v_q(E)\, \|x\|}{\|x_q\|} \leqq \frac{2K}{\|x_q\|} \cdot \|x\| .$$

Hence,

$$\|F_q\|_{E^*} \leqq \frac{2K}{\|x_q\|} . \tag{2.8}$$

From (2.7) and (2.8) we now get

$$Q_E(M_n, [B_j]) \leqq 2K \sum_{s=1}^{n+1} \frac{\|x_s\|}{\|x_q\|} .$$

Since $\mathfrak{B}$ is conservative and $q \geqq n + 1$, we have $\|x_s\| \leqq \|x_q\|$, $s = 1, 2, \dots, n + 1$, and

$$Q_E(M_n, [B_j]) \leqq 2K(n + 1) ,$$

which completes the proof.

Remark 2.1. It can be seen from the proof of Theorem 2.1 that we could assume $\|x_j\| \leqq \|x_k\|$ whenever $1 \leqq j \leqq n + 1 \leqq k \leqq r$ instead of the conservativeness to obtain the same result for a fixed n.

Remark 2.2. A basis $\{x_j\}$ of a B-space E is said to be *monotone* if we have

$$\|\sum_{i=1}^{n} \alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m} \alpha_i x_i\|$$

for all finite sequences of complex numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+m}$. Monotone bases satisfy the condition (2.3) of Theorem 2.1 trivially since their norm is $v(E) = 1$ in any B-space [8]. In Hilbert spaces, monotonicity is equivalent to orthogonality.

We now give some examples of classes $\mathfrak{B}$ satisfying the hypotheses of Theorem 2.1.

Example 2.1. *The spaces $L^p([0, 1])$ and the Haar functions.* It is well-known [8] that the sequence of equivalence classes $\{\tilde{y}_j\}$, where y_j are the Haar functions, i.e. the functions defined on $[0, 1]$ by

$$(2.9) \qquad y_1(t) \equiv 1 ,$$

$$y_{2^k+l}(t) = \begin{cases} \sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-2}{2^{k+1}}, \dfrac{2l-1}{2^{k+1}}\right), \\ -\sqrt{2^k} & \text{for} \quad t \in \left[\dfrac{2l-1}{2^{k+1}}, \dfrac{2l}{2^{k+1}}\right), \\ 0 & \text{for the other} \quad t , \end{cases}$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$ constitutes a basis of the space $L^p([0, 1])$ $(p \geqq 1)$. Further, it may be shown that this basis is monotone [8]. An easy computation yields

$$(2.10) \qquad \|\tilde{y}_1\|_p = 1 , \quad \|\tilde{y}_{2^k+l}\|_p = (2^{(p-2)/2p})^k ,$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$ where $\|\cdot\|_p$ denotes the norm in $L^p([0, 1])$. From (2.10) we see that the class $\mathfrak{B}_1$ of the spaces $L^p([0, 1])$, $p \geqq 2$, with the Haar basis is conservative. Hence, $\mathfrak{B}_1$ satisfies the assumptions of Theorem 2.1 with $K = 1$.

Example 2.2. *General separable Orlicz spaces with the Haar basis.* Let $M(u)$ be an even convex continuous function defined on $(-\infty, +\infty)$ with the following properties:

$$(2.11) \qquad \text{a)} \quad \lim_{n \to 0} \frac{M(u)}{u} = 0 ,$$

$$\text{b)} \quad \lim_{n \to \infty} \frac{M(u)}{u} = \infty ,$$

c) there exist constants $k > 0$, $u_0 \geqq 0$ such that $M(2u) \leqq k\, M(u)$ for $u \geqq u_0$.

The general separable Orlicz space*) $L_M([0, 1])$ is then the space of the equivalence classes $\tilde{u}$ given by real-valued functions $u(t)$ defined on $[0, 1]$ for which

$$(2.12) \qquad \int_0^1 M(u(t))\, dt < \infty .$$

*) Proofs of the properties of Orlicz spaces and functions $M(u)$ used in this example can be found e.g. in [4].

The norm $\|\tilde{u}\|_M$ can be introduced by the relation

$$(2.13) \qquad \int_0^1 M\left[\frac{u(t)}{\|\tilde{u}\|_M}\right] \mathrm{d}t = 1 .$$

It can be proved that $L_M([0, 1])$ with this norm is a separable B-space and, moreover, the equivalence classes $\{\tilde{y}_j\}$ where $y_j(t)$ are the Haar functions defined by (2.9) constitute a monotone basis of $L_M([0, 1])$ [9].

For example, $M(u) = |u|^p (p > 1)$ satisfies (2.11) and this choice yields $L_M([0, 1]) = L^p([0, 1])$. Another possible choice of $M(u)$ is

$$(2.14) \qquad M(u) = |u|^p (|\ln |u|| + 1)$$

with $p > 1$; the resulting Orlicz spaces are different from the L^p-spaces.

According to (2.13), the norms of the Haar functions are

$$(2.15) \qquad \|\tilde{y}_1\|_M = \frac{1}{M^{-1}(1)},$$

$$\|\tilde{y}_{2^k+l}\|_M = \frac{\sqrt{2^k}}{M^{-1}(2^k)},$$

$(l = 1, 2, \ldots, 2^k;\ k = 0, 1, 2, \ldots)$, where $M^{-1}(v)$ is the inverse function for the function $M(u)$ considered on $[0, +\infty)$. (It may be shown that every $M(u)$ that satisfies (2.11) is increasing on $[0, +\infty)$.)

Let $\mathfrak{B}_2$ be the class of separable Orlicz spaces $L_M([0, 1])$ whose $M(u)$ satisfy

$$(2.16) \qquad M(u \sqrt{2}) \geqq 2\, M(u)$$

for $u \geqq M^{-1}(1)$. The class $\mathfrak{B}_2$ contains e.g. the spaces $L^p([0, 1])$ for $p \geqq 2$ and the spaces $L_M([0, 1])$ with $M(u)$ given by (2.14) for $p \geqq 2$.

We now show that $\mathfrak{B}_2$ with the Haar basis is conservative. It is sufficient to prove that (2.16) implies

$$(2.17) \qquad 2^{1/2} M^{-1}(v) \geqq M^{-1}(2v)$$

for $v \geqq 1$. Denote $v = M(u)$. We can write (2.16) as $2v \leqq M(u \sqrt{2})$. Since $M(u)$ is increasing for $u \geqq 0$, $M^{-1}(v)$ is increasing for $v \geqq 0$ and we have

$$M^{-1}(2v) \leqq u \sqrt{2} = 2^{1/2} M^{-1}(v),$$

which is (2.17). (2.17) yields the conservativeness immediately.

We recall that the Haar basis is monotone and conclude that the class $\mathfrak{B}_2$ satisfies the assumptions of Theorem 2.1 with $K = 1$.

It is a priori clear that e.g. in the case described by Theorem 2.1 more than one universal approximation exist. For example, the approximation $[\tilde{B}_j]$ with the same $[g_k(j)]$ as $[B_j]$ and $a_k = F_k + c_k F_{q_k}$, $q_k \geqq n + 1$, c_k arbitrary complex numbers, $k = 1, 2, \ldots, n$, is also universal with respect to the class $\mathfrak{B}$ described in the above theorem. It is reasonable, therefore, to search for the universal approximations with minimum error.

To be able to do this we need a characterization of the set $U_n \subset M_n$ of all the approximations universal with respect to a given class $\mathfrak{B}$. We shall describe U_n by means of some conditions on $[g_k(j)]$ which the universal approximations satisfy necessarily. Such results are also of interest in answering the question of the proper choice of $[g_k(j)]$. In order to find the necessary properties of matrices $[g_k(j)]$ we must suppose, however, that the class $\mathfrak{B}$ considered is sufficiently wide.

Theorem 3.1. *Let $\mathfrak{B}$ be a conservative class of B-spaces. Let $v_r(E) \leqq K$ for every $E \in \mathfrak{B}$ (K independent of E) and let n be an integer, $1 \leqq n \leqq r$, such that for any D there exists a space $E_D \in \mathfrak{B}$ in which*

$$\frac{\|x_{n+1}\|_{E_D}}{\|x_n\|_{E_D}} > D\,. \tag{3.1}$$

Then the matrices $[g_k(j)]$ of a universal approximation $[G_j] \in U_n$ have the following two properties:

$$\text{a) } g_k(j) = 0\,, \quad k = 1, 2, \ldots, n\,, \quad j = n + 1, \ldots, r\,, \tag{3.2}$$

$$\text{b) } \operatorname{rank}\left([g_k(j)]_{k,j=1}^{n}\right) = n\,.$$

Proof is exactly parallel to that of Theorem 5.6 in [5] and will be only sketched.

For every s such that $n + 1 \leqq s \leqq r$ denote by $[g_k(j)]_s$ the $n \times (n + 1)$ submatrix of $[g_k(j)]$ consisting of the columns $1, 2, \ldots, n, s$. We shall investigate the solutions $\boldsymbol{\lambda}^{(s)} = [\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)}]^T$ of

$$[g_k(j)]_s \, \boldsymbol{\lambda}^{(s)} = 0 \tag{3.3}$$

satisfying

$$\sum_{j=1}^{n+1} |\lambda_j^{(s)}| = 1\,. \tag{3.4}$$

Denote $|\boldsymbol{\lambda}^{(s)}| = [|\lambda_1^{(s)}|, |\lambda_2^{(s)}|, \ldots, |\lambda_{n+1}^{(s)}|]^T$ and let $\mathbf{e}_k$ be the k-th unit vector.

The proof is based on Lemma 5.1 of [5], which we present in a somewhat modified form:

Lemma 3.1. *The conditions (3.2) are equivalent to the following statement:*

There exists a unique system of vectors $\{|\boldsymbol{\lambda}^{(s)}|\}_{s=n+1}^{r}$ such that $\boldsymbol{\lambda}^{(s)}$, $s = n + 1$, $n + 2, \ldots, r$, satisfy (3.3) and (3.4), namely $|\boldsymbol{\lambda}^{(n+1)}| = |\boldsymbol{\lambda}^{(n+2)}| = \ldots = |\boldsymbol{\lambda}^{(r)}| = \mathbf{e}_{n+1}$.

The proof of the theorem is by contradiction. If the conditions (3.2) are violated, then using Lemma 3.1 we conclude that for some s, $n + 1 \leqq s \leqq r$, there exists a vector $\boldsymbol{\lambda}^{(s)}$ satisfying (3.3) and (3.4) whose p-th component, $1 \leqq p \leqq n$, is not zero. According to Theorem 1.2, we have for the approximation $[G_j]$ violating (3.2)

$$\omega([G_j]) \geqq \sum_{j=1}^{n} \frac{|\lambda_j^{(s)}|^2}{\|x_j\|} + \frac{|\lambda_{n+1}^{(s)}|^2}{\|x_s\|} \geqq \frac{|\lambda_p^{(s)}|^2}{\|x_p\|}. \tag{3.5}$$

To complete the proof we need an appropriate upper bound for $\Omega(M_n)$. It is sufficient to make use of the trivial fact that $\Omega(M_n) \leqq \omega([I_j])$ for any approximation $[I_j] \in M_n$. Choosing for $[I_j]$ the approximation $[B_j]$ from Theorem 2.1 and using (2.8) we obtain

$$\Omega(M_n) \leqq \frac{2K}{\|x_q\|}, \tag{3.6}$$

where $n + 1 \leqq q \leqq r$. (3.5) and (3.6) now yield for the approximation $[G_j]$

$$Q(M_n, [G_j]) \geqq \frac{|\lambda_p^{(s)}|^2}{2K} \cdot \frac{\|x_{n+1}\|}{\|x_n\|}$$

and, in view of (3.1), $[G_j]$ is not universal.

The classes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ from Examples 2.1 and 2.2 do not satisfy (3.1). It is easy, however, to construct classes of Hilbert spaces with orthogonal bases [5], [6] satisfying the assumptions of Theorem 3.1. The strongly periodic spaces described in [5] may serve as an example.

Theorem 1.2 and Lemma 3.1 imply immediately that the error of an optimal approximation from U_n is bounded by

$$\Omega(U_n) \geqq \frac{1}{\|x_{n+1}\|} \tag{3.7}$$

in all spaces satisfying the assumptions of Theorem 3.1.

Let us consider again the approximation $[B_j]$ from Theorem 2.1. This approximation belongs to U_n and we can compare its error with $\Omega(U_n)$.

Theorem 3.2. *Let $\mathfrak{B}$ be a conservative class of B-spaces. Let $v_r(E) \leqq K$ for every $E \in \mathfrak{B}$ and let n be an integer, $1 \leqq n \leqq r$, such that for any D there exists a space $E_D \in \mathfrak{B}$ in which* (3.1) *holds. Then*

$$Q_E(U_n, [B_j]) \leqq 2K \tag{3.8}$$

in every $E \in \mathfrak{B}$.

If, moreover, the basis $\{x_j\}$ is orthogonal); then $[B_j]$ is an optimal universal approximation in every $E \in \mathfrak{B}$, i.e. $Q_E(U_n, [B_j]) = 1$ in every $E \in \mathfrak{B}$.*

*) A basis $\{x_j\}$ of E is orthogonal, if every permutation of $\{x_j\}$ is a monotone basis of E. In Hilbert spaces this is the orthogonality in the usual sense [8].

Proof. From (2.6) we have

$$\omega_E([B_j]) = \|F_q\|_{E^*}, \quad n + 1 \leqq q \leqq r .$$

Using (3.7), (2.8) we obtain

$$Q_E(U_n, [B_j]) \leqq \|F_q\|_{E^*} \cdot \|x_{n+1}\| \leqq 2K \frac{\|x_{n+1}\|}{\|x_q\|} \leqq 2K . \tag{3.9}$$

The orthogonality implies (cf. [8], p. 556) $\|F_j\|_{E^*} = \|x_j\|_E^{-1}$ and, instead of (3.9), we have

$$Q_E(U_n, [B_j]) \leqq \|x_q\|^{-1} \cdot \|x_{n+1}\| \leqq 1 .$$

The theorem is proved.

Remark 3.1. The conclusions on the advantage of the approximation $[B_j]$ can be given more practical meaning by replacing the functionals F_k in (2.4) by sequences of functionals assumed to be convergent to F_k in every $E \in \mathfrak{B}$. The whole procedure would be the same as in [5] where this was done for a special class of Hilbert spaces. So, asymptotic results analogous to the above "theoretical" ones could be obtained having computational character. Since the procedure would bring nothing new as compared with [5] we have omitted this aspect in the present paper.

References

[1] *I. Babuška:* Über optimale Formeln zur numerischen Berechnung linearer Funktionale, Aplikace matematiky *10*, pp. 441—443 (1965).

[2] *I. Babuška:* Über universal optimale Quadraturformeln, Aplikace matematiky *13*, pp. 304—338, 388—404 (1968).

[3] *I. Babuška, S. L. Sobolev:* Оптимизация численных методов, Aplikace matematiky *10*, pp. 96—129 (1965).

[4] *М. А. Красносельский, Я. Б. Рутицкий:* Выпуклые функции и пространства Орлича. Государственное издательство физико-математической литературы, Москва 1958.

[5] *P. Přikryl:* Optimal universal approximations of Fourier coefficients in spaces of continuous periodic functions, Čas. pěst. mat. *97*, pp. 259—296 (1972).

[6] *P. Přikryl:* О наилучших универсальных приближениях коэффициентов Фурье. In: Применение функциональных методов к краевым задачам математической физики. Институт математики СО АН СССР, Новосибирск 1972, pp. 191—198.

[7] *P. Přikryl:* Universal approximations of certain functionals in Banach spaces, Acta Universitatis Carolinae — Math. et Phys. *15*, pp. 135—136 (1974).

[8] *I. Singer:* Bases in Banach spaces I. Springer-Verlag, Berlin—Heidelberg—New York 1970.

[9] *М. З. Соломяк:* Об ортогональном базисе в пространстве Банаха, Вестник Ленинградского ун-та, № 1, серия математики, механики и астрономии, вып. 1 (1957).

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

COMPACT ELEMENTS OF THE LATTICE OF CONGRUENCES IN AN ALGEBRA

JITKA ŠEVEČKOVÁ, Brno

(Received June 16, 1976)

In [5] I, a basic information about partitions in a set and congruences in an algebra can be found. Here, only necessary concepts will be introduced. A *partition* A *in* a set G is a system of pairwise disjoint nonempty subsets of G. These subsets will be called *blocks* of the partition A, its union $\bigcup A$ the *domain* of A. Of course, A is a partition *on* the set $\bigcup A$. Partitions in G are in a 1-1-correspondence with the symmetric and transitive binary relations (ST-relations) in G, analogously as partitions *on* G correspond to equivalence relations in G. For this reason, we shall sometimes not distinguish partitions and ST-relations. If (G, F) is a partial algebra then the ST-relations in the set G which are stable with respect to F are called *congruences in* (G, F). For the sake of completeness we give the definition of a stable binary relation A in a partial algebra (G, F): Let $f \in F$ be n-ary $(n \geqq 1)$ and $a_i A b_i$ $(i = 1, 2, \ldots, n)$, let $f(a_1, \ldots, a_n)$ and $f(b_1, \ldots, b_n)$ exist. Then $f(a_1, \ldots, a_n)\, A\, f(b_1, \ldots, b_n)$.

The theory of partitions in a set and of congruences in an algebra has been an object of systematic study only recently even though the concepts appeared in the literature not less than forty years ago [2, 3, 4, 5, 7]. Nonetheless, the congruences "in" actually acted latently much earlier, already in the classical group theory, e.g. in connection with the Schreier-Zassenhaus theorem in which congruences on subgroups are considered and not only those on the whole group. It was in this domain where "in" approach yielded formal as wel as matter-of-fact means for generalizing this theorem to algebras [7].

The sets $R(G)$ of all binary relations in a set G, $P(G)$ of all partitions in G and $\mathscr{K}(G, F)$ of all congruences in a partial algebra (G, F) are complete lattices under set inclusion. In all these cases the infimum of a system of relations – elements of the corresponding lattice – is equal to their set-intersection [4, 5]. Also the lattices $\Pi(G)$ of all partitions on a set G and $\mathscr{C}(G, F)$ of all congruences on a partial algebra (G, F) are complete, the latter being a closed sublattice of the former which is not true in the situation "in". $\Pi(G)$ is a closed sublattice of $P(G)$. The lattices $\Pi(G)$ and $\mathscr{C}(G, F)$, are algebraic. In Section 1, we shall prove the same property for the lattices $R(G)$

$P(G)$ and $\mathscr{K}(G, F)$ (1.3, 1.4, 1.6, 1.13). It is shown that the compact elements of $\mathscr{K}(G, F)$ are precisely the upper $\mathscr{K}$-modifications (see Def. 1.5) of compact elements of $P(G)$ (or of $R(G)$) and the compact elements of $P(G)$ and $R(G)$ are exactly the finite relations in G (1.3, 1.4, 1.6, 1.14).

In Section 2, we construct the upper $\mathscr{K}$-modification Ψ_A of a binary relation A in a partial algebra (G, F) (2.7). The construction is similar to that of the upper $\mathscr{C}$-modification Θ_A of a relation A given in [6] 5.3, 5.4. It is identical with it if we replace the algebra (G, F) in the construction of Θ_A by its subalgebra $(\bigcup \Psi_A, F)$ (2.14). For this purpose we need to know the set $\bigcup \Psi_A$; this is established in 2.11.

1. PROPERTIES OF LATTICES $R(G)$, $P(G)$ AND $\mathscr{K}(G, F)$

1.1. ([5] I 1.2). *Let (G, F) be an algebra, and $\{A_n\} \subseteq \mathscr{K}(G, F)$. Then $\bigvee_{\mathscr{K}\,\alpha} A_\alpha = \bigvee_{P\,\beta} B_\beta$, where B_β stands for the congruence $A_{\alpha_1} \vee_{\mathscr{K}} \ldots \vee_{\mathscr{K}} A_{\alpha_n}$ for an arbitrary finite choice $A_{\alpha_1}, \ldots, A_{\alpha_n}$ in $\{A_\alpha\}$.*

In general, the theorem does not hold for partial algebras.

1.2 ([5] I 1.2.0). *Let (G, F) be an algebra and $\{A_\alpha\}$ an up-directed subset of $\mathscr{K}(G, F)$. Then $\bigvee_{\mathscr{K}\,\alpha} A_\alpha = \bigvee_{P\,\alpha} A_\alpha = \bigcup_\alpha A_\alpha$.*

Proof. The first equality is proved in [5]. The other is obvious.

1.3. Theorem. *The set $R(G)$ of all binary relations in a set G is an algebraic lattice with respect to inclusion. The compact elements of $R(G)$ are exactly the finite relations in G.*

Proof. Evidently, $R(G)$ is a complete lattice. Infima are intersections and suprema are unions.

Let $T \in R(G)$, $T = \{x_1, \ldots, x_n\}$ and let n be a positive integer. Suppose that a system $\{T_\alpha : \alpha \in I\}$ satisfies $\bigcup_{\alpha \in I} T_\alpha \supseteq T$. For each $x_i \in T$ there exists $\alpha_i \in I$ with $x_i \in T_{\alpha_i}$ $(i = 1, \ldots, n)$. Thus $\bigcup_{i=1}^{n} T_{\alpha_i} \supseteq T$.

Let $T \in R(G)$ be an infinite relation. Define $T_x = \{x\}$ for each $x \in T$. Then $T = \bigcup_{x \in T} T_x$ and $T \nsupseteq \bigcup_{x \in T_1} T_x$ for all $T_1 \subseteq T$ and $T_1 \neq T$.

Finally, let $T \in R(G)$. Then $T = \bigcup_{x \in T} \{x\}$ and $\{x\}$ is compact in $R(G)$ for all $x \in T$.

1.4. Theorem. *The lattice $P(G)$ of all partitions in a set G is algebraic. A partition is compact in $P(G)$ if and only if it contains only finitely many blocks, each of them being a finite set.*

Proof. $P(G)$ is a complete lattice by [2]. First, we shall prove that a partition $A = \{A^1\}$ with one finite block $A^1 = \{x_1, \ldots, x_n\}$ is compact in $P(G)$. If $\mathfrak{A} = \{A_\delta : \delta \in \Delta\} \subseteq P(G)$ and $\bigvee \mathfrak{A} \geqq A$ then a certain block $B^1 \in \bigvee \mathfrak{A}$ contains A^1. Given $x_i, x_j \in A^1$ there exist elements $y_1, \ldots, y_{m-1}$ of G and indices $\delta_1, \ldots, \delta_m$ of Δ with $x_i A_{\delta_1} y_1 \ldots y_{m-1} A_{\delta_m} x_j$.

Denote $\mathfrak{A}_{i,j} = \{A_{\delta_k} : k = 1, \ldots, m\}$ and $\mathfrak{B}_1 = \bigcup_{i,j=1}^{n} \mathfrak{A}_{i,j}$. Then $\bigvee \mathfrak{B}_1 \geqq A$ and $\mathfrak{B}_1$ is a finite subsystem of $\mathfrak{A}$. If the partition A consists of finite blocks $A^1, \ldots, A^k$ (k a positive integer) we construct a (finite) system $\mathfrak{B}_t \subseteq \mathfrak{A}$ for every A^t $(1 \leqq t \leqq k)$ in the described manner; then $\bigvee \mathfrak{B} \geqq A$ for $\mathfrak{B} = \bigcup_{t=1}^{k} \mathfrak{B}_t$ and $\mathfrak{B}$ is a finite subsystem of $\mathfrak{A}$.

Next, we shall prove that a partition A 1) with at least one infinite block or 2) with infinite many blocks fails to be compact.

1) Let A^1 be infinite, $A^1 \in A$, $x, y \in A^1$. Denote by $A_{x,y}$ the partition in G which we obtain from A taking the block $\{x, y\}$ instead of A^1 (the other blocks of A remain unchanged). The join of the system $\mathfrak{A}$ of all partitions $A_{x,y}$ $(x, y \in A^1)$ equals A. The blocks of the join of an arbitrary finite subsystem $\mathfrak{A}_1$ of $\mathfrak{A}$ are all blocks of the partition A except A^1 and in addition some blocks which together cover only a finite part of A^1. Thus $\bigvee \mathfrak{A}_1 > A$.

2) Let $A = \{A^\delta : \delta \in \Delta\}$, card $\Delta \geqq \aleph_0$. Define one-block partitions $A_\delta = \{A^\delta\}$, $\delta \in \Delta$. Then $\bigvee \{A_\delta : \delta \in \Delta\} = A$. It is evident that none of the finite subsystems of $\{A_\delta : \delta \in \Delta\}$ has supremum $\geqq A$.

It remains to prove that an arbitrary element of $P(G)$ is the join of compact ones. Given $A \in P(G)$, $A^1 \in A$ and $x, y \in A^1$ we construct a one-block partition $A_{x,y} = \{\{x, y\}\}$. All these partitions are compact elements of $P(G)$ and its supremum is equal to A. The theorem is proved.

1.5. Definition. Let L be a partially ordered set, $\emptyset \neq K \subseteq L$ and $a \in L$. An element $b \in K$ is said to be *an upper K-modification of a* if b is the least element of K containing a.

1.6. Theorem. *Let (G, F) be an algebra. Then $\mathscr{K}(G, F)$ is an algebraic lattice. The upper $\mathscr{K}$-modifications of compact elements of $P(G)$ are compact in $\mathscr{K}(G, F)$.*

1.7. Remark. In 1.14 we shall prove that all compact elements of $\mathscr{K}(G, F)$ are of the above mentioned form.

Proof. Let T be a compact element of $P(G)$ and K the upper $\mathscr{K}$-modification of T. Let $\{K_\alpha : \alpha \in I\} \subseteq \mathscr{K}(G, F)$ and $\bigvee_{\mathscr{K}, \alpha \in I} K_\alpha \geqq K$. By 1.1, $\bigvee_{P, \beta \in J} L_\beta = \bigvee_{\mathscr{K}, \alpha \in I} K_\alpha$, where L_β runs through the $\mathscr{K}$-suprema of all finite subsets of $\{K_\alpha : \alpha \in I\}$. We have $\bigvee_{P, \beta \in J} L_\beta = \bigvee_{\mathscr{K}, \alpha \in I} K_\alpha \geqq K \geqq T$. There exists a finite subset J_1 of J with $\bigvee_{P, \beta \in J_1} L_\beta \geqq T$. Therefore

$\bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq \bigvee_{P, \beta \in J_1} L_\beta \geqq T$ and thus $\bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq K$. For each $\beta \in J_1$ there exists a finite subset $I(\beta)$ of I such that L_β is a $\mathscr{K}$-supremum of the system $\{K_\delta\}$ $(\delta \in I(\beta))$. Let I_1 be the join of all sets $I(\beta)$ with β running over J_1. Then I_1 is finite and $\bigvee_{\mathscr{K}, \gamma \in I_1} K_\gamma \geqq \geqq \bigvee_{\mathscr{K}, \beta \in J_1} L_\beta \geqq K$. Consequently, K is a compact element of $\mathscr{K}(G, F)$.

The lattice $\mathscr{K}(G, F)$ is complete by [5] I 1.1. It remains to prove that it is compactly generated. An arbitrary congruence K is a partition, hence it is $\bigvee_P$ of a set of compact elements of $P(G)$, say $\mathfrak{B}$. For $B \in \mathfrak{B}$ let A be the upper $\mathscr{K}$-modification of B; let $\mathfrak{A}$ be the set of these modifications A. Evidently $K = \bigvee_P \mathfrak{B} \leqq \bigvee_P \mathfrak{A} \leqq \bigvee_{\mathscr{K}} \mathfrak{A} \leqq K$. Thus $K = \bigvee_{\mathscr{K}} \mathfrak{A}$.

1.8. In what follows we shall need some known concepts definitions of which will be introduced now for convenience of the reader (see e.g. [1], [6]).

The closure operation on a partially ordered set L is a mapping $\lambda : L \to L$ with the following properties: 1) $a \leqq \lambda a$ $(a \in L)$, 2) $a \leqq b \Rightarrow \lambda a \leqq \lambda b$, 3) $\lambda\lambda a = \lambda a$ $(a \in L)$, 4) $\lambda 0 = 0$ (provided 0 exists). The set of all compact elements of L will be denoted by L^*. The closure operation λ of L will be called *algebraic* if every $a \in L^*$ satisfies the following condition: If $a \leqq \lambda x$ then there exists $x' \in L^*$ with $x' \leqq x$ and $a \leqq \lambda x'$.

1.9 ([6] 4.7). *A closure operation λ of an algebraic lattice L is algebraic if and only if it fulfils $\bigvee_L S \in \lambda L$ for every directed subset S of λL.*

1.10. Definition. Let G be a set. Then $\lambda_1 : R(G) \to P(G)$ is defined as follows: $\lambda_1(A)$ is the upper P-modification of $A \in R(G)$. If (G, F) is an algebra we define the mappings $\lambda_2 : P(G) \to \mathscr{K}(G, F)$ and $\lambda_3 : R(G) \to \mathscr{K}(G, F)$ analogously.

1.11. Theorem. *The maps λ_i $(i = 1, 2, 3)$ from Definition* 1.10 *are algebraic closure operations.*

Proof. It is clear that λ_i $(i = 1, 2, 3)$ is a closure operation. Further, by 1.9, it is enough to fulfil the condition $\bigcup \mathfrak{A} \in P(G)$ (as for λ_1) or $\bigcup \mathfrak{A} \in \mathscr{K}(G, F)$ (as for λ_2 and λ_3), for an arbitrary directed subset $\mathfrak{A}$ of $P(G)$ (as for λ_1) or of $\mathscr{K}(G, F)$ (as for λ_2 and λ_3), respectively.

λ_1: Let $\mathfrak{A} = \{A_\alpha : \alpha \in I\}$ be a directed subset of $P(G)$. It suffices to prove that $\bigcup_\alpha A_\alpha$ is symmetric and transitive. The first property is evident, the other follows from the fact that for $x, y \in G$ we have $x(\bigcup \mathfrak{A})\, y$ if and only if xAy for some $A \in \mathfrak{A}$ (since $\mathfrak{A}$ is directed). The assertions for λ_2 and λ_3 follow from 1.2.

1.12 ([6] 4.3). *If λ is an algebraic closure operation of an algebraic lattice L then λL is again an algebraic lattice, and it holds $\lambda(L^*) = (\lambda L)^*$.*

1.13. *Now, the property to be algebraic for $P(G)$ (G a set) and $\mathscr{K}(G, F)$ ((G, F) an algebra) follows by virtue of* 1.11 *and* 1.12. In fact, $P(G) = \lambda_1 R(G)$ and λ_1 is algebraic by 1.11. Thus by 1.12, $P(G)$ is algebraic. Analogously for $\mathscr{K}(G, F)$ with aid of λ_2 or λ_3.

In the following theorem the characterization of $\mathscr{K}(G, F)^*$ will be completed. Simultaneously, we discover the structure of $P(G)^*$.

1.14. Theorem. *Let G be a set. Compact elements of the lattice $P(G)$ are exactly the upper P-modifications of compact elements of $R(G)$ (i.e. of finite subsets of $G \times G$). Analogously for $\mathscr{K}(G, F)$ if (G, F) is an algebra.*

Also, compact elements of $P(G)$ (G a set) are exactly the finite partitions whose blocks are finite sets and if (G, F) is an algebra then compact elements of $\mathscr{K}(G, F)$ are exactly the upper $\mathscr{K}$-modifications of compact elements of $P(G)$.

Proof. According to 1.12, the first assertion follows from the fact that λ_1 and λ_3 are algebraic (1.11) and that the compact elements of $R(G)$ are precisely the finite subsets of $G \times G$ (1.3).

To obtain the other description of compact elements of $P(G)$ it suffices to verify that the upper P-modification B of a finite relation A in G is finite again. It holds $A \subseteq C \times C$, where $C = \bigcup A \cup \bigcup A^{-1}$, so that $B \subseteq C \times C$ (as $C \times C$ is a partition in G) and $C \times C$ is finite.

The last assertion follows from 1.12 since λ_2 is algebraic (1.11).

2. DETERMINATION OF THE UPPER $\mathscr{K}$-MODIFICATION OF AN ARBITRARY BINARY RELATION IN A PARTIAL ALGEBRA

The aim of this section is the determination of the upper $\mathscr{K}$-modification Ψ_A of an arbitrary relation A in a partial algebra (G, F). The construction is similar to that of the upper $\mathscr{C}$-modification Θ_A of A given in [6] 5.3 and 5.4. It is identical with it if we replace the algebra (G, F) in the construction of Θ_A by its subalgebra $(\bigcup \Psi_A, F)$ (2.14). Therefore we need to know the set $\bigcup \Psi_A$; this is established in 2.11.

2.1. Definition. (See [6] 2 and 5) *Let (G, F) be a partial algebra and X a non-empty set. For every pair of positive integers i, n ($i \leqq n$) we define the n-ary operation $e^{n,i}(x_1, \ldots, x_n)$ on G by*

$$e^{n,i}(a_1, \ldots, a_n) = a_i \quad \text{for all} \quad a_1, \ldots, a_n \in G .$$

Further, we put $F^* = F \cup \{e^{n,i}\}_{n,i}$.

If $w = w(x_1, \ldots, x_n)$ is a word over X generated by F^* and if we substitute k ($0 \leqq k \leqq n$) of its variables (e.g. $x_{n-k+1}, \ldots, x_n$) by fixed elements $a_{n-k+1}, \ldots, a_n$ of G then the resulting symbol

$$w(x_1, \ldots, x_{n-k}, a_{n-k+1}, \ldots, a_n) =: p(x_1, \ldots, x_{n-k})$$

defines an $(n - k)$*-ary operation in* G*. It will be called an algebraic function in* (G, F)*. For* $k = n - 1$*,* $p(x)$ *is a unary operation which is said to be a unary algebraic function.*

2.2. *The symmetric-transitive hull* A^T *of a binary relation* A *in a set* G *is, evidently,* $A^T = \bigcup_{n=1}^{\infty} B^n$*, where* $B = A \cup A^{-1}$*.*

2.3. Definition. [6] Let (G, F) be a partial algebra and $A \in R(G)$. We define the following relations in G: A^H, A^F, A^U as follows:

A^H is the set of all $(u, v) \in G \times G$ to which there exist a word $w(x_1, \ldots, x_n)$ generated by F^* and elements $(a_i, b_i) \in A$ $(i = 1, \ldots, n)$ such that $u = w(a_1, \ldots, a_n)$, $v = w(b_1, \ldots, b_n)$.

A^F and A^U is obtained by replacing the term "word" in the above definition of A^H by "an algebraic function" and "a unary algebraic function", respectively.

Remark. *If* $A \neq \emptyset$ *then* A^F *is a reflexive relation. (If* $a \in G$ *and* $a_1 A b_1$ *then* $a = e^{2,2}(a_1, a)\, A^F\, e^{2,2}(b_1, a) = a$*.)*

2.4. Proposition. [6] *If* S *denotes any of the symbols* T, H, F *and* U *then the map* $\lambda : R(G) \to R(G)$*, defined by* $\lambda A = A^S$*, is a closure operation in* $R(G)$*.*

2.5. Denote

$$A_0 = A,\ A_1 \doteq A_0^H,\ A_2 = A_1^T,\ A_3 = A_2^H, \ldots, A_{2i} = A_{2i-1}^T,$$
$$A_{2i-1} = A_{2i-2}^H \quad (i = 1, 2, \ldots).$$

Evidently, it holds $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$.

Denote

$$A' = \bigcup_{n=0}^{\infty} A_n .$$

2.6. Definition. Let (G, F) be a partial algebra and $A \in R(G)$. Then Ψ_A and Θ_A, denote the upper $\mathscr{K}$-modification and the upper $\mathscr{C}$-modification of A, respectively.

2.7. Theorem. *If* (G, F) *is a partial algebra and* $A \in R(G)$ *then* $\Psi_A = A'$*.*

Proof. By induction, let us prove $A' \subseteq \Psi_A$. Evidently $A \subseteq \Psi_A$. Now, we shall show $A_{2i-1} \subseteq \Psi_A \Rightarrow A_{2i}, A_{2i+1} \subseteq \Psi_A$. The first inclusion is evident because of $A_{2i} = A_{2i-1}^T \subseteq \Psi_A$. Let $A_{2i} \subseteq \Psi_A$ and $(u, v) \in A_{2i+1} = A_{2i}^H$. There exist a word $w(x_1, \ldots, x_n)$ generated by F^* and elements $(a_j, b_j) \in A_{2i}$ $(j = 1, \ldots, n)$ such that $u = w(a_1, \ldots, a_n)$, $v = w(b_1, \ldots, b_n)$. Hence the congruence Ψ_A contains $(u, v) = (w(a_1, \ldots, a_n), w(b_1, \ldots, b_n))$ because of $(a_j, b_j) \in \Psi_A$ $(j = 1, \ldots, n)$. So $A' \subseteq \Psi_A$.

The equality will follow if we prove that A' is a congruence. A' is symmetric since every $(u, v) \in A'$ belongs to A_{2i} $(= A_{2i-1}^{T})$ for some i and this is symmetric. By a similar argument, A' is transitive. Analogously, A' is stable since A_{2i+1} $(= A_{2i}^{H})$ – – for all i – is stable.

2.8 ([6] 5.3). *Let (G, F) be a partial algebra and $A \in R(G)$. Then Θ_A is the union of the sequence of relations $A \subseteq A^F \subseteq A^{FT} \subseteq A^{FTF} \subseteq \ldots$.*

2.9. Proposition. *Let A be a congruence in a partial algebra (G, F). Then $(\bigcup A, F)$ is a subalgebra of (G, F) and A is a congruence on $(\bigcup A, F)$.*

Proof. Evidently, A is a partition on the set $\bigcup A$. Let $(a_1, \ldots, a_n) \in D(f, G) \cap \cap (\bigcup A)^n$.*) It is $a_i A a_i$ $(i = 1, \ldots, n)$ hence $f(a_1, \ldots, a_n) \, A \, f(a_1, \ldots, a_n)$ and therefore $f(a_1, \ldots, a_n) \in \bigcup A$.

2.10. Definition [2] 2.3. Let A be a binary relation in a set G and $B \subseteq G$. The intersection of the relation A and the subset B is the relation $B \sqcap A = \{(a, b) \in A: a, b \in B\}$.

2.11. Theorem. *Let (G, F) be a partial algebra and $A \in R(G)$. Then $\bigcup \Psi_A$ is the subalgebra $\langle \bigcup A \cup \bigcup A^{-1} \rangle$ **) of (G, F) generated by the set $\bigcup A \cup \bigcup A^{-1}$.*

Proof. From the symmetry of Ψ_A it follows that $\bigcup A \cup \bigcup A^{-1} \subseteq \Psi_A$ and consequently $\langle \bigcup A \cup \bigcup A^{-1} \rangle \subseteq \bigcup \Psi_A$ by 2.9. Conversely, the intersection $\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A$ is a congruence containing A, hence $\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A \supseteq \supseteq \Psi_A$. The reverse inclusion is evident so that $\bigcup(\langle \bigcup A \cup \bigcup A^{-1} \rangle \sqcap \Psi_A) = \bigcup \Psi_A$. Thus $\bigcup \Psi_A = \langle \bigcup A \cup \bigcup A^{-1} \rangle \cap \bigcup \Psi_A = \langle \bigcup A \cup \bigcup A^{-1} \rangle$.

2.12 [6] 5.5 and 5.4. *Let (G, F) be an algebra and $A \in R(G)$. Then $A^{UT} = A^{FT}$ and $A^{UT} = A^{UTU}$. Consequently, $\Theta_A = A^{UT}$ if $A \neq \emptyset$.*

2.13. Let (G, F) be a partial algebra, (B, F) a subalgebra of (G, F) and $A \subseteq B \times B$. We need to distinguish the least congruence in (G, F) containing A from the least congruence in (B, F) containing A. We shall denote the latter by $\Psi_A(B)$ and the former by $\Psi_A(G)$. Similarly, we distinguish $\Theta_A(B)$ from $\Theta_A(G)$ and $A^{S(B)}$ from $A^{S(G)}$ for $S = H, F$ and U.

2.14. Theorem. *Let (G, F) be a partial algebra, $A \in R(G)$, and $B = \bigcup \Psi_A(G)$ $(= \langle \bigcup A \cup \bigcup A^{-1} \rangle)$. Then $\Psi_A(G) = \Theta_A(B) = A \cup A^{F(B)} \cup A^{F(B)T} \cup A^{F(B)TF(B)} \cup \ldots$. If (G, F) is an algebra then $\Psi_A(G) = \Theta_A(B) = A^{U(B)T}$.*

Proof follows from 2.8, 2.9 and 2.12.

---

*) By $D(f, G)$ the set of all $(a_1, \ldots, a_n) \in G^n$ is denoted for which $f(a_1, \ldots, a_n)$ exists.

**) $\bigcup A = \{y \in G: \exists x \in G, \, yAx\}$, [5] III Df. 3.5.

2.15. Corollary. *Let (G, F) be an algebra, $A \in R(G)$ and $B = \langle \bigcup A \cup \bigcup A^{-1} \rangle$. Then $(u, v) \in \Psi_A$ if and only if there exist a sequence $u = z_0, z_1, \ldots, z_n = v$ of elements of B, elements $(a_i, b_i) \in A$ $(i = 1, \ldots, n)$ and unary algebraic functions $p_1(x), \ldots, p_n(x)$ in (B, F) such that $p_i(a_i) = z_{i-1}$, $p_i(b_i) = z_i$ or $p_i(b_i) = z_{i-1}$, $p_i(a_i) = z_i$, $i = 1, \ldots, n$.*

2.16. Denotation. *If $A = \{(a, b)\}$ is a one-element relation we put $\Psi_{a,b}$ instead of $\Psi_{\{(a,b)\}}$.*

2.17. Corollary (see [6] 5.5). *Let (G, F) be an algebra, $A = \{(a, b)\}$ a one-element relation in G and $B = \bigcup \Psi_A$. Then $(u, v) \in \Psi_{a,b}$ if and only if there exist a sequence $u = z_0, z_1, \ldots, z_n = v$ of elements of B and unary algebraic functions $p_1(x), \ldots \ldots, p_n(x)$ such that $z_{i-1} = p_i(a)$, $z_i = p_i(b)$ or $z_{i-1} = p_i(b)$, $z_i = p_i(a)$.*

This is a special case of 2.15.

References

[1] *Birkhoff, G.:* Lattice theory (Russian translation), Moskva 1952.

[2] *Borůvka, O.:* The theory of partitions in a set (Czech). Publ. Fac. Sci. Univ. Brno, No. 278 (1946), 1—37.

[3] *Borůvka, O.:* Foundations of the theory of groupoids and groups. Berlin 1974, (Czech) Praha 1962, (German) Berlin 1960.

[4] *Draškovičová, H.:* The lattice of partitions in a set. Acta Fac. Rer. Nat. Univ. Comen., Math. *24* (1970), 37—65.

[5] *Mai, T. D.:* Partitions and congruences in algebras. Arch. Math. (Brno), *10* (1974) I. 111—122, II, 159—172, III, 173—188, IV, 231—254.

[6] *Schmidt, E. T.:* Kongruenzrelationen algebraischer Strukturen. Berlin 1969.

[7] *Šik, F.:* Complements of congruences in an algebra. 1974 (preprint).

[8] *Šik, F.:* Schreier-Zassenhaus theorem for sets and universal algebras. 1975 (preprint).

Author's address: 662 73 Brno, Čechyňská 16, Výzkumný ústav tvářecích strojů a technologie tváření).

STRUČNÉ CHARAKTERISTIKY ČLÁNKŮ OTIŠTĚNÝCH V TOMTO ČÍSLE V CIZÍM JAZYKU

IVAN CHAJDA, Přerov, BOHDAN ZELINKA, Liberec: *Permutable tolerances.* (Permutovatelné tolerance.)

Kompatibilní tolerance na algebře je definována analogicky jako kongruence, pouze je vynechán požadavek transitivnosti. V článku se studují kompatibilní tolerance na algebrách, které jsou permutovatelné vzhledem k součinu binárních relací.

MILAN MEDVEĎ, Bratislava: *On a class of nonlinear evasion games.* (O jednej triede nelineárnej hry vyhýbania sa.)

V článku je dokázaná existencia stratégie vyhýbania sa pre diferenciálnu hru opísanú systémom diferenciálnych rovníc $z^{(n)} + A_1 z^{(n-1)} + \ldots + A_{n-1} z' + A_n z = f(u,v) + \mu g(z, z', \ldots, z^{(n-1)}, u, v)$, kde $z \in R^m$, $f: R^p \times R^q \to R^n$, A_i, $i = 1,2,\ldots,n$ sú konštantné matice, pre $\mu \in R^1$ dostatočne malé.

WŁADYSŁAW WILCZYŃSKI, Łódź: *Remark on the theorem of Egoroff.* (Poznámka k Jegorovově větě.)

Autor dokazuje větu, která je v jistém smyslu zostřením Jegorovovy věty. Podobné zostření Luzinovy věty bylo publikováno v tomto časopise 96 (1971), 225—228, v článku I. Vrkoče: „Remark about the relation between measurable and continuous functions".

BOHDAN ZELINKA, Liberec: *A remark on isotopies of digraphs and permutation matrices.* (Poznámka o isotopiích orientovaných grafů a o maticích permutací.)

Jsou-li G, G' orientované grafy, pak isotopie G na G' je uspořádaná dvojice $\langle f_1, f_2 \rangle$ bijekc množiny vrcholů G na množinu vrcholů G' s touto vlastností: pro každé dva vrcholy u, v z G je existence hrany $\overrightarrow{f_1(u) f_2(v)}$ v G' ekvivalentní s existencí hrany $\overrightarrow{uv}$ v G. V článku jsou ukázány aplikace matic permutací při zkoumání isotopií orientovaných grafů.

MIROSLAV SOVA, Praha: *On Hadamard's concepts of correctness.* (O Hadamardových pojmech korektnosti.)

Článek je věnován studiu pojmu korektnosti lineárních diferenciálních rovnic v Banachových prostorech, tj. rovnic typu $u^{(n)}(t) + A_1 u^{(n-1)}(t) + \ldots + A_n u(t) = 0$, kde $A_1, A_2, \ldots, A_n$ jsou (obecně neohraničené) operátory z Banachova prostoru do sebe. Je zaveden zeslabený pojem korektnosti, který je porovnán s obvyklou definicí.

V. SATHYABHAMA, Waterloo: *Generalized LC-identity on GD-groupoids.* (Zobecněná LC-identita na GD-grupoidech.)

Autor vyšetřuje funkcionální rovnici $A_1(A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z)))$, která je zobecněním tzv. LC-identity $(x . (x . y)) . z = x . (x . (y . z))$ na lupě. Nalézá řešení zobecněné LC-identity na GD-grupoidech, přičemž GD-grupoidem rozumí čtveřici $(G_1, G_2, G; A)$, kde A: $G_1 \times G_2 \to G$ a rovnice $A(a, y) = c$, $A(x, b) = c$ pro x, y jsou řešitelné. V závěru článku autor aplikuje dosažený výsledek na kvazigrupy definované na stejné množině.

ŠTEFAN SCHWABIK, Praha: *Note on Volterra-Stieltjes integral equations.* (Poznámka k Volterrovým-Stieltjesovým integrálním rovnicím.)

Podmínka regularity matice $I - (K(t, t) - K(t, t-))$ pro každé $t \in (0, 1\rangle$ je nutná a stačí k tomu, aby Volterrova-Stieltjesova integrální rovnice $x(t) - \int_t^0 d_s[K(t, s)]\, x(s) = f(t)$, $t \in \in [0, 1]$ měla jediné řešení pro každé $f \in BV_n$. Pro rovnice splňující podmínku regularity je dána rezolventní formule.

JAROMÍR DUDA, Brno, IVAN CHAJDA, Přerov: *Ideals of binary relational systems.* (Ideály binárních relačních systémů.)

Pojem svazového ideálu lze zobecnit i pro případ obecného binárního relačního systému. V práci je ukázáno, že tyto tzv. ϱ-ideály mají většinu důležitých vlastností požadovaných pro svazové ideály a dále, některá tvrzení o svazových ideálech lze zesílit i pro případ obecných binárních systémů. V další části práce je dána nová charakterizace často užívaných relací, jako jsou uspořádání, kvaziuspořádání, ekvivalence a úplná relace pomocí ϱ-ideálů a je odvozena jednoduchá metoda pro vnoření jistých binárních relačních systémů do třídy částečně uspořádaných množin.

PETR PŘIKRYL, Praha: *Universal simultaneous approximations of the coefficient functionals.* (Universální simultánní aproximace koeficientů rozvoje podle dané báze.)

Článek pojednává o aproximaci koeficientů rozvoje podle dané báze společné pro určitou třídu Banachových prostorů. Tyto koeficienty se chápou jako lineární funkcionály nad danými prostory a studují se aproximace konečné množiny koeficientů pomocí lineárních kombinací menšího počtu jiných lineárních funkcionálů. Formulují se podmínky na třídu prostorů, za nichž je jistá jednoduchá aproximace tohoto typu pro uvažovanou třídu universální ve smyslu Babušky a Soboleva. Udávají se postačující podmínky pro to, aby tato universální aproximace byla optimální.

JITKA ŠEVEČKOVÁ, Brno: *Compact elements of the lattice of congruences in an algebra.* (Kompaktní prvky svazu kongruencí v algebře.)

Svazy $R(G)$ všech binárních relací v množině G, $P(G)$ všech rozkladů v G (tj. všech symetrických a transitivních relací v G) a $\mathscr{K}(G, F)$ všech kongruencí v algebře (G, F) (tj. stabilních rozkladů v (G, F)) jsou algebraické. Jsou popsány množiny jejich kompaktních prvků; pro poslední z nich je to množina všech horních $\mathscr{K}$-modifikací konečných relací v G. Jsou dány dva způsoby konstrukce těchto modifikací.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

RECENSE

Oscar Zariski - Pierre Samuel: COMMUTATIVE ALGEBRA, Volume 2. Graduate Texts in Mathematics, Vol. 29. Springer-Verlag, New York—Heidelberg—Berlin 1975. Stran X + 414, cena DM 36,20.

O charakteru Zariského a Samuleovy knihy o komutativní algebře a o jejím prvním dílu bylo referováno v Čas. pěst. mat. *102* (1977), 208. Proto se tu jen stručně zmíním o obsahu druhého dílu. Tento svazek je reprintem svého prvního vydání z roku 1960. Tvoří přímé pokračování prvního dílu knihy; obsahuje tři kapitoly a sedm dodatků. Algebraicko geometrické aspekty tu vystupují už mnohem víc do popředí než v prvním dílu.

První kapitola 2. dílu (číslovaná jako šestá kapitola celého díla) je věnována teorii ohodnocení. Čtenář zajímající se o algebraickou geometrii tu nalezne mnoho užitečného materiálu, např. i výklad o Riemannově varietě nadtělesa K nad tělesem k a o normálních modelech variet nad k.

Tématem druhé kapitoly je teorie okruhů polynomů a formálních mocninných řad a její aplikace pro algebraickou geometrii. Kromě klasických výsledků pojednává kapitola o graduovaných okruzích a modulech a jejich charakteristických funkcích.

Třetí kapitola se zabývá lokální algebrou. V první elementární části se čtenář doví o teorii zúplnění, o základních vlastnostech úplných modulů, o Zariského okruzích, Henselově lemmatu; další část se pak věnuje teorii dimenze a násobnosti v lokálních okruzích, studiu vlastností regulárních lokálních okruhů a aplikací výsledků v algebraické geometrii (analyticky ireducibilní a analyticky normální variety). V celé této kapitole je opět zdůrazněna těsná souvislost probírané látky se studiem lokálních vlastností algebraických variet.

Obsahem jednotlivých dodatků je vyšetřování některých speciálnějších témat navazujících na předchozí látku knihy: např. ohodnocení v noetherovských okruzích, Macaulayovy okruhy, jednoznačnost rozkladu v prvočinitele v regulárních lokálních okruzích.

K tomu, co bylo řečeno v referátu o prvním dílu knihy dodejme už jen, že zejména tento druhý díl je velmi užitečný každému, kdo se chce zabývat algebraickou geometrií: tvoří pro její studium potřebný algebraický základ.

Václav Vilhelm, Praha

Gheorghe Mihoc, Mariana Craiu: INFERENŢĂ STATISTICĂ PENTRU VARIABILE DEPENDENTE. (Statistická inference pro závislé veličiny.) Editura Academiei Republicii Socialiste România, Bukurešť, stran 301, cena 13 lei.

Kniha má tři části: 1. Kapitola I o statistice nezávislých náhodných veličin. 2. Kapitoly II—IV o statistice v Markovových řetězcích jednoduchých, vícenásobných a s obecnou množinou stavů. 3. Kapitola V o řetězcích s úplnou vazbou.

1. V kapitole I i v pojetí celé knihy je předlohou Cramérovo dílo *Mathematical Methods of Statistics* (1946). Odtud jsou převzaty např. věta 2§2 o asymptotické vydatnosti odhadů metodou maximální věrohodnosti a věta 4§2 o odhadech modifikovanou metodou minimálního χ^2, jejichž důkazy jsou téměř doslovným překladem Cramérova textu. Všimněme si podrobněji věty 2§2. V jejím předpokladu 4 je vypuštěna důležitá podmínka, že střední hodnota kvadrátu logaritmické derivace hustoty je kladná, i když je na to v důkaze přímá odvolávka. V závěru důkazu se říká,

že odhad $\hat{t}_n$ je asymptoticky normální $N(\Theta_0, 1/nk^2)$. Odtud se vyvozuje Var $(\hat{t}_n) = 1/nk^2$ a tedy vydatnost odhadu, nikoliv pouze asymptotická vydatnost. Tento pojem není v knize definován. Definice vydatnosti zahrnuje i nevychýlenost odhadu. Kapitola I obsahuje rovněž odstavce o testování hypotéz, o χ^2-testu a o testu založeném na podílu věrohodností.

2. Ve statistice v Markovových řetězcích je stále základním dílem monografie P. Billingsleye *Statistical Inference for Markov Processes* (1961). Novější zpracování této problematiky, napsané s ohledem zejména na uživatele matematicko-statistických metod, ve světové literatuře dosud chybí. Recensovaná kniha je i v této hlavní části přehledem výsledků a důkazových postupů různých autorů, někdy bez náležité důslednosti v označení a s tiskovými chybami. Nejasnosti se vyskytují v používání symbolu pro střední hodnotu. Může to být, bez bližšího vysvětlení, střední hodnota, podmíněná střední hodnota i střední hodnota vzhledem ke stacionárnímu rozložení. Nedělá se také rozdíl mezi tvrzením „řešení (věrohodnostní) rovnice existuje" a tvrzením „řešení rovnice existuje s pravděpodobností libovolně blízkou 1 při $n \to \infty$", apod. Je pojednáno o odhadech metodou maximální věrohodnosti, metodou minima χ^2, o testech založených na podílu věrohodností, o Whittleově formuli a o sekvenční analýze.

3. Podrobně jsou vysvětleny řetězce s úplnou vazbou klasické (Onicescu-Mihocova typu) i zobecněné. Jsou uvedeny základní výsledky, zejména v oblasti ergodických vět. Statistika v řetězcích s úplnou vazbou je doposud málo rozvinuta, nechceme-li za takové řetězce prohlašovat libovolné posloupnosti náhodných veličin, zadané hustotami sdruženého rozložení.

Kniha je psána rumunsky a nečiní si jistě nároků být monografií světové úrovně.

Petr Mandl, Praha

Hans-Jakob Lüthi: KOMPLEMENTARITÄTS- UND FIXPUNKTALGORITHMEN IN DER MATHEMATISCHEN PROGRAMMIERUNG, SPIELTHEORIE UND ÖKONOMIE. Lecture Notes in Economics and Mathematical Systems. Springer-Verlag, Berlin—Heidelberg—New York 1976. Str. 145, cena DM 18,—.

Nechť f je zobrazení n rozměrného euklidovského prostoru R^n do sebe. Problémem komplementarity rozumíme úlohu najít $z \in R^n$ tak, že $f(z) \geqq 0$, $z \geqq 0$ a $f(z) . z = 0$. (. značí skalární součin). Je-li f tvaru $f(z) = q + Mz$, kde $q \in R^n$ je daný vektor a $M \in R^{n \times n}$ daná matice, mluvíme o lineárním problému komplementarity.

Kniha je rozdělena do dvou částí: první část (88 stran) se zabývá problémy komplementarity, druhá část (45 stran) je věnována pevným bodům spojitých a polospojitých zobrazení v R^n. Struktura obou částí je zhruba stejná. V úvodu se řeší otázky existence, dále se popisují výpočetní postupy a uvádějí se možnosti aplikací.

V části o komplementaritě je zvláštní pozornost věnována lineárním problémům komplementarity. Lineární problémy jsou početně dobře zvládnutelné a nacházejí uplatnění v oblasti kvadratického programování a při řešení dvoumaticových her. Autor v této části přináší i svoje původní výsledky, týkající se zejména zobecnění teorie komplementarity na nelineární případ a některých výpočetních postupů.

V části o pevných bodech najdeme především klasickou Brouwerovu větu, Scarfův výpočetní postup, Kakutaniho větu o pevném bodě a odstavec, kde se ukazuje, jak lze dokázat existenci řešení ve Walrasově modelu ekonomické rovnováhy pomocí věty o pevném bodě, dále jak lze tuto větu aplikovat na řešení úlohy nelineárního programování a v důkazu existence rovnovážného bodu v nekooperativní hře n hráčů. V závěru je uveden asi čtyřstránkový seznam literatury vztahující se k tématu knihy.

Práce je napsána s německou důkladností, v jednotném stylu a její rozčlenění umožňuje udržet si přehled po vykládané látce. Je cenná tím, že podává souhrn současných teoretických výsledků v oblasti komplementarity a pevných bodů. Pokud jde o popis algoritmů, neuvažuje se problematika jejich výpočetní efektivnosti. Pak vlastně není ani nutné studovat odděleně algoritmy pro problémy komplementarity a pro hledání pevných bodů, neboť problém komplementarity lze převést na problém vyhledání pevných bodů vhodně zvoleného zobrazení a obráceně. V ukázkách aplikací se uvádějí většinou věci již několikrát publikované, takže čtenář asi ocení hlavně skutečnost, že je zde najde v souhrnu.

Miroslav Maňas, Praha

J. Dénes - A. D. Keedwell: LATIN SQUARES AND THEIR APPLICATIONS. Akadémia Kiadó, Budapest 1974. Stran 547, cena neudána.

Nechť je tento rozbor vřelým doporučením znamenitého díla, svědectvím opět jednoho šťastného setkání dvou významných matematiků.

Recensovaná kniha vznikla za spolupráce dvou universit, tj. maďarské Loránd Eötvös Egyetem v Budapešti v osobě J. Dénese a britské University of Surrey, kde působí A. D. Keedwell. Autoři předkládají čtenářům rozsáhlou a hlubokou studii a učebnici o latinských čtvercích a příbuzných strukturách, již lze pokládat za první tohoto druhu. Doposud se objevovaly jen kapitoly o latinských čtvercích v pracích z kombinatoriky.

Pojem latinského čtverce je zhruba starý 200 let a byl vzápětí sledován Eulerovou úlohou o 36 důstojnících, o níž teprve na začátku tohoto století bylo dokázáno, že nemá řešení. Poměrně nedávno se staly latinské čtverce opět předmětem vážného studia a byla otištěna ohromná řada prací. Příčinu nového rozmachu studia latinských čtverců lze hledat jednak v objevech souvislostí tohoto pojmu s algebrou zobecněných binárních systémů a s kombinatorickými úvahami, jež se týkají zejména konečných geometrií, jednak v užití latinských čtverců při vytváření schémat statistických pokusů a v teorii informace. Právě závažnost těchto souvislostí a aplikací, zároveň s tím, že práce tohoto oboru jsou značně roztroušeny po časopisech, vedla autory k myšlence shromáždit literaturu ke všem známým problémům a vytvořit vyčerpávající studii o výsledcích této teorie.

Ze zběžného pohledu vidíme, že naše kniha má v podstatě dvě fáze. V první z nich běží o studium vlastností jednoduchých latinských čtverců v úzké souvislosti s teorií kvazigrup a lúp, dále v menší míře také s teorií grafů. Ve druhé fázi máme studii množiny vzájemně ortogonálních latinských čtverců. Zde pak následuje souvislost s teorií konečných projektivních rovin a konstrukce statistických schemat. Obě stránky studovaného pojmu se však nezkoumají pedanticky odděleně. Čtenář snadno nalezne mnoho vzájemných styčných bodů, jak ani jinak, v tak živě podaném textu, nemůže být. Zřetelně jsou sledovány oba základní rysy latinského čtverce, a to jak kombinatorický, tak i algebraický.

Latinským čtvercem rozumíme čtvercovou matici řádu n, vytvořenou z n různých prvků, z nichž každý se vyskytuje právě jednou v každém řádku a v každém sloupci matice. V kapitole 1 se ihned interpretuje latinský čtverec jako multiplikativní tabulka kvazigrupy. Postupně následují isotopie kvazigrup, definice transversály latinského čtverce, úplné zobrazení kvazigrup a latinské subčtverce. 2. kapitola začíná identitami kvazigrup, pokračuje zmínkou o Steinerově systému trojic a končí úplnými latinskými čtverci. Kapitola 3. obsahuje věty o latinských obdélnících, řádkových a sloupcových latinských čtvercích, dále některé věty o existenci latinských čtverců a pojem neúplného latinského čtverce. V této kapitole běží o novou látku, připisovanou autorům. 4. kapitola podává klasifikaci latinských čtverců a vyčerpávající rozklad známých výsledků o počtu latinských čtverců daného řádu. Dva latinské čtverce $\|a_{ij}\|$, $\|b_{ij}\|$ téhož řádu n a vytvořené týmiž n symboly, se nazývají ortogonální, když každý uspořádaný pár těchto symbolů se vyskytuje právě jednou mezi všemi páry (a_{ij}, b_{ij}). Tento pojem se studuje v kapitole 5. spolu s jeho rozšířením a zobecněním na latinské krychle a hyperkrychle, řecko-latinské čtverce a pravoúhlá schemata. V kapitole 6 nacházíme popis konstrukce diagonálních latinských čtverců, magických čtverců a typu magických čtverců pojmenovaných podle T. G. Rooma. Opět následuje větší mírou původní látka připisovaná autorům a to v kapitole 7. Zde se diskutují jednotlivé metody konstrukce párů ortogonálních latinských čtverců, na příklad permutací řádků nebo sloupců a další. V kapitole 8. se v podstatě probírá už tradiční látka o konečných geometriích, hovoří se totiž o k-tkáních a projektivních rovinách, spolu se zavedením souřadnic, dále o existenci nedesarguesovských rovin a souvislost ortogonálních latinských čtverců s projektivní a afinní rovinou. Poznámky k teorii grafů v souvislosti s latinskými čtverci nacházíme v kapitole 9., kde se shledáváme s nutnou podmínkou o neexistenci transversály R. H. Brucka a s nutnou podmínkou o doplnění množiny vzájemně ortogonálních latinských čtverců na úplnou S. S. Shrikhanda. Z užití teorie latinských čtverců v teorii informace a ve statistice jsou uvedeny v kapitole 10 jen úlohy o kódech, jež vyhledávají a opravují chyby a úlohy o plánování pokusů. Kniha vrcholí 11. kapitolou a to vyvrácením Eulerovy domněnky (L. Euler se domníval, že pro $n = 4t + 2$ neexistují páry ortogonálních latinských čtverců) a odvozením všech známých dolních hranic funkce $N(n)$, tj. maximálního počtu ortogonálních latinských čtverců řádu n. Následující dvě kapitoly pojednávají o dalších konstrukcích ortogonálních latinských čtverců a o příbuzných tématech. Zejména poslední kapitola uvádí přehled prací, jež využívají výpočetní techniky k získání jak latinských čtverců, tak i ortogonálních latinských čtverců. Poznamenejme ještě, že i tam, kde to nebylo výslovně řečeno, se řada otištěných výsledků vyskytuje poprvé.

Uvedené důkazy vět jsou věcně stručné a úplné, radost je je číst. Navíc se čtenář nikterak nevyhne vzrušení. Je totiž každá kapitola doprovázená v průběhu textu historickým přehledem a komentářem prací, jež se vztahují k tématu kapitoly. Rozsah historických poznámek sahá od magických čtverců dávných dob až po zprávy ze současnosti. V bibliografii jsou zachyceny články až do roku 1974 a autoři se nevyhýbají citacím prací, jež byly ještě v tisku. U prací je uveden také odkaz na rešerše v Mathematical Reviews. Odkazů je více než 600. Pohledem do budoucna je seznam 73 formulací dosud nerozřešených problémů.

Kniha je určena čtenářům, kteří už prošli kursem matematiky na universitě. Je dobrým úvodem k zahájení studia v této problematice, neméně však také počátkem k další badatelské práci v tomto oboru. Pro svůj rozsah, hodnotu a přehled poslouží výtečně nejen jako kniha na níž se lze odvolávat, ale i k rychlé orientaci v problematice oboru. Pro poslední vlastnost, lze očekávat, že bude vyhledávána pracovníky jiných oborů, kteří latinské čtverce aplikují.

Význam a kvalitu této knihy podtrhuje P. Erdös, který kromě účasti na ní, napsal také další předmluvu. Mimo něj byli radou nápomocni význační matematici oboru, jako V. D. Bělousov, D. E. Knuth, C. C. Lindner, H. B. Mann, N. S. Mendelsohn, A. Sade, J. Schönheim a další.

Věroslav Jurák, Poděbrady

Jon Barwise: ADMISSIBLE SETS AND STRUCTURES. (An approach to definability theory.) Springer-Verlag, Berlin—Heidelberg—New York 1975, XIV + 394 str., cena DM 72,—.

Barwisova kniha je první kniha nové série Perspectives in mathematical logic, kterou založila a řídí tak zvaná skupina Ω (R. O. Gandy, H. Hermes, A. Levy, G. H. Müller, G. Sacks a D. S. Scott), působící od r. 1969 a formálně sídlící v Heidelbergu. Cílem série je „zmapovat" složitý terén matematické logiky, přičemž v každé knize má být položen důraz na určité důležité téma, tak aby kniha byla něčím víc než pouhou sbírkou výsledků a metod.

Barwisova kniha plně odpovídá těmto záměrům. Pojem přípustných množin (admissible sets) se bezesporu stal jedním z velmi důležitých pojmů soudobé matematické logiky; autorovi však nejde jen o shrnutí nepřehledné hory časopisecké literatury, nýbrž pojímá teorii přípustných množin jako společný základ pro různé zdánlivě nesouvisející partie matematické logiky (části teorie množin, zejména teorie konstruktivních množin (constructible sets), teorie modelů včetně modelů speciálních teorií (Peanovy aritmetiky teorie množin), zobecněné logiky, zejména logiky s nekonečně dlouhými formulemi, zobecněná teorie rekurse). Pojem přípustné množiny je technickým pojmem pro toto sjednocení; tématem, na které se celá kniha soustřeďuje, je pojem definovatelnosti v jeho nejrůznějších podobách.

Přípustné množiny (resp. přípustné ordinály) zavedli nezávisle na sobě Kripke a Platek. Přípustné množiny jsou jisté „standardní“ modely velice slabého systému axiomů teorie množin, zvaného Kripkova-Platkova teorie množin (KP); lze říci, že jsou to jisté rozumné „počáteční úseky“ universa množin. Přípustné ordinály jsou systémy ordinálních čísel přípustných množin. (Nejmenší přípustná množina je množina všech dědičně konečných množin, nejmenší přípustný ordinál je ω — první nekonečný ordinál.) Novinkou Barwisova přístupu je, že studuje od samého začátku přípustné množiny nad systémem urelementů (= prvků, které nejsou množinami), což mu umožňuje vybudovat nad libovolnou matematickou strukturou hierarchii přípustných množin. Příslušný axiomatický systém se nazývá KPU (Kripke-Platek surelementy); lze říci, že celou knihou se táhne vzájemná souhra (interakce) toho, co lze dokázat uvnitř teorie KPU a toho, co lze o modelech této teorie říci z hlediska celého universa množin.

Kniha se dělí na tři části (A, B, C) a ty celkem na osm kapitol (a dodatek). Kniha má XIV + + 394 stran. Názvy částí (základní teorie, absolutní teorie, k obecné teorii) mnoho neříkají; mnohem užitečnější jsou názvy kapitol a graf závislostí kapitol (str. XIV). Zmíním se stručně o jednotlivých kapitolách.

Kap. I (Teorie přípustných množin). Zde se zavádí axiomatický systém KPU a v něm se definují základní pojmy a dokazují základní věty (a schémata vět). Je zavedena důležitá třída $\sum$-formulí, odvodí se princip definování $\sum$-rekursí a probírají se dvě (v KPU neekvivalentní) formy Mostowského věty o kolapsu.

V kap. II (Některé přípustné množiny) se studují důležité modely teorie KPU; vedle dědičně konečných množin se studuje přípustná množina $\mathbf{HYP}_{\mathfrak{M}}$ (kde $\mathfrak{M}$ je nějaká relační struktura), což je nejmenší přípustná množina $\mathbf{A}$ taková, že nosič struktury $\mathfrak{M}$ tvoří množinu urelementů množiny $\mathbf{A}$ a (zhruba řečeno) struktura $\mathfrak{M}$ jako celek je prvkem množiny $\mathbf{A}$. $\mathbf{HYP}$ má připomínat hyperaritmetické množiny přirozených čísel, protože ty jsou v jistém přirozeném vztahu k prvkům množiny $\mathbf{HYP}_{\mathfrak{N}}$, kde $\mathfrak{N}$ značí strukturu přirozených čísel se sčítáním a násobením.

Kap. III (Spočetné fragmenty logiky $\mathbf{L}_{\infty\omega}$) studuje logiku s nekonečně dlouhými formulemi ve vztahu k přípustným množinám, zejména ke spočetným přípustným množinám. Jde především o fundamentální Barwisovy věty o úplnosti a kompaktnosti a o interpolační teorém pro spočetné přípustné množiny. Důkaz se opírá o pojem konsistenčních vlastností (consistency properties) zavedený Keislerem; autorovi se podařilo vypreparovat základní etapy důkazů s velikou dokonalostí a průzračností.

Kap. IV (Elementární výsledky o $\mathbf{HYP}_{\mathfrak{M}}$) se především zabývá teorií $\prod_1^1$ a $\sum_1^1$ predikátů, tj. predikátů (relací) definovaných na $\mathfrak{M}$ formulemi 2. řádu s jedním blokem universálních resp. existenčních kvantifikátorů 2. řádu. Základní věta praví, že pro každou spočetnou strukturu $\mathfrak{M}$ a každou relaci R na $\mathfrak{M}$ platí: R je $\prod_1^1$, právě když R je $\sum$-definovatelná část struktury $\mathbf{HYP}_{\mathfrak{M}}$. (Srv. poznámku o $\mathbf{HYP}_{\mathfrak{N}}$ výše; připomínám, že množina přirozených čísel je hyperaritmetická, právě když ona i její komplement jsou $\prod_1^1$-definovatelné.)

V kap. V (Teorie rekurse pro $\sum$-predikáty na přípustných množinách) se studují $\sum$-predikáty na přípustných množinách jakožto analogon rekursivně spočetných množin přirozených čísel a buduje se rozsáhlá analogie (věty o rekursi, normální forma, věta o separaci atd.). Dále se studují tzv. „rekursivně velké“ ordinály (rekursivně nedosažitelné, stabilní atd.).

Kap. VI (Induktivní definice) je věnována studiu induktivně definovaných relací na libovolné (ne nutně spočetné) přípustné množině $\mathbf{A}$ vzhledem k částem množiny $\mathbf{A}$ $\sum$-definovatelným v $\mathbf{HYP}_{\mathbf{A}}$.

Také kap. VII (Více o $\mathbf{L}_{\infty\omega}$) a VIII (Striktní $\prod_1^1$-predikáty a Königovy principy) se soustřeďuje převážně na výsledky o přípustných množinách, které nevyžadují předpoklad spočetnosti. Dodatek je věnován pojmu přípustného pokrytí (admissible cover) modelů teorie množin, zejména nestandardních.

Kniha je psána velmi elegantně, je dobře srozumitelná, autor se snaží soustřeďovat sebe i čtenáře na myšlenky a ne na technické detaily. To se mu také daří; jediná daň, kterou musí zaplatit za to, že buduje přípustné množiny nad strukturami, je značné technické zkomplikování definice konstruktivních množin. I zde však autor vypreparuje potřebné myšlenky a celou technickou ,,lopotu" soustředí do důkazu jednoho lemmatu, který zabere jeden celý paragraf. Tento paragraf je zakončen receptem na zákusek, který se má podávat, pokud se tento paragraf bude probírat na nějakém semináři. Tento vtip je charakteristický pro styl knihy. Velice cenná jsou cvičení, kterými je kniha hojně vybavena.

Kniha poněkud trpí některými drobnými nedůslednostmi a nepřesnými referencemi. Čtenáři, oprav si např. toto: Str. 37_2 místo Feferman [1975] má být Feferman [1974]; str. 105^{10} místo Barwise [1973] má být Barwise [1973c]; str. 123_9 místo for all $\mathfrak{M}$, R, F má být for all $\mathfrak{M}'$, R', F. Str. 126_9: zde se — jako na mnoha jiných místech — vyskytuje označení ,,the $\overline{YY}$-compactness theorem" místo ,,the Barwise compactness theorem". Srv. str. 102, kde autor (správně) říká, že označení ,,Barwisova věta o kompaktnosti" je tak vžité, že by bylo nemístnou skromností (a matoucí) zavádět nový pojem. Přitom $\overline{YY}$ je zřejmě nutno číst ,,bar-Y's", tedy opět ba:əwais; jde patrně o pozůstatek dřívější nemístné autorovy skromnosti. Str. 141_{10}: místo II.8.6 má být II.8.7. Str. 365^{13}: místo Barwise [1974] má být Barwise [1974a].

Barwisova kniha je bezesporu velice cenným přínosem; po léta byla očekávána monografie, která teorii přípustných množin učiní běžně dostupnou k prospěchu všech logiků. Dočkali jsme se velmi dobré knihy, která důstojně zahajuje novou sérii. Lze se jen těšit na připravované další svazky (např. Hinman: Inductive definitions and higher types, Scott a Kraus: Languages and structures, Levy: Basic set theory, Smorynski: Metamathematics of arithmetics aj.).

Petr Hájek, Praha

Diocles: ON BURNING MIRRORS. The Arabic Translation of the Lost Greek Original. Edited, with English Translation and Commentary by G. J. Toomer. Sources in the History of Mathematics and Physical Sciences 1. Springer Verlag, Berlin—Heidelberg—New York 1976. Stran IX + 249, cena DM 68,—.

Podle svědectví řeckého matematika Eutokia z Askalónu (žil kolem r. 500; k Archimédovým a Apolloniovým spisům připojil komentáře, které jsou důležitým pramenem pro dějiny matematiky) napsal Diokles pojednání *περι πυριων* (O zápalných zrcadlech), věnované parabolickým a sférickým zrcadlům a zdvojení krychle. Matematická část Dioklova díla, z něhož Eutokius zachytil dva úryvky, je založena na teorii kuželoseček. Překladatel Dioklova pojednání do arabštiny není znám.

Kniha obsahuje tyto části:

Úvod (str. 1—33) zahrnuje Dioklovu biografii a rozbory jeho díla, zvláště teorie kuželoseček až po Diokla a kuželoseček v díle samém. Následuje zhodnocení vlivu Dioklova pojednání a soupis rukopisů i textů.

Na str. 34—113 je vždy na pravé straně arabský text Dioklova díla, na levé anglický překlad.

Na str. 114—137 jsou fotografie arabského textu, který byl základem pro vydání.

Str. 138—175 zaplňují editorské poznámky.

Na str. 177—216 jsou čtyři dodatky: Řecký text s anglickým překladem výše zmíněných Eutokiových úryvků; starověké a středověké důkazy fokální vlastnosti paraboly; konečně dva kratší příspěvky O. Neugebauera, historika matematiky, fyziky a astronomie.

Obsáhlá bibliografie (str. 217—223), seznam technických termínů a obecný index zakončují toto kritické vydání.

Zbyněk Nádeník, Praha

Wendell H. Fleming, Raymond W. Rishel: DETERMINISTIC AND STOCHASTIC OPTIMAL CONTROL. Springer-Verlag, Berlin—Heidelberg—New York 1975, 222 str., cena DM 60,60.

Kniha sestává ze dvou částí. První část pojednává o teorii optimální regulace a variačním počtu v deterministickém případě, druhá je věnována stochastické optimální regulaci pro difúzní procesy.

První kapitola obsahuje krátký úvod do klasického variačního počtu. Je popsána nejjednodušší úloha minimalizace integrálního funkcionálu podél křivek s pevnými koncovými body s cílem vysvětlit postupy variačního počtu v extremálních úlohách. Ve druhé kapitole se autoři věnují formulaci úlohy minimalizace funkcionálu ve třídě funkcí, které splňují jistou diferenciální rovnici s parametrem a počáteční podmínky. Pro takové úlohy optimální regulace formulují Pontrjaginův princip maxima a diskutují jeho důsledky pro různé případy a úlohy. V závěru této kapitoly je uveden důkaz principu maxima metodou variace trajektorií pomocí abstraktního pravidla multiplikátorů. Předmětem III. kapitoly je otázka existence optimální regulace ve třídě integrovatelných funkcí pro běžné regulační úlohy a v závěru kapitoly autoři studují otázku, za jakých podmínek existuje spojitá regulace pro danou úlohu. Čtvrtá kapitola je věnována metodě dynamického programování a otázkám syntézy regulace (je vyložen postup V. G. Boltjanského pro konstrukci tzv. regulární syntézy) spolu s postačujícími podmínkami pro optimalitu pro jisté speciální případy úloh. Je také uvedeno srovnání výsledků metody dynamického programování s výsledky, které dává princip maxima.

Druhá část knihy je uvedena pátou kapitolou. Je přehledem té části teorie stochastických procesů, která je potřebná pro matematicky přesné zpracování teorie regulace difúzních procesů. Ve zhuštěné, ale přehledné podobě je v ní zpracována teorie spojitých stochastických procesů, stochastických diferenciálních rovnic a difúzních procesů. Poslední šestá kapitola se převážně zabývá teorií regulovaných difúzních procesů. Využívá se přitom metoda dynamického programování a tím se problém převede na zkoumání jisté nelineární parciální diferenciální rovnice. Má-li tato parciální diferenciální rovnice rozumné vlastnosti, lze nalézt syntézu optimální regulace. Zásadní je přitom stejnoměrná paraboličnost rovnice dynamického programování.

Výklad je doplněn bohatým příkladovým materiálem a úlohami. Dodatky v závěru knihy připomínají použitá fakta z jiných oblastí matematiky (konvexní množiny a funkce, základy pravděpodobnosti, parabolické parciální diferenciální rovnice apod.).

Štefan Schwabik, Ivo Vrkoč, Praha

MATHEMATICAL SYSTEMS THEORY. Proceedings of the international symposium Udine, Italy, June 16—27, 1975. Edited by G. Marchesini and S. K. Mitter. Lecture Notes in Economics and Mathematical Systems 131, Springer-Verlag, Berlin—Heidelberg—New York 1976, X + 408 str., cena DM 35,—.

Sborník symposia věnovaného matematické teorii systémů. Teorie systémů je bouřlivě se rozvíjející disciplína, která objevila přednosti matematického, analytického přístupu před inženýrským syntetickým postupem, který v případech složitých systémů selhává zákonitě, a obecně vede k výsledkům kratší, ekonomičtější cestou. Matematické metody, které tvoří základ analytické teorie systémů mají velmi široké spektrum od lineární algebry, algebry kategorií, diferenciální geometrie, kvalitativní teorie diferenciálních rovnic až k hlubším aspektům teorie Hilbertova prostoru a funkcionální analýze.

Tento sborník je věnován jenom některým bodům tohoto spektra, kterým odpovídá také jeho rozdělení do částí: teorie automatů, konečnědimenzionální lineární systémy, bilineární a nelineární systémy, lineární nekonečnědimenzionální systémy, teorie kódování a filtrování pro sekvenciální systémy, obecné dynamické systémy a kategoriální přístup k systémům. Obsahuje 27 příspěvků zařazených do těchto částí. Jsou zajímavé nejenom pro systémové teoretiky-inženýry, také matematik v nich najde mnoho materiálu, který s pochopením přečte z hlediska své specializace.

Štefan Schwabik, Praha