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ON PARTITION GRAPHS AND GENERALIZATIONS OF LINE GRAPHS

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By a graph we mean a graph in the sense of BEHZAD and CHARTRAND [1] or HARARY [2]. If G is a graph, then we denote by $V(G)$, $E(G)$, $\delta(G)$ and $L(G)$ its vertex set, edge set, minimum degree and line graph, respectively. If G is a graph and $u \in V(G)$, then we denote

$$E(G, u) = \{x \in E(G);\ x \text{ is incident with } u\}\ .$$

Let G and H be graphs, let $E(G) \neq \emptyset$, and let M be a graph-theoretical property (of graphs). We shall say that H is an M-extension of G if there exists a 1-1-mapping g from $E(G)$ onto $V(H)$ such that the following conditions hold:

($1_{G,g,H}$) if $x, y \in E(G)$ and $g(x)\,g(y) \in E(H)$, then x and y are adjacent edges of G;

($2_{G,g,H}/M$) if $u \in V(G)$ and $E(G, u) \neq \emptyset$, then the subgraph of H induced by $\{g(z); z \in E(G, u)\}$ has the property M.

We denote by A_1, A_2, and A_3 the properties
"either to be trivial or to contain no vertex of degree 0",
"to be connected",
and
"to be complete",
respectively.

It is clear that for every graph G with $E(G) \neq \emptyset$, $L(G)$ is the only A_3-extension of G. This means that the concepts of an A_1-extension and an A_2-extension are generalizations of the concept of the line graph of a graph.

Let F and G be graphs. We say that G is a partition graph of F if there exists a mapping f from $V(F)$ onto $V(G)$ such that the following condition holds:

($3_{F,f,G}$) if u and v are distinct vertices of G, then u and v are adjacent if and only if there exist $r \in f^{-1}(u)$ and $s \in f^{-1}(v)$ such that $rs \in E(F)$.

We say that G is a contraction of F if there exists a mapping f from $V(F)$ onto $V(G)$ such that $(3_{F,f,G})$ and the following condition holds:

$(4_{F,f,G})$ if $w \in V(G)$, then the subgraph of F induced by $f^{-1}(w)$ is connected.

The concept of a partition graph of a graph was studied by E. SAMPATHKUMAR and V. N. BHAVE [3]. (The concept of a contraction of a graph can be found in [1], p. 92.)

The following theorem is the main result of the present note:

Theorem. *Let G, H, and J be graphs, and let $\delta(G) \geqq 2$. Then*

(I) *if H is an A_1-extension of G, and J is an A_2-extension of H, then G is a partition graph of J;*

(II) *if H is an A_2-extension of G, and J si the A_3-extension of H, then G is a contraction of J.*

Proof. Let H be an A_1-extension of G (resp. an A_2-extension of G), and let J be an A_2-extension of H (resp. the A_3-extension of H). There exists a 1-1-mapping $g: E(G) \to V(H)$ such that $(1_{G,g,H})$ and $(2_{G,g,H}/A_1)$ (resp. $(2_{G,g,H}/A_2)$) hold. Similarly, there exists a 1-1-mapping $h: E(H) \to V(J)$ such that $(1_{H,h,J})$ and $(2_{H,h,J}/A_2)$ (resp. $(2_{H,h,J}/A_3)$) hold.

First, we assume that $(2_{G,g,H}/A_1)$ and $(2_{H,h,J}/A_2)$ hold.

Let r be an arbitrary vertex of G. We denote by $H(r)$ the subgraph of H induced by $\{g(x_r);\ x_r \in E(G, r)\}$. Since $\delta(G) \geqq 2$, we have that $H(r)$ is nontrivial. From $(2_{G,g,H}/A_1)$ it follows that $\delta(H(r)) \geqq 1$. We denote by $J(r)$ the subgraph of J induced by $\{h(y_r);\ y_r \in E(H(r))\}$.

We introduce a mapping f from $V(J)$ into $V(G)$. Let v be an arbitrary vertex of J. Then there are adjacent vertices t and u of H such that $h^{-1}(v) = tu$. From $(1_{G,g,H})$ it follows that $g^{-1}(t)$ and $g^{-1}(u)$ are adjacent edges of G. We denote by $f(v)$ the vertex of G incident both with $g^{-1}(t)$ and $g^{-1}(u)$. Since t and u are vertices of $H(f(v))$, we have that v is a vertex of $J(f(v))$.

Let s be an arbitrary vertex of G. It is easy to see that $f(w) = s$ for each $w \in V(J(s))$. This means that f is a mapping from $V(J)$ onto $V(G)$, and that $f^{-1}(s_0) = V(J(s_0))$ for every $s_0 \in V(G)$.

Let v_1 and v_2 be adjacent vertices of J and let $f(v_1) \neq f(v_2)$. There exist $y_1, y_2 \in E(H)$ such that $h(y_1) = v_1$ and $h(y_2) = v_2$. From $(1_{H,j,J})$ it follows that y_1 and y_2 are adjacent. This means that there exist distinct vertices u_0, u_1 and u_2 of H such that $y_1 = u_0u_1$ and $y_2 = u_0u_2$. It is clear that $f(v_1)$ is incident both with $g^{-1}(u_0)$ and $g^{-1}(u_1)$, and that $f(v_2)$ is incident both with $g^{-1}(u_0)$ and $g^{-1}(u_2)$. Since $f(v_1) \neq f(v_2)$, we have that $g^{-1}(u_0) = f(v_1)f(v_2)$. Hence $f(v_1)$ and $f(v_2)$ are adjacent.

Let s_1 and s_2 be adjacent vertices of G. We shall prove that there exist $w_1 \in f^{-1}(s_1)$ and $w_2 \in f^{-1}(s_2)$ such that w_1 and w_2 are adjacent vertices of J. Denote $x_0 = s_1s_2$. Obviously, $V(H(s_1)) \cap V(H(s_2)) = \{g(x_0)\}$ and $E(H(s_1)) \cap E(H(s_2)) = \emptyset$. From $(2_{G,h,H}/A_1)$ it follows that there exist $u' \in V(H(s_1))$ and $u'' \in V(H(s_2))$ such that