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ON INVERSION OF LAPLACE TRANSFORM (I)

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The aim of this note is to show how a complex inversion theorem may be deduced from the general Post-Widder inversion theorem.

1. We denote by R and C respectively the real and complex number fields and by R^+ the set of all positive numbers. Further, if M_1, M_2 are two arbitrary sets, then $M_1 \to M_2$ will denote the set of all mappings of the set M_1 into the set M_2.

2. Lemma. *For every $\alpha \geqq 0$ and $r \in \{1, 2, \ldots\}$ such that $r > \alpha$, we have*

$$\left(\frac{r}{r-\alpha}\right)^r \leqq e^{(r/r-\alpha)\alpha} = e^{\alpha} e^{\alpha^2/(r-\alpha)}.$$

Proof. Under our assumptions we have

$$\log\left(\frac{r}{r-\alpha}\right)^r = \log\left(\frac{1}{1-\frac{\alpha}{r}}\right)^r = -r\log\left(1-\frac{\alpha}{r}\right) = r\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha}{r}\right)k \leqq$$

$$\leqq r\sum_{k=1}^{\infty}\left(\frac{\alpha}{r}\right)^k = r\frac{\alpha}{r}\frac{1}{1-\frac{\alpha}{r}} = \alpha\frac{r}{r-\alpha} = \alpha + \frac{\alpha^2}{r-\alpha}$$

and our result follows.

3. Lemma. *For every $z \in C$, $(1 + z/q)^q \to e^z (q \to \infty)$.*

Proof. According to the binomial theorem, we can write

$$(1)\qquad \left(1+\frac{z}{q}\right)^q=\sum_{k=0}^{q}\binom{q}{k}\frac{z^k}{q^k}=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{k!}\frac{z^k}{q^k}=$$

$$=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{q^k}\frac{z^k}{k!}=$$

$$=1+z+\sum_{k=2}^{q}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z^k}{k!}$$

for every $q \in \{2, 3, \dots\}$.

Let now $z \in C$ and $\varepsilon > 0$. Then there exists a $k_0 \in \{2, 3, \dots\}$ such that

$$(2)\qquad \sum_{k=k_0+1}^{\infty}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}.$$

It follows from (2) that

$$(3)\qquad \left|e^z-\sum_{k=0}^{k_0}\frac{z^k}{k!}\right|\leqq\frac{\varepsilon}{3}.$$

Further by (1) and (2),

$$(4)\qquad \left|\left(1+\frac{z}{q}\right)^q-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|=$$

$$=\left|\sum_{k=k_0+1}^{q}\left(1-\frac{1}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right|\leqq\sum_{k=k_0+1}^{k}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}$$

for every $q \geqq k_0 + 1$.

Finally it is easy to see that there exists a $q_0 \in \{k_0 + 1, k_0 + 2, \dots\}$ such that

$$(5)\qquad \left|\sum_{k=0}^{k_0}\frac{z^k}{k!}-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|\leqq\frac{\varepsilon}{3}$$

for every $q \geqq q_0$.

Now we have immediately from (3), (4) and (5)

$$\left|e^z-\left(1-\frac{z}{q}\right)^q\right|\leqq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

for every $q \geqq q_0$ and this gives the assertion.

4. Theorem (Post-Widder). *Let $f \in R^+ \to E$ and let M, ω be two nonnegative constants. If*

(α) *the function f is measurable over R^+,*

(β) $|f(t)| \leqq Me^{\omega t}$ *for almost all $t \in R^+$, then*

(a) $$\int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \text{ exists for every } t \in R^+ \text{ and } p + 1 > \omega t,$$

(b) $$\left|\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\int_0^\infty e^{-(p+1)/\tau}\tau^p f(\tau)\,\mathrm{d}\tau\right| \leqq Me^{\omega t}e^{\omega^2 t^2/(p+1-\omega t)}$$

for every $t \in R^+$ and $p + 1 > \omega t$,

(c) $$\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \to f(t) \quad (p \to \infty,\ p + 1 > \omega t)$$

for almost all $t \in R^+$.

5. Lemma. *Let α be a nonnegative constant and $J \in \{z : \mathrm{Re}\, z \geqq \alpha\} \to C$. If*

(α) *J is continuous on $\{z : \mathrm{Re}\, z \geqq \alpha\}$,*

(β) *J is analytic on $\{z : \mathrm{Re}\, z > \alpha\}$,*

(γ) $J(\lambda) \to 0$ $(\lambda \geqq \alpha,\ \lambda \to \infty)$,

(δ) *there exist a constant K and a number $k \in \{0, 1, \ldots\}$ so that for every $\mathrm{Re}\, z \geqq \alpha$, we have $|J(z)| \leqq K(1 + |z|)^k$,*

(ε) $$\int_{-\infty}^\infty \frac{|J(\alpha + i\beta)|}{1 + |\beta|}\,\mathrm{d}\beta < \infty,$$

then for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$,

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} J(\lambda) = (-1)^p \frac{p!}{2\pi i}\int_{-\infty}^\infty \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\,\mathrm{d}\beta.$$

Proof. Let us first fix a $\lambda > \alpha$.

Moreover, we choose fixed numbers K, k so that the assumption (δ) holds.

By virtue of Cauchy's integral theorem, we obtain from (α), (β) that

(1) $$\begin{aligned}\frac{2\pi}{p!} J^{(p)}(\lambda) = &-\int_{-N}^N \frac{1}{(\alpha + i\beta - \lambda)^{p+1}} J(\alpha + i\beta)\,\mathrm{d}\beta + \\ &+ \int_{-N}^N \frac{1}{(\alpha + 2N + i\beta - \lambda)^{p+1}} J(\alpha + 2N + i\beta)\,\mathrm{d}\beta - \\ &- i\int_0^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta + iN)\,\mathrm{d}\eta + \\ &+ i\int_0^{2N} \frac{1}{(\alpha + \eta - iN - \lambda)^{p+1}} J(\alpha + \eta - iN)\,\mathrm{d}\eta\end{aligned}$$

for every $p \in \{0, 1, \ldots\}$ and $N > \frac{1}{2}\lambda$.

Using (δ), we obtain

$$(2)\qquad \left|\int_{-N}^{N}\frac{1}{(\alpha+2N+i\beta-\lambda)^{p+1}}\,J(\alpha+2N+i\beta)\,\mathrm{d}\beta\right| \leqq$$

$$= \int_{-N}^{N}\frac{1}{((\lambda-\alpha+2N)^2+\beta^2)^{(p+1)/2}}\,[1+((\alpha+2N)^2+\beta^2)^{1/2}]^k\,\mathrm{d}\beta =$$

$$= \int_{-N}^{N}\frac{1}{(\lambda-\alpha+2N)^{p+1}}\,[1+((\alpha+2N)^2+N^2)^{1/2}]^k =$$

$$= \frac{2N}{(\lambda-\alpha+2N)^{p+1}}\,[1+((\alpha+2N)^2+N^2)^{1/2}]^k,$$

$$\left|\int_{0}^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta+iN)\,\mathrm{d}\eta\right| \leqq$$

$$\leqq \int_{0}^{2N}\frac{1}{((\lambda+\eta-\alpha)^2+N^2)^{(p+1)/2}}\,[1+(\alpha^2+N^2)^{1/2}]^k\,\mathrm{d}\eta \leqq$$

$$\leqq \int_{0}^{2N}\frac{1}{N^{p+1}}\,[1+(\alpha^2+N^2)^{1/2}]^k \leqq \frac{2}{N^p}\,[1+(\alpha^2+N^2)^{1/2}]^k,$$

$$\int_{0}^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta-iN)\,\mathrm{d}\eta \leqq \frac{2}{N^p}\,[1+(\alpha^2+N^2)^{1/2}]^k$$

for every $p\in\{0, 1, \ldots\}$ and $N > \frac{1}{2}\lambda$.

Letting $N\to\infty$, we see from (2) that

$$(3)\qquad \int_{-N}^{N}\frac{1}{(\alpha+2N+i\beta-\lambda)^{p+1}}\,J(\alpha+2N+i\beta)\,\mathrm{d}\beta \to_{N\to\infty} 0,$$

$$\int_{0}^{2N}\frac{1}{(\alpha+\eta+iN-\lambda)^{p+1}}\,J(\alpha+\eta+iN)\,\mathrm{d}\eta \to_{N\to\infty} 0,$$

$$\int_{0}^{2N}\frac{1}{(\alpha+\eta-iN-\lambda)^{p+1}}\,J(\alpha+\eta-iN)\,\mathrm{d}\eta \to_{N\to\infty} 0$$

for every $p\in\{k+1, k+2, \ldots\}$.

Now we conclude from (1) and (3) by means of (ε) that

$$(4)\qquad J^{(p)}(\lambda) = -\frac{p!}{2\pi}\int_{-\infty}^{\infty}\frac{J(\alpha+i\beta)}{(\alpha+i\beta-\lambda)^{p+1}}\,\mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p\in\{k+1, k+2, \ldots\}$.

On the other hand, let us define, on the basis of (ε)

$$J_0(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)}\, d\beta \quad \text{for every} \quad \lambda > \alpha .$$

It is easy to verify that

(6) the function J_0 is infinitely differentiable on (α, ∞),

(7) $$J_0^{(p)}(\lambda) = -\frac{p!}{2\pi}\int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)^{p+1}}\, d\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$,

(8) $$J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty) .$$

By (4)–(7),

(9) $$J^{(k+1)}(\lambda) = J_0^{(k+1)}(\lambda)$$

for every $\lambda > \alpha$. Consequently, by (9)

(10) $$J - J_0 \quad \text{is a polynomial} .$$

Taking (γ) and (8) into account, we see that

(11) $$J(\lambda) - J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty) .$$

Hence (10) and (11) imply $J = J_0$ and the conclusion of Lemma 5 follows immediately from (7).

6. Theorem. *Let $f \in R^+ \to C$ and $\alpha > 0$. If*

(α) *the function f is measurable,*

(β) *there exist two nonnegative constants M, ω so that $\omega < \alpha$ and $|f(t)| \leqq Me^{\omega t}$ for almost all $t \in R^+$,*

(γ) $$\int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{-(\alpha + i\beta)\tau} f(\tau)\, d\tau \right| d\beta < \infty ,$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha + i\beta)t} \left(\int_0^{\infty} e^{-(\alpha + i\beta)\tau} f(\tau)\, d\tau \right) d\beta .$$

for almost all $t \in R^+$.

Proof. Let us first fix the constants M, ω so that the assumption (β) holds.

Further let us define a function $F \in (\omega, \infty) \to C$ by

(1) $$F(\lambda) = \int_0^{\infty} e^{-\lambda\tau} f(\tau)\, d\tau \quad \text{for} \quad \lambda > \omega .$$

By Theorem 4 we have

$$(2)\qquad \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to f(t) \quad (p \to \infty,\ p+1 > \alpha t)$$

for almost all $t \in R^+$.

On the other hand, let J be the function defined by

$$(3)\qquad J(z) = \int_0^\infty e^{-z\tau} f(\tau)\, \mathrm{d}\tau$$

for every $\operatorname{Re} z \geqq \alpha$.

It is easy to deduce from our assumptions that

(4) the function J has the properties 5 $(\alpha)-(\varepsilon)$.

Hence by (4), we obtain from Lemma 5 that

$$(5)\qquad J^{(p)}(\lambda) = (-1)^p \frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\, \mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$.

Now it follows from (1), (3) and (5) that

(6)

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\left(\frac{p+1}{t}\right)^{p+1}}{\left(\frac{p+1}{t} - \alpha - i\beta\right)^{p+1}} J(\alpha + i\beta)\, \mathrm{d}\beta =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(1 - \frac{(\alpha + i\beta)\, t}{p+1}\right)^{p+1}} J(\alpha + i\beta)\, \mathrm{d}\beta$$

for every $t \in R^+$ and $p + 1 > \alpha t$.

By Lemma 2 we see that

$$(7)\qquad \left|\frac{1}{\left(1 - \frac{(\alpha + i\beta)\, t}{p+1}\right)^{p+1}}\right| = \frac{1}{\left[\left(1 - \frac{\alpha t}{p+1}\right)^2 + \left(\frac{\beta t}{p+1}\right)^2\right]^{(p+1)/2}} \leqq$$

$$\leqq \frac{1}{\left(1 - \frac{\alpha t}{p+1}\right)^{p+1}} \leqq e^{\alpha t} e^{\alpha t/(p+1-\alpha t)}$$

for every $t \in R^+$ and $p + 1 > \alpha t$.