Then there exist two numbers $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ and $\varrho > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon}_1]$ there is a unique $x^\varepsilon \in B(x_0, \varrho)$ satisfying ${}^\varepsilon G(x^\varepsilon) = 0$. Moreover, $\lim_{\varepsilon \to 0} x^\varepsilon = x_0$.

Proof. Let $\bar{\varepsilon}_1$, α and ϱ be the numbers chosen in the proof of Lemma 3.3. Let $x_1^\varepsilon, x_2^\varepsilon \in B(x_0, \varrho)$, $0 < \varepsilon \leqq \bar{\varepsilon}_1$ satisfy ${}^\varepsilon G(x_1^\varepsilon) = {}^\varepsilon G(x_2^\varepsilon) = 0$. Then we can write

$$x_1^\varepsilon - x_2^\varepsilon = -T^\varepsilon({}^\varepsilon G(x_1^\varepsilon) - {}^\varepsilon G(x_2^\varepsilon) - {}^\varepsilon G'(x_0)(x_1^\varepsilon - x_2^\varepsilon)) .$$

Using [2] (relation 8.6.2), we obtain

$$\|x_1^\varepsilon - x_2^\varepsilon\|_X \leqq \bar{m}\|x_1^\varepsilon - x_2^\varepsilon\|_X \,.$$
$$. \sup \{\|{}^\varepsilon G'(x) - {}^\varepsilon G'(x_0)\|_{[X,Y]};\ x \in B(x_0, \varrho),\ \varepsilon \in (0, \bar{\varepsilon}_1]\} \leqq \alpha\|x_1^\varepsilon - x_2^\varepsilon\|_X \,.$$

As $\alpha < 1$, we have $x_1^\varepsilon = x_2^\varepsilon$. This completes the proof.

Lemma 3.5. *Let X and Y be Banach spaces. Let $A \in [X, Y]$ and $B \in [Y, X]$ satisfy $AB = I_Y$. Then for every $\Delta \in [X, Y]$, $\|\Delta\|_{[X,Y]} \leqq (2\|B\|_{[Y,X]})^{-1}$ there exists $B_\Delta \in \in [Y, X]$ such that*

$$(A + \Delta) B_\Delta = I_Y \,, \tag{3.10}$$

$$\|B_\Delta\|_{[Y,X]} \leqq 2\|B\|_{[Y,X]} \,. \tag{3.11}$$

If in addition the operators A and B fulfil $BA = I_X$, then B_Δ satisfies (3.10), (3.11) *and $B_\Delta(A + \Delta) = I_X$.*

Proof is easy.

References

[1] *Fink J. P., Hall W. S.:* Entrainment of frequency in evolution equations, J. Diff. Eq. *14* (1973), 9—41.

[2] *Dieudonné J.:* Foundations of Modern Analysis, Academic Press, New York, 1969.

[3] *Lovicarová H.:* Periodic solutions of a weakly nonlinear wave equation in one dimension, Czechoslovak Math. J. *19 (94)*, (1969), 324—342.

[4] *Rabinowitz P.:* Periodic solutions of nonlinear hyperbolic partial differential equations, Comm. Pure Appl. Math. *20* (1967), 145—205.

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).