PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION

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INTRODUCTION

In this paper the existence of a solution to the equations

$$(0.1) \qquad u_{tt}(t, x) - u_{xx}(t, x) = \varepsilon F_\varepsilon(u)(t, x), \quad t \in R^+, \quad x \in R,$$

$$(0.2) \qquad u(t, x) = u(t, x + 2\pi) = -u(t, -x), \quad t \in R^+, \quad x \in R,$$

$$(0.3) \qquad u(t + 2\pi + \varepsilon\lambda, x) = u(t, x), \quad t \in R^+, \quad x \in R$$

is investigated for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. The number $\varepsilon_0 > 0$ is supposed to be sufficiently small and the number $\lambda > 0$ is supposed to be fixed. The operator F_ε has the form

$$(0.4) \qquad F_\varepsilon(u)(t, x) = f_\varepsilon(t, x, u(t, x), u_t(t, x), u_x(t, x)).$$

The function f_ε is assumed to satisfy the next two conditions:

$$(0.5) \qquad f_\varepsilon(t, x, y_0, y_1, y_2) = f_\varepsilon(t, x + 2\pi, y_0, y_1, y_2) =$$
$$= -f_\varepsilon(t, -x, -y_0, -y_1, y_2) = f_\varepsilon(t + 2\pi + \varepsilon\lambda, x, y_0, y_1, y_2)$$

for every $(t, x, y_0, y_1, y_2) \in R^+ \times R^4$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

(0.6) If the derivative

$$D \equiv D_x^\alpha D_{y_0}^{\beta_0} D_{y_1}^{\beta_1} D_{y_2}^{\beta_2}$$

satisfies $\alpha + \beta_0 + \beta_1 + \beta_2 \leqq 2$, $\alpha \leqq 1$, then the function Df_ε is continuous on $R^+ \times R^4$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$\lim_{\varepsilon \to 0} \sup \{|Df_\varepsilon(t, x, y_0, y_1, y_2) - Df_0(t, x, y_0, y_1, y_2)| ;$$
$$t \in [0, 2\pi + 1], \ x \in R, \ |y_0|, |y_1|, |y_2| \leqq \varrho\} = 0$$

for every $\varrho > 0$ and

$$\lim_{r \to 0+} \sup \{ |Df_\varepsilon(t, x, y_0, y_1, y_2) - Df_\varepsilon(t, x, \bar{y}_0, \bar{y}_1, \bar{y}_2)| ;$$
$$t \in [0, 2\pi + 1], \ x \in R, \ |y_i - \bar{y}_i| \leqq r, \ i = 0, 1, 2, \ \varepsilon \in [-\varepsilon_0, \varepsilon_0] \} = 0$$

for every $(y_0, y_1, y_2) \in R^3$.

The first section of this paper contains two assertions on the existence of periodic solutions of the problem described (Theorems 1.1 and 1.2) which are deduced under some additional assumptions on F_ε. This part is modelled by [1].

In the second section it is shown that a solution to (0.1)–(0.3) with F_ε given by

(0.7) $$F_\varepsilon(u)(t, x) = g(u, u_t, u_x) + h_\varepsilon(t, x)$$

exists for every ε with $|\varepsilon|$ sufficiently small provided

(0.8) the second derivatives of g are continuous on R^3,

(0.9) $$g(y_0, y_1, y_2) = -g(-y_0, -y_1, y_2) \quad \text{for} \quad (y_0, y_1, y_2) \in R^3,$$

(0.10) $$g_{y_1}(y_0, y_1, y_2) \geqq \gamma_1, \quad |g_{y_0}(y_0, y_1, y_2)| \leqq \gamma_0,$$
$$|g_{y_2}(y_0, y_1, y_2)| \leqq \gamma_2 \quad \text{for} \quad (y_0, y_1, y_2) \in R^3,$$

(0.11) $$\gamma_1 - \gamma_2 - 2\gamma_0 > 0,$$

(0.12) $h_\varepsilon = h_\varepsilon(t, x) : R^+ \times R \to R$ and $(h_\varepsilon)_x$ are continuous for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$h_\varepsilon(t, x) = h_\varepsilon(t, x + 2\pi) = -h_\varepsilon(t, -x) = h_\varepsilon(t + 2\pi + \varepsilon\lambda, x) \quad \text{for} \quad (t, x) \in R^+ \times R$$

and

$$\lim_{\varepsilon \to 0} \sup \{ |D_x h_\varepsilon(t, x) - D_x h_0(t, x)|; \ t \in [0, 2\pi + 1], \ x \in R \} = 0.$$

These assumptions, from which (0.11) describes "some sort of monotonicity of F_ε", are similar to those in [3] where 2π-periodic solutions were investigated. Eventually, Section 2 contains a brief discussion of the existence of a $(2\pi + \varepsilon\lambda)$-periodic solution to

(0.13) $$u_{tt} - u_{xx} = \varepsilon(3u^2 u_t + h_\varepsilon(t, x))$$

for every ε from a neighbourhood of 0 provided (0.12) is satisfied and

$$\int_0^{2\pi} h_0(\vartheta, x - \vartheta) \, d\vartheta \neq 0 \text{ for some } x \in R.$$

Section 3 contains some auxiliary assertions.

The problem analogous to (0.1)–(0.3) was investigated by J. P. FINK and W. S. HALL in [1]. These authors developed a general theory for a system of first order equations and as a by-product they obtained the existence of periodic solutions for one special type of the wave equation (cf. (0.13)). In their paper the difficulties connected with the existence of periodic solutions whose periods depend on a parameter were also thoroughly discussed and therefore everybody who wants to be informed in detail is referred to [1].

The author is grateful to O. VEJVODA who attracted his attention to paper [1].

1. GENERAL THEOREMS

Let H_k be the space of all real valued 2π-periodic functions s which have generalized derivatives up to order k and satisfy

$$\int_0^{2\pi} s(\xi)\,\mathrm{d}\xi = 0 \quad \text{and} \quad \int_0^{2\pi} (s^{(k)}(\xi))^2\,\mathrm{d}\xi < +\infty\,.$$

The space H_k endowed with the inner product

$$(r, s)_k = \int_0^{2\pi} r^{(k)}(\xi)\, s^{(k)}(\xi)\,\mathrm{d}\xi$$

is a real Hilbert space. The norm in the space H_k will be denoted by $|\cdot|_k$. Putting

$$\mathscr{H}_k = \{s \in H_k;\ s(x) = -s(-x) \text{ for all } x \in R\}$$

and endowing $\mathscr{H}_k$ with the norm $|\cdot|_k$, we set

$$U_\infty = C^2([0, \infty); \mathscr{H}_0) \cap C^1([0, \infty); \mathscr{H}_1) \cap C^0([0, \infty); \mathscr{H}_2)$$

and

$$U_T = C^2([0, T]; \mathscr{H}_0) \cap C^1([0, T]; \mathscr{H}_1) \cap C^0([0, T]; \mathscr{H}_2)$$

for $0 < T < \infty$. The space U_T equipped with the norm

$$\|u\|_{U_T} = \sum_{i=0}^{2} \|u\|_{C^{2-i}([0,T];\mathscr{H}_i)}$$

is a Banach space. For the sake of simplicity we fix $T = 2\pi + 1$ and introduce an operator $Z : H_2 \to U_\infty$ by

$$Z\, s(t, x) = s(t + x) - s(t - x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

The space of all linear continuous mappings from X into Y will be denoted by $[X, Y]$. For $A \in [X, Y]$ we put

$$\|A\|_{[X,Y]} = \sup\,\{\|Ax\|_Y;\ x \in X,\ \|x\| \leqq 1\}\,.$$

Using Lemmas 3.1 and 3.2, we verify that a function $u \in U_\infty$ satisfying (0.1)–(0.3) for $\varepsilon \neq 0$ exists if and only if there is a pair of functions $(u, s) \in U_T \times H_2$ such that

$$(1.1) \qquad {}^{\varepsilon}G_1(u, s)(t, x) \equiv -u(t, x) + Z\, s(t, x) + \\ + \frac{\varepsilon}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta = 0\,, \quad t \in [0, T]\,, \quad x \in R\,,$$

$$(1.2) \qquad {}^{\varepsilon}G_2(u, s)(x) \equiv \frac{1}{\varepsilon}(s'(x) - s'(x - \varepsilon\lambda)) + \\ + \frac{1}{2}\int_0^{2\pi + \varepsilon\lambda} F_\varepsilon(u)(\vartheta, x - \vartheta)\, d\vartheta = 0\,, \quad x \in R\,.$$

Sufficient conditions under which a solution of (1.1) and (1.2) exists are described in the following two theorems.

Theorem 1.1. *Let $\lambda > 0$ and let a function f_ε satisfy (0.5) and (0.6). Let the following assumptions be satisfied:*

(i) *There exists $s_0 \in H_3$ such that $Ms_0 = 0$ where*

$$(1.3) \qquad Ms(x) = \lambda s''(x) + \frac{1}{2}\int_0^{2\pi} F_0(Zs)(\vartheta, x - \vartheta)\, d\vartheta = 0\,, \quad x \in R\,.$$

(ii) *There exists a constant m and a family of operators $Y^\varepsilon \in [H_1, H_2]$ such that*

$$(1.4) \qquad V^\varepsilon Y^\varepsilon = I_{H_1} \quad \text{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0\,,$$

$$(1.5) \qquad \|Y^\varepsilon\|_{[H_1, H_2]} \leqq m \quad \text{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0$$

where

$$(1.6) \quad V^\varepsilon \sigma(x) = |\varepsilon|^{-1}(\sigma'(x) - \sigma'(x - |\varepsilon|\,\lambda)) + \frac{1}{2}\int_0^{2\pi} F_0'(Zs_0)\, Z\sigma(\vartheta, x - \vartheta)\, d\vartheta\,, \ x \in R\,.$$

Then there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for every ε, $0 < |\varepsilon| \leqq \varepsilon_1$ there is $u \in U_\infty$ satisfying (0.1)–(0.3). Moreover, denoting this u by u^ε, we have

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Let us suppose that*

$$Y^\varepsilon V^\varepsilon = I_{H_2} \quad \text{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0\,.$$

Then there exist two numbers $r > 0$ and $\varepsilon_2 \in (0, \varepsilon_0]$ such that for every ε, $0 < |\varepsilon| \leqq \varepsilon_2$ there is a unique $u \in U_\infty$ satisfying (0.1)–(0.3) and $\|u - Zs_0\|_{U_T} \leqq r$.

Moreover, denoting this u by u^ε, we have

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Proof of Theorem 1.1. Let us put $X = U_T \times H_2$, $Y = U_T \times H_1$ and ${}^\varepsilon G(u, s) = ({}^\varepsilon G_1(u, s), {}^\varepsilon G_2(u, s))$ where ${}^\varepsilon G_1$ and ${}^\varepsilon G_2$ are given by (1.1) and (1.2) respectively. Assuming $\varepsilon \in (0, \varepsilon_0]$, we shall prove that the mapping ${}^\varepsilon G$ satisfies the assumptions of Lemma 3.3. Routine but lengthy calculations show that the mapping ${}^\varepsilon G : X \to Y$ is continuous for every fixed $\varepsilon \in (0, \varepsilon_0]$. The derivative ${}^\varepsilon G'$ of ${}^\varepsilon G$ with respect to (u, s) is given by

$${}^\varepsilon G'(u, s) = ({}^\varepsilon G_1'(u, s), {}^\varepsilon G_2'(u, s))$$

where

$$({}^\varepsilon G_1'(u, s)(v, \sigma))(t, x) = -v(t, x) + Z\,\sigma(t, x) + \frac{\varepsilon}{2}\int_0^t\int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon'(u)\, v(\vartheta, \xi)\,\mathrm{d}\xi\,\mathrm{d}\vartheta\,, \quad t\in[0, T]\,, \quad x\in R\,,$$

$$({}^\varepsilon G_2'(u, s)(v, \sigma))(x) = \varepsilon^{-1}(\sigma'(x) - \sigma'(x - \varepsilon\lambda)) + \frac{1}{2}\int_0^{2\pi+\varepsilon\lambda} F_\varepsilon'(u)\, v(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta\,, \quad x \in R$$

for $(v, \sigma) \in X$. These relations imply that ${}^\varepsilon G'(u, s) \in [X, Y]$ for every $(u, s) \in X$ and $\varepsilon \in (0, \varepsilon_0]$. Denoting $u_0 = Zs_0$, we obtain

$$\limsup_{\varrho\to 0_+} \{\|{}^\varepsilon G'(u, s) - {}^\varepsilon G'(u_0, s_0)\|_{[X,Y]}\,;\ \varepsilon\in(0, \varepsilon_0],\ \|(u, s) - (u_0, s_0)\|_X \leqq \varrho\} = 0\,.$$

The assumption (i) yields

$$\lim_{\varepsilon\to 0_+} \|{}^\varepsilon G(u_0, s_0)\|_Y = 0\,.$$

We shall now define a pair of operators by

$$(A_1(v, \sigma))(t, x) = -v(t, x) + Z\,\sigma(t, x)\,, \quad t\in[0, T]\,, \quad x\in R\,,$$

$$({}^\varepsilon A_2(v, \sigma))(x) = \varepsilon^{-1}(\sigma'(x) - \sigma'(x - \varepsilon\lambda)) + \frac{1}{2}\int_0^{2\pi} F_0'(u_0)\, v(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta\,, \quad x\in R\,.$$

Putting ${}^\varepsilon A = (A_1, {}^\varepsilon A_2)$, we easily verify

$$\lim_{\varepsilon\to 0_+} \|{}^\varepsilon G'(u_0, s_0) - {}^\varepsilon A\|_{[X,Y]} = 0\,. \tag{1.7}$$

We shall show that there exists a constant m_1 and a family of operators $B^\varepsilon \in [Y, X]$, $0 < \varepsilon \leqq \varepsilon_0$ satisfying

$${}^\varepsilon A B^\varepsilon = I_Y\,, \tag{1.8}$$

$$\|B^\varepsilon\|_{[Y,X]} \leqq m_1 \tag{1.9}$$

for every $\varepsilon \in (0, \varepsilon_0]$. For the sake of simplicity we put

$$P\, v(x) = \frac{1}{2} \int_0^{2\pi} F_0'(u_0)\, v(\vartheta, x - \vartheta)\, d\vartheta\,.$$

Then we set

$$B_2^\varepsilon(w, \eta) = Y^\varepsilon(\eta + Pw)\,, \quad B_1^\varepsilon(w, \eta) = -w + Z\, B_2^\varepsilon(w, \eta)$$

for $(w, \eta) \in Y$. The assumptions (1.4) and (1.5) show that the operator $B^\varepsilon = (B_1^\varepsilon, B_2^\varepsilon)$ satisfies (1.8) and (1.9). In virtue of (1.7) we can apply Lemma 3.5 to the operator ${}^\varepsilon G'(u_0, s_0)$. Hence there are $\bar{m} > 0$, $\bar{\varepsilon} \in (0, \varepsilon_0]$ and a family of operators T^ε, $0 < \varepsilon \leqq \bar{\varepsilon}$ such that ${}^\varepsilon G'(u_0, s_0)\, T^\varepsilon = I_Y$ and $\|T^\varepsilon\|_{[Y,X]} \leqq \bar{m}$. Thus all the assumptions of Lemma 3.3 are satisfied and therefore the theorem is proved for ε positive. The case $\varepsilon \in [-\varepsilon_0, 0)$ can be treated in the same way if Lemma 3.3 is applied to the pair of operators $({}^{-\varepsilon}G_1(u, s), {}^{-\varepsilon}G_3(u, s))$ where ${}^\varepsilon G_3(u, s)(x) = {}^\varepsilon G_2(u, s)(x + \varepsilon\lambda)$. This completes the proof.

Theorem 1.2 can be proved analogously to Theorem 1.1 if Lemma 3.4 is applied.

2. APPLICATIONS

We start by proving the following assertion:

Theorem 2.1. *Let two functions g and h_ε satisfy (0.8)–(0.12). Then there exist $\varepsilon_1 \in (0, \varepsilon_0]$, $r > 0$ and $s_0 \in H_3$ with the following property: For every ε, $0 < |\varepsilon| \leqq \varepsilon_1$ there is unique $u \in U_\infty$ satisfying $\|u - Zs_0\|_{U_T} \leqq r$ and (0.1)–(0.3) with F_ε given by (0.7). Moreover, denoting this u by u^ε, we have*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Proof. The theorem will follow from Theorem 1.2 if we prove:
(a) There is $s_0 \in H_3$ which satisfies

$$s_0''(x) + (2\lambda)^{-1} \int_0^{2\pi} F_0(Zs_0)(\vartheta, x - \vartheta)\, d\vartheta = 0\,, \quad x \in R\,. \tag{2.1}$$

(b) There is $(V^\varepsilon)^{-1} \in [H_1, H_2]$ satisfying

$$\|(V^\varepsilon)^{-1}\|_{[H_1,H_2]} \leqq m$$

for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\varepsilon \neq 0$.

Here V^ε is given by (1.6). Firstly, we shall show that (a) is valid. Let us denote by K the linear operator from $[H_1, H_2]$ given by

$$Ks(x) = (2\lambda)^{-1} \left(\int_0^x s(\xi)\, d\xi + (2\pi)^{-1} \int_0^{2\pi} \xi\, s(\xi)\, d\xi \right), \quad x \in R$$

and by Φ the continuous and bounded operator from H_1 into itself given by

$$\Phi\, \sigma(x) = \int_0^{2\pi} F_0(2\,\lambda Z K \sigma)(\vartheta, x - \vartheta)\, d\vartheta =$$

$$= \int_0^{2\pi} g \left(\int_{-x+2\vartheta}^{x} \sigma(\xi)\, d\xi,\ \sigma(x) - \sigma(-x + 2\vartheta),\ \sigma(x) + \sigma(-x + 2\vartheta) \right) d\vartheta +$$

$$+ \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\, d\vartheta\,, \quad x \in R\,.$$

The operator K is a linear compact mapping from H_1 into itself which satisfies

$$(Ks, s)_1 = (s, s')_0 = 0\,.$$

Denoting

$$g_j(x, \xi) = g_{y_j}\left(\int_\xi^x \sigma(\eta)\, d\eta,\ \sigma(x) - \sigma(\xi),\ \sigma(x) + \sigma(\xi) \right), \quad j = 0, 1, 2\,,$$

$$\bar{h}(x) = \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\, d\vartheta$$

we have

$$(\Phi\sigma)'\,(x) = \int_0^{2\pi} g_0(x, \xi)\, \sigma(x) + (g_1(x, \xi) + g_2(x, \xi))\, \sigma'(x)\, d\xi + \bar{h}(x)\,.$$

Thus

$$((\Phi\sigma)', \sigma')_0 = \int_0^{2\pi} \int_0^{2\pi} (g_1(x, \xi) + g_2(x, \xi))\, (\sigma'(x))^2 + g_0(x, \xi)\, \sigma(x)\, \sigma'(x)\, d\xi\, dx +$$

$$+ \int_0^{2\pi} \bar{h}'(\xi)\, \sigma'(\xi)\, d\xi \geqq 2\pi(\gamma_1 - \gamma_2)\, |\sigma'|_0^2 - 2\pi\, \gamma_0 |\sigma|_0\, |\sigma'|_0 - |\bar{h}'|_0\, |\sigma'|_0\,.$$

As $|\sigma|_0 \leqq |\sigma'|_0$, the preceding inequality yields

$$(\Phi\sigma, \sigma)_1 > 0$$

for all $\sigma \in H_1$, $|\sigma|_1 = R$ where $R = 1 + (2\pi(\gamma_1 - \gamma_2 - \gamma_0))^{-1}\, |\bar{h}'|_0$. Hence there do not exist $t \in [0, 1]$ and $\sigma \in H_1$, $|\sigma|_1 = R$ such that

$$\sigma + tK\Phi\sigma = 0\,.$$

Really, if there were such t and σ, then they should satisfy

$$0 = (\sigma + tK\Phi\sigma, \Phi\sigma)_1 = (\sigma, \Phi\sigma)_1 > 0 .$$

But this is a contradiction. Therefore the Leray-Schauder theorem implies that there is $\sigma_0 \in H_1$, $|\sigma_0|_1 < R$ satisfying

$$\sigma_0 + K\Phi\sigma_0 = 0 .$$

Let us set $s_0 = 2\lambda K\sigma_0$. Then $s_0 \in H_2$ and s_0 satisfies (2.1). In virtue of (0.8) and (0.12) we obtain $s_0 \in H_3$. Thus (a) is satisfied.

Secondly, we shall show that (b) is satisfied. Putting

$$\bar{g}_j(x, \xi) = g_{y_j}(s_0(x) - s_0(\xi), s_0'(x) - s_0'(\xi), s_0'(x) + s_0'(\xi)) ,$$

$j = 0, 1, 2$ we can write

$$V^\varepsilon \sigma(x) = |\varepsilon|^{-1} (\sigma'(x) - \sigma'(x - |\varepsilon| \lambda)) +$$
$$+ \frac{1}{2}\int_0^{2\pi} (\bar{g}_1(x, \xi) (\sigma'(x) - \sigma'(\xi)) + \bar{g}_2(x, \xi) (\sigma'(x) + \sigma'(\xi)) +$$
$$+ \bar{g}_0(x, \xi) (\sigma(x) - \sigma(\xi))) \,\mathrm{d}\xi .$$

Let us denote by $C^\infty_{2\pi}$ the space of infinitely differentiable 2π-periodic functions on R. Let $\eta \in C^\infty_{2\pi} \cap H_0$. Then

$$(V^\varepsilon\eta, -\eta''')_0 = |\varepsilon|^{-1} \left(|\eta''|_0^2 - \int_0^{2\pi} \eta''(x)\, \eta''(x - |\varepsilon| \lambda) \,\mathrm{d}x\right) +$$
$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} ((\bar{g}_1(x, \xi) + \bar{g}_2(x, \xi)) (\eta''(x))^2 + \bar{g}_0(x, \xi)\, \eta'(x)\, \eta''(x) \,\mathrm{d}\xi \,\mathrm{d}x +$$
$$+ \frac{1}{2}\int_0^{2\pi} \left(\int_0^{2\pi} (\bar{g}_{1x}(x, \xi) (\eta'(x) - \eta'(\xi)) + \bar{g}_{2x}(x, \xi) (\eta'(x) + \eta'(\xi)) +\right.$$
$$\left. + \bar{g}_{0x}(x, \xi) (\eta(x) + \eta(\xi))) \,\mathrm{d}\xi \right) \eta''(x) \,\mathrm{d}x .$$

As $|\eta|_0 \leqq |\eta'|_0 \leqq |\eta''|_0$ and

$$\int_0^{2\pi} \eta''(x)\, \eta''(x - |\varepsilon| \lambda) \,\mathrm{d}x \leqq |\eta''|_0^2$$

we have

(2.2$_1$)
$$(V^\varepsilon\eta, -\eta''')_0 \geqq \pi(\gamma_1 - \gamma_2 - \gamma_0) |\eta''|_0^2 - c_1|\eta'|_0 |\eta''|_0 \geqq$$
$$\geqq 2^{-1}\pi(\gamma_1 - \gamma_2 - \gamma_0) |\eta''|_0^2 - c_1^2(2\pi(\gamma_1 - \gamma_2 - \gamma_0))^{-1} |\eta'|_0^2 .$$

The constant c_1 does not depend on η. Similarly,

$$(V^\varepsilon \eta, \eta')_0 \geqq \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \gamma_1(\eta'(x) - \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\bar{g}_1(x, \xi) - \gamma_1)(\eta'(x) - \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \bar{g}_2(x, \xi)(\eta'(x) + \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \bar{g}_0(x, \xi)(\eta(x) - \eta(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi = I_1 + I_2 + I_3 + I_4\,.$$

Interchanging the variables x and ξ in I_2 and using the relations $\bar{g}_1(x, \xi) = \bar{g}_1(\xi, x)$ and $\bar{g}_1(x, \xi) \geqq \gamma_1$ we can write

$$2I_2 = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\bar{g}_1(x, \xi) - \gamma_1)(\eta'(x) - \eta'(\xi))^2\,\mathrm{d}x\,\mathrm{d}\xi \geqq 0\,.$$

Thus simple estimations of I_3 and I_4 yield

$$(V^\varepsilon \eta, \eta')_0 \geqq \pi\,\gamma |\eta'|_0^2 \tag{2.2$_2$}$$

where $\gamma = \gamma_1 - \gamma_2 - 2\gamma_0$. Let Λ be an operator defined by

$$\Lambda\eta = -\eta''' + c_2\eta'\,, \quad c_2 = c_1^2(2\pi^2\,\gamma(\gamma_1 - \gamma_0 - \gamma_2))^{-1}\,.$$

By (2.2) Λ satisfies

$$(V^\varepsilon \eta, \Lambda\eta)_0 \geqq \gamma_3 |\eta''|_0^2 = \gamma_3 |\eta|_2^2 \tag{2.3}$$

with $\gamma_3 = 2^{-1}\,\pi(\gamma_1 - \gamma_2 - \gamma_0)$. Let

$$(V^\varepsilon)^*\,\varphi(x) = |\varepsilon|^{-1}(-1)(\varphi'(x) - \varphi'(x + |\varepsilon|\,\lambda)) -$$

$$- \frac{1}{2}\int_0^{2\pi} (\bar{g}_1(x, \xi)(\varphi(x) - \varphi(\xi)))_x\,\mathrm{d}\xi - \frac{1}{2}\int_0^{2\pi} (\bar{g}_2(x, \xi)(\varphi(x) - \varphi(\xi)))_x\,\mathrm{d}\xi +$$

$$+ \frac{1}{2}\int_0^{2\pi} \bar{g}_0(x, \xi)(\varphi(x) - \varphi(\xi))\,\mathrm{d}\xi\,.$$

Then

$$(V^\varepsilon \eta, \varphi)_0 = (\eta, (V^\varepsilon)^*\,\varphi)_0 \tag{2.4}$$

for every $\eta, \varphi \in C_{2\pi}^{\infty} \cap H_0$. Using the negative norms (cf. [4], p. 165–167), we complete the proof. The negative norm $|\cdot|_{-k}$, k positive integer is defined by

$$|v|_{-k} = \sup \{|(v, w)_0| \, |w|_k^{-1};\ 0 \neq w \in H_k\} .$$

The completion of H_0 with respect to the norm $|\cdot|_{-k}$ will be denoted by H_{-k}. Applying Fourier series, we easily show that for every $\varphi \in C_{2\pi}^{\infty} \cap H_0$ there exists a unique $\eta \in C_{2\pi}^{\infty} \cap H_0$ such that $\Lambda\eta = \varphi$. By (2.3) and (2.4),

$$|\eta|_2 \, |(V^{\varepsilon})^* \varphi|_{-2} \geqq (\eta, (V^{\varepsilon})^* \varphi)_0 = (V^{\varepsilon}\eta, \Lambda\eta)_0 \geqq \gamma_3 |\eta|_2^2 .$$

Hence

$$(2.5) \qquad |(V^{\varepsilon})^* \varphi|_{-2} \geqq \gamma_3 |\eta|_2 .$$

By definition,

$$|\varphi|_{-1} = \sup \{|(\varphi, w)_0| \, |w|_1^{-1};\ 0 \neq w \in H_1\} =$$
$$= \sup \{|(-\eta''' + c_2\eta', w)_0| \, |w|_1^{-1};\ 0 \neq w \in H_1\} \leqq (1 + c_2) \, |\eta|_2 .$$

This inequality together with (2.5) yields

$$(2.6) \qquad |(V^{\varepsilon})^* \varphi|_{-2} \geqq \gamma_3 (1 + c_2)^{-1} \, |\varphi|_{-1}$$

for every $\varphi \in C_{2\pi}^{\infty} \cap H_0$. Finally, let $g \in H_1$. Let us put $Q = (V^{\varepsilon})^* (C_{2\pi}^{\infty} \cap H_0)$. To every $\psi \in Q$ we assign the value

$$l(\psi) = (\varphi, g)_0$$

where $\psi = (V^{\varepsilon})^* \varphi$. This is possible because by (2.6) the function φ is uniquely determined for every ψ. Using (2.6), we conclude

$$|l(\psi)| \leqq |\varphi|_{-1} \, |g|_1 \leqq (\gamma_3^{-1}(1 + c_2) \, |g|_1) \, |\psi|_{-2} .$$

Hence l is a linear functional on $Q \subset H_{-2}$. According to the Hahn-Banach theorem, there is a linear functional l' on H_{-2} such that l' is an extension of l and the norm of l' equals that of l. By Lax's theorem ([4], p. 167) there exists a unique $v \in H_2$ such that

$$l'(\psi) = (\psi, v)_0$$

and

$$(2.7) \qquad |v|_2 \leqq \gamma_3^{-1}(1 + c_2) \, |g|_1 .$$

Putting $\psi = (V^{\varepsilon})^* \varphi$ for $\varphi \in C_{2\pi}^{\infty} \cap H_0$, we have

$$l'(\psi) = (\varphi, g)_0 = ((V^{\varepsilon})^* \varphi, v)_0 = (\varphi, V^{\varepsilon}v)_0 ,$$

i.e. $(\varphi, g - V^\varepsilon v)_0 = 0$. As $g, V^\varepsilon v \in H_0$, the last equality yields $V^\varepsilon v = g$. This implies that $(V^\varepsilon)^{-1} \in [H_1, H_2]$ exists. By (2.7),

$$\|(V^\varepsilon)^{-1}\|_{[H_1,H_2]} \leqq \gamma_3^{-1}(1 + c_2).$$

Hence the condition (b) is satisfied. This completes the proof.

In the second part of this section we show that for every ε from a neighbourhood of 0 there is a solution $u \in U_\infty$ to the equation

$$u_{tt}(t, x) - u_{xx}(t, x) = \varepsilon(3u^2 u_t + h_\varepsilon(t, x)), \quad t \in R^+, \quad x \in R \tag{2.8}$$

satisfying the conditions (0.2) and (0.3). We shall suppose that the function h_ε fulfils (0.12) and that the function

$$\bar{h}(x) = \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\, d\vartheta$$

does not vanish identically. The existence of solutions follows from Theorem 1.2 if the next two conditions are satisfied.

(c) There is a function $s \in H_3$, $s \neq 0$ such that

$$s''(x) + (2\lambda)^{-1} \int_0^{2\pi} 3(s(x) - s(\xi))^2 (s'(x) - s'(\xi))\, d\xi + \bar{h}(x) = 0, \quad x \in R. \tag{2.9}$$

(d) The operator $V^\varepsilon \in [H_2, H_1]$ given by

$$V^\varepsilon \sigma(x) = |\varepsilon|^{-1} (\sigma'(x) - \sigma'(x - |\varepsilon| \lambda)) + \\ + \frac{3}{2} \int_0^{2\pi} (s(x) - s(\xi))^2 (\sigma'(x) - \sigma'(\xi))\, d\xi + \\ + 3 \int_0^{2\pi} (s(x) - s(\xi)) (s'(x) - s'(\xi)) (\sigma(x) - \sigma(\xi))\, d\xi, \quad x \in R$$

has an inverse $(V^\varepsilon)^{-1} \in [H_1, H_2]$ whose norm is bounded by a constant independent of ε.

The existence of solutions to (2.8), (0.2) and (0.3) was proved in [1] under the assumption that h_ε is a function π-antiperiodic in the variable x. The authors obtained this result as a by-product when investigating a system of two first order equations. The same theorems as in [1] have to be applied to complete the proofs of (c) and (d) which are indicated below. They can however be applied after simpler calculations and without the assumption of π-antiperiodicity of the function h_ε.

Firstly, we shall treat (c). Let L_p be the space of all 2π-periodic real functions s satisfying

$$\int_0^{2\pi} s(\xi)\, d\xi = 0 \quad \text{and} \quad \int_0^{2\pi} s^p(x)\, dx < \infty.$$

Let us denote by K the linear compact operator from $L_{4/3}$ into L_4 given by

$$K\,s(x) = \int_0^x s(\xi)\,\mathrm{d}\xi + (2\pi)^{-1}\int_0^{2\pi} \xi\,s(\xi)\,\mathrm{d}\xi\,, \quad x \in R\,.$$

As

$$\int_0^{2\pi}\left\{\int_0^{2\pi}(s(x) - s(\xi))^3\,\mathrm{d}\xi\right\} s(x)\,\mathrm{d}x \geqq 2\pi\int_0^{2\pi} s^4(x)\,\mathrm{d}x$$

we can use the theorem which was applied in the corresponding step in [1]. Thus there is $s \in L_4$ such that

$$s + (2\lambda)^{-1} K\left(\int_0^{2\pi}(s(\cdot) - s(\xi))^3\,\mathrm{d}\xi\right) + K^2\bar{h} = 0\,.$$

Differentiating this equation, we can show that $s \in H_3$. Clearly $s \neq 0$ and (2.9) is satisfied.

In the end we shall show how to treat (d). Let $g \in H_1$. Let us denote $W^\varepsilon = KV^\varepsilon$. Then

$$W^\varepsilon\,\sigma(x) = |\varepsilon|^{-1}\,(\sigma(x) - \sigma(x - |\varepsilon|\,\lambda)) +$$
$$+ \frac{3}{2}\int_0^{2\pi}(s(x) - s(\xi))^2\,(\sigma(x) - \sigma(\xi))\,\mathrm{d}\xi$$

and the equation $W^\varepsilon\sigma = Kg$ is equivalent to $V^\varepsilon\sigma = g$. Let us put $I = \frac{3}{2}\int_0^{2\pi} s^2(\xi)\,\mathrm{d}\xi$. Then we immediately verify

$$(W^\varepsilon\sigma, \sigma)_0 \geqq I|\sigma|_0^2\,,$$
$$((W^\varepsilon\sigma)', \sigma')_0 \geqq I|\sigma'|_0^2 - M_1|\sigma|_0\,|\sigma'|_0\,,$$
$$((W^\varepsilon\sigma)'', \sigma'')_0 \geqq I|\sigma''|_0^2 - M_2|\sigma'|_0\,|\sigma''|_0$$

for every $\sigma \in H_2$ with M_1 and M_2 independent of σ and ε. Using the Lax-Milgram theorem in the same way as in [1], we see that (d) is satisfied.

3. AUXILIARY ASSERTIONS

Lemma 3.1. *Let $\varepsilon \neq 0$ satisfy $0 < 2\pi + \varepsilon\lambda < T$. Let $u \in U_\infty$ and $s \in H_2$ satisfy*

(3.1) $$u(t, x) = Z\,s(t, x) + \frac{\varepsilon}{2}\int_0^t\int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)\,(\vartheta, \xi)\,\mathrm{d}\xi\,\mathrm{d}\vartheta\,, \quad t \in R^+\,, \quad x \in R$$

and

(3.2) $$u(t, x) = u(t + 2\pi + \varepsilon\lambda, x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

Then the pair of functions consisting of the restriction of the function u to $[0, T] \times R$ *and the function s satisfies* (1.1) *and* (1.2).

Proof. (3.1) implies that (1.1) holds. Thus only (1.2) has to be shown. Let us put $\omega = 2\pi + \varepsilon\lambda$. Inserting u from (3.1) into (3.2) and making use of the obvious relations

$$\int_{-x+t+\omega-\vartheta}^{x-t-\omega+\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi = 0\,,$$

$$\int_0^{t+\omega} \int_{x-t-\omega+\vartheta}^{x+t+\omega-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta = \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta + $$
$$+ \int_0^{\omega} \int_{x-t-\omega+\vartheta}^{x+t+\omega-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta\,,$$

we obtain

$$s(t + x + \omega) - s(t + x) + \frac{\varepsilon}{2}\int_0^{\omega}\int_0^{t+x} F_\varepsilon(u)(\vartheta, \xi + \omega - \vartheta)\, d\xi\, d\vartheta =$$
$$= s(t - x + \omega) - s(t - x) + \frac{\varepsilon}{2}\int_0^{\omega}\int_0^{t-x} F_\varepsilon(u)(\vartheta, \xi + \omega - \vartheta)\, d\xi\, d\vartheta$$

for every $t \in R^+$ and $x \in R$. From here (1.2) follows immediately.

Lemma 3.2. *Let* $\varepsilon \neq 0$ *satisfy* $0 < 2\pi + \varepsilon\lambda < T$. *Let* $u \in U_T$ *and* $s \in H_2$ *satisfy* (1.1) *and* (1.2). *Let us denote by* $\bar{u}$ *the function satisfying*

(3.3) $$\bar{u}(t, x) = u(t, x)\,, \quad t \in [0, 2\pi + \varepsilon\lambda)\,, \quad x \in R$$

and

(3.4) $$\bar{u}(t + 2\pi + \varepsilon\lambda, x) = \bar{u}(t, x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

Then $\bar{u} \in U_\infty$ *and*

(3.5) $$\bar{u}(t, x) = Z\, s(t, x) + \frac{\varepsilon}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi)\, d\xi\, d\vartheta$$

for every $t \in R^+$ *and* $x \in R$.

Proof. From (1.1) it follows that

$$u_t(t, x) = s'(t + x) - s'(t - x) + \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x + t - \vartheta)\, d\vartheta +$$
$$+ \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x - t + \vartheta)\, d\vartheta\,,$$

$$u_x(t, x) = s'(t + x) + s'(t - x) + \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x + t - \vartheta)\, d\vartheta -$$
$$- \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x - t + \vartheta)\, d\vartheta$$

for $t \in [0, T)$ and $x \in R$. Let $\omega = 2\pi + \varepsilon\lambda$. Using (1.2) we obtain

$$u_t(t + \omega, x) - u_t(t, x) = \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x + t - \vartheta) - F_\varepsilon(u)(\vartheta, x + t - \vartheta)\} \, d\vartheta$$

$$+ \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x - t + \vartheta) - F_\varepsilon(u)(\vartheta, x - t + \vartheta)\} \, d\vartheta,$$

$$u_x(t + \omega, x) - u_x(t, x) = \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x + t - \vartheta) - F_\varepsilon(u)(\vartheta, x + t - \vartheta)\} \, d\vartheta -$$

$$- \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x - t + \vartheta) - F_\varepsilon(u)(\vartheta, x - t + \vartheta)\} \, d\vartheta$$

for $t \in [0, T - \omega)$ and $x \in R$. In virtue of (0.2) we have

$$|u(t, x)| \leqq \int_0^{|x|} |u_x(t, \xi)| \, d\xi.$$

By Gronwall's lemma we deduce from the last three relations:

$$u(t, x) = u(t + \omega, x)$$

for $t \in [0, T - \omega)$ and $x \in R$. This shows that there is a function $\bar{u} \in U_\infty$ satisfying (3.3) and (3.4). Induction will be used to prove (3.5). Let $n \geqq 1$ be an integer such that (3.5) holds for $t \in [0, n\omega]$. Let $\tau \in (n\omega, (n + 1)\omega]$. Then we have

$$\bar{u}(\tau, x) = \bar{u}(\tau - \omega, x) = Z\, \dot{s}(\tau - \omega, x) + \frac{\varepsilon}{2}\int_0^{\tau - \omega}\int_{x - \tau + \omega + \vartheta}^{x + \tau - \omega - \vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi) \, d\xi \, d\vartheta =$$

$$Z\, s(\tau, x) + \frac{\varepsilon}{2}\int_0^{\tau}\int_{x - \tau + \vartheta}^{x + \tau - \vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi) \, d\xi \, d\vartheta + \Xi(\tau, x)$$

where

$$\Xi(\tau, x) = Z\, s(\tau - \omega, x) - Z\, s(\tau, x) - \frac{\varepsilon}{2}\int_0^{\omega}\int_{x - \tau + \vartheta}^{x + \tau - \vartheta} F_\varepsilon(u)(\vartheta, \xi) \, d\xi \, d\vartheta.$$

By (1.2), $\Xi(\tau, x) = 0$. Thus (3.5) holds for $t \in [0, (n + 1)\omega]$. This completes the proof.

The next two lemmas are modifications of the implicit function theorem and are closely related to Theorems 2.3 and 2.4 in [1].

Lemma 3.3. *Let X, Y be Banach spaces, $\bar{m}$, $\bar{\varepsilon}$ positive numbers and $x_0 \in X$. Let a family of mappings ${}^\varepsilon G : X \to Y$, $\varepsilon \in (0, \bar{\varepsilon}]$ satisfy the following assumptions:*

(i) *The mapping ${}^\varepsilon G : X \to Y$ is continuous and its derivative ${}^\varepsilon G' : X \to [X, Y]$ exists for every $\varepsilon \in (0, \bar{\varepsilon}]$.*

(ii) $\lim_{\varrho \to 0_+} \sup \{\|{}^\varepsilon G'(x) - {}^\varepsilon G'(x_0)\|_{[X,Y]};\ \varepsilon \in (0, \bar{\varepsilon}],\ \|x - x_0\|_X < \varrho\} < 1/\bar{m}$.

(iii) $\lim_{\varepsilon \to 0+} \|{}^{\varepsilon}G(x_0)\|_Y = 0$.

(iv) *For every* $\varepsilon \in (0, \bar{\varepsilon}]$ *there exists* $T^{\varepsilon} \in [Y, X]$ *satisfying* ${}^{\varepsilon}G'(x_0)\, T^{\varepsilon} = I_Y$, $\|T^{\varepsilon}\|_{[Y,X]} \leqq \bar{m}$.

Then there exists $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon}_1]$ *there is* $x^{\varepsilon} \in X$ *satisfying* ${}^{\varepsilon}G(x^{\varepsilon}) = 0$. *Moreover*, $\lim_{\varepsilon \to 0} x^{\varepsilon} = x_0$.

Proof. Let us choose $\alpha \in (0, 1)$ and $\varrho > 0$ such that

$$\sup \{\|{}^{\varepsilon}G'(x) - {}^{\varepsilon}G'(x_0)\|_{[X,Y]};\ x \in B(x_0, \varrho),\ \varepsilon \in (0, \bar{\varepsilon}]\} < \alpha/\bar{m}\,.$$

Let $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ be such that $\varepsilon \in (0, \bar{\varepsilon}_1]$ implies

$$\|{}^{\varepsilon}G(x_0)\|_Y \leqq (1 - \alpha)\, \varrho/\bar{m}\,.$$

Let us put $x_0^{\varepsilon} = x_0$ and $x_{n+1}^{\varepsilon} = x_n^{\varepsilon} - T^{\varepsilon}\, {}^{\varepsilon}G(x_n^{\varepsilon})$ for $\varepsilon \in (0, \bar{\varepsilon}_1]$, $n = 0, 1, \ldots$. We easily obtain

$$\|x_{k+1}^{\varepsilon} - x_k^{\varepsilon}\|_X \leqq \bar{m}\|{}^{\varepsilon}G(x_k^{\varepsilon})\|_Y \tag{3.6}$$

for $k = 0, 1, \ldots$. If for an integer $n \geqq 1$ we have $x_k^{\varepsilon} \in B(x_0, \varrho)$, $k = 1, 2, \ldots, n$, then by [2] (relation 8.6.2),

$$\begin{aligned} \|{}^{\varepsilon}G(x_k^{\varepsilon})\|_Y &= \|{}^{\varepsilon}G(x_k^{\varepsilon}) - {}^{\varepsilon}G(x_{k-1}^{\varepsilon}) - {}^{\varepsilon}G'(x_0)\,(x_k^{\varepsilon} - x_{k-1}^{\varepsilon})\|_Y \leqq \\ &\leqq \|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X \sup \{\|{}^{\varepsilon}G'(x) - {}^{\varepsilon}G'(x_0)\|_{[X,Y]};\ \varepsilon \in (0, \bar{\varepsilon}_1],\ x \in B(x_0, \varrho)\} \leqq \\ &\leqq \alpha\|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X/\bar{m}\,. \end{aligned} \tag{3.7}$$

This estimate together with (3.6) implies

$$\|x_{k+1}^{\varepsilon} - x_k^{\varepsilon}\|_X \leqq \alpha\|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X \tag{3.8}$$

for $k = 1, 2, \ldots, n$. Using (3.6) for $k = 0$ and (3.8), we obtain

$$\|x_{k+1}^{\varepsilon} - x_0\|_X \leqq \bar{m}\|{}^{\varepsilon}G(x_0)\|_Y/(1 - \alpha)\,. \tag{3.9}$$

Thus $x_n^{\varepsilon} \in B(x_0, \varrho)$ for all $\varepsilon \in (0, \bar{\varepsilon}_1]$ and all positive integers n. By (3.8) we can put $x^{\varepsilon} = \lim_{n \to \infty} x_n^{\varepsilon}$. ${}^{\varepsilon}G(x^{\varepsilon}) = 0$ and $\lim_{\varepsilon \to 0} x^{\varepsilon} = x_0$ are consequences of (3.7) and (3.9) respectively.

Lemma 3.4. *Let all the assumptions of Lemma* 3.3 *be satisfied. Let*

$$T^{\varepsilon}\, {}^{\varepsilon}G'(x_0) = I_X \quad \textit{for} \quad \varepsilon \in (0, \bar{\varepsilon}]\,.$$