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EXTENSION OF A HOMEOMORPHISM OF A TOPOLOGICAL CIRCUMFERENCE

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In the present paper we prove that any homeomorphic mapping h of a topological circumference $T \subset \mathbf{S}$ into $\mathbf{S}$ ($\mathbf{S}$ being the closed Gaussian plane) can be extended to a homeomorphic mapping of the whole $\mathbf{S}$ onto $\mathbf{S}$. The only more advanced results used from the topology of the plane are the Jordan theorem and the theorem on the θ-curves.

The result just formulated is of crucial importance for the topology of the plane as well as for its applications e.g. in the theory of functions of a complex variable. It implies immediately a.o. the theorem on accessibility of all points of the boundary $\partial\Omega$ of a Jordan region Ω from the region as well as other analogous theorems (concerning the outer topological properties of topological circumferences) useful in the theory of conformal mappings and other fields.

However, the application of the above theorem in the mentioned direction in elementary courses of the theory of functions is essentially hindered by the fact that the usual proofs given in the literature are based a.o. on the theorem on accessibility of boundary points of a Jordan region Ω from Ω (cf. e.g. [1], pp. 374–381). The present paper offers an almost elementary proof of the theorem, showing how the proof from [1] can be modified not only to remove the above mentioned drawback but also to avoid the use of other theorems, not widely known and difficult to prove.

Let us first introduce the necessary notation, definitions and theorems. The *closed Gaussian plane* is denoted by $\mathbf{S}$, the *open Gaussian plane* by $\mathbf{E}$. The *boundary* of a set $M \subset \mathbf{S}$ will be denoted by ∂M, the *closure* by $\overline{M}$. If $\emptyset \neq M \subset \mathbf{E}$, then

$$\operatorname{diam} M = \sup_{z',z'' \in M} |z' - z''| ;$$

besides, we put $\operatorname{diam} \emptyset = 0$. If $\operatorname{diam} M < \infty$, then M is said to be a *bounded* set. If $\emptyset \neq M_i \subset \mathbf{E}$ for $i = 1, 2$, then

$$\operatorname{dist}(M_1, M_2) = \inf_{z' \in M_1, z'' \in M_2} |z' - z''| .$$

If for instance $M_1 = \{a\}$ is a one-point set, then we write dist (a, M_2) instead of dist $(\{a\}, M_2)$.

If $M \subset \mathbf{S}$ is either a closed or an open set and if $p, q \in \mathbf{S} - M$ are two points, we say *M separates the points p, q in $\mathbf{S}$*, if p, q belong to different components of the set $\mathbf{S} - M$. (Cf. [2], p. 108.)

Let us recall the Janiszewski theorem (see e.g. [2], p. 172): *If the sets M_1, M_2 $(\subset \mathbf{S})$ are either both closed or both open, if p, q are two points from $\mathbf{S} - (M_1 \cup M)$, if neither M_1 nor M_2 separates the points p, q in $\mathbf{S}$ and the intersection $M_1 \cap M_2$ is connected, then the set $M_1 \cup M_2$ does not separate the points p, q in $\mathbf{S}$, either.*

For any $\varepsilon \in (0, \infty)$ and $z \in \mathbf{E}$ the set

$$U(z, \varepsilon) = \{z' \in \mathbf{E};\ |z' - z| < \varepsilon\}$$

is called an *ε-neighbourhood* (briefly: a *neighbourhood*) of the point z; for each $M \subset \mathbf{E}$ we put

$$U(M, \varepsilon) = \bigcup_{z \in M} U(z, \varepsilon)\,.$$

A *segment* with *end-points* $a \neq b$ from $\mathbf{E}$ is the set

$$u(a, b) = \{z;\ z = a + t(b - a),\ t \in \langle 0, 1 \rangle\}\ ;$$

the corresponding *open segment* is defined to be

$$o(a, b) = u(a, b) - \{a, b\}\ ^{1})\,.$$

The points $z \in o(a, b)$ are called the *interior points* of the segment $u(a, b)$.

An *arc* is a homeomorphic image of a segment. The images of the end-points of the segment in the corresponding homeomorphic mapping are called the *end-points* of the arc. If L is an arc with end-points a, b, then

$$\tilde{L} = L - \{a, b\}$$

defines the corresponding *open arc*. The term *topological circumference* stands for a homeomorphic image of a circumference. Under a *polygonal line* we shall mean here an arc or a topological circumference which is the union of a finite number of segments.

A region (i.e., a connected open set) whose boundary is a topological circumference is called a *Jordan region*. We shall use the term *polygon* for a bounded Jordan region whose boundary is a polygonal line, or for the closure of such a region; the actual meaning will be always clear from the context. Any maximal segment contained in $\partial\Omega$ will be called a *side* of the polygon Ω while each end-point of any one of its sides will be called its *vertex*.

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[1]) $\{a, b\}$ is the two-point set consisting of the points a, b.

The *Jordan theorem* (see [2], p. 108) asserts that for every topological circumference $T \subset \mathbf{S}$, the set $\mathbf{S} - T$ is the union of two disjoint Jordan regions whose common boundary is T. If $T \subset \mathbf{E}$ then one of these regions contains the point ∞ (we shall denote it by Ext T) while the other one is bounded (and will be denoted by Int T).

The theorem on θ-curves will be applied in the following form (see [2], p. 184): *If Ω is a Jordan region, L an arc with end-points a, b in $\partial\Omega$ and satisfying $\tilde{L} \subset \Omega$, if M_1, M_2 are arcs with the same end-points a, b for which $M_1 \cup M_2 = \partial\Omega$, then $\Omega - L$ is the union of two disjoint Jordan regions Ω_1, Ω_2 which fulfil $\partial\Omega_i = M_i \cup L$ for $i = 1, 2$.*

The concept of a net (cf. [1], p. 374) will be of importance for us. Let Ω be a bounded Jordan region, let T_k $(2 \leqq k \leqq n,\ n \geqq 1)$ be arcs with end-points a_k, b_k. The sequence

$$(1) \qquad T_1 = \partial\Omega\,,\ T_2, \ldots, T_n$$

is called a *(n-term) net* in $\bar{\Omega}$ if

$$(2) \qquad \tilde{T}_k \subset \Omega$$

and

$$(3) \qquad T_k \cap (T_1 \cup \ldots \cup T_{k-1}) = \{a_k, b_k\}$$

for each $k = 2, \ldots, n$.

The following assertion is easily proved (by induction with respect to n) in virtue of the theorem on θ-curves (cf. [1], p. 374):

Lemma 1. *Let (1) be a net in $\bar{\Omega}$. Then the set $\Omega - \bigcup_{k=2}^{n} T_k$ has exactly n components, say $\Omega_1, \ldots, \Omega_n$, and*

$$(4) \qquad \bigcup_{k=1}^{n} \partial\Omega_k = \bigcup_{k=1}^{n} T_k\,.$$

The boundary of the only unbounded component of the set $\mathbf{S} - \bigcup_{k=2}^{n} T_k$ is $\partial\Omega$. Each point $z \in \partial\Omega - \bigcup_{k=2}^{n} T_k$ belongs to the closure of exactly one component of the set $\Omega - \bigcup_{k=2}^{n} T_k$.

Another concept needed in the sequel is the *linear accessibility*: Let Ω be a region, $a \in \partial\Omega$. We say that the point a is linearly accessible from Ω, if there exists a segment $u(a, b)$ such that $u(a, b) - \{a\} \subset \Omega$.

The following lemma is easily verified (cf. [2], p. 527):

Lemma 2. *If Ω is a bounded region then the set Z of all points from $\partial\Omega$ which are linearly accessible from Ω is dense in $\partial\Omega$.*

Let us recall another fact which we believe to be well-known:

Lemma 3. *Each point of the boundary of a polygon is linearly accessible from it.*

The following result will be of importance as well:

Lemma 4. *Let* (1) *be a net in* $\bar{\Omega}$ *with* $T_2, \ldots, T_n$ *polygonal lines. Then every point* $z \in \bigcup_{k=2}^{n} T_k \cap \Omega$ *is linearly accessible from each component of the set* $\Omega - \bigcup_{k=2}^{n} T_k$ *to whose boundary it belongs.*

Proof. If $z \in \bigcup_{k=2}^{n} T_k \cap \Omega$, then for every sufficiently small neighbourhood $U(z, \varepsilon)$ of the point z the set $U(z, \varepsilon) - \bigcup_{k=2}^{n} T_k$ is the union of a finite number of open circular sectors and each component of the set $\Omega - \bigcup_{k=2}^{n} T_k$ whose boundary contains the point z includes also one of these sectors. Obviously, the point z is linearly accessible from any one of these sectors.

The following assertion plays an important role in the sequel:

Lemma 5. *Let* $\varepsilon > 0$. *Let us denote by* $\mathscr{S}$ *the family of all squares*

$$C_{m,n} = \{z; (m-1)\,\varepsilon \leqq \operatorname{Re} z \leqq m\varepsilon, (n-1)\,\varepsilon \leqq \operatorname{Im} z \leqq n\varepsilon\}$$ [2])

where m, n *are integers. Let* Ω *be a bounded Jordan region,* M *a straigt line intersecting* Ω. *Then there exists a finite family* $\mathscr{L}$ *of segments which satisfies the following four conditions:*

(4_1) $L \in \mathscr{L} \Rightarrow L \subset M$;

(4_2) $L \in \mathscr{L},\ L = u(a, b) \Rightarrow o(a, b) \subset \Omega,\ a, b \in \partial\Omega$;

(4_3) $L_1, L_2 \in \mathscr{L},\ L_1 \neq L_2 \Rightarrow L_1 \cap \tilde{L}_2 = \emptyset$;

(4_4) *none of the components of the set* $\Omega - \bigcup_{L \in \mathscr{L}} L$ *intersects simultaneously two squares* $C', C'' \in \mathscr{S}$ *which lie in different components of the set* $\mathbf{E} - M$.

Proof. Let $\mathscr{L}^*$ be the family of all segments which are closures of the components of the set $M \cap \Omega$. The family $\mathscr{L}^*$ is either finite or denumerable and satisfies the implications (4_1)–(4_3) with $\mathscr{L}$ replaced by $\mathscr{L}^*$.

If the family $\mathscr{L}^*$ is finite put $\mathscr{L} = \mathscr{L}^*$. Then the implication (4_4) is obvious as well: every connected set included in Ω and intersecting both components of the set $\mathbf{E} - M$ intersects also the set $M \cap \Omega = \bigcup_{L \in \mathscr{L}} \tilde{L}$.

It remains to prove the assertion under the assumption that the family $\mathscr{L}^*$ is infinite. In this case let us arrange the segments of the family $\mathscr{L}^*$ in a sequence $L_1, \ldots, L_n, \ldots$ with mutually different terms. Let the endpoints of the segment L_n

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[2]) Re z, Im z stand for the real and imaginary parts of a number $z \in \mathbf{E}$, respectively.

be a_n, b_n. With regard to the fact that the open segments $\tilde{L}_n$ are disjoint and contained in the bounded part $\Omega \cap M$ of the straight line M we have

$$(5) \qquad |a_n - b_n| \to 0 .$$

Let us choose an arc X of the topological circumference $\partial\Omega$ which does not intersect M, and let Y be the arc of the topological circumference $\partial\Omega$ with the same endpoints as X, which satisfies $X \cup Y = \partial\Omega$. Let φ be a homeomorphic mapping of the interval $\langle 0, 1 \rangle$ onto Y. Denoting

$$(6) \qquad \alpha_n = \varphi_{-1}(a_n), \quad \beta_n = \varphi_{-1}(b_n)$$ [3])

we can assume that $\alpha_n < \beta_n$ for all n since the denotation of the endpoints of the segments L_n is immaterial in the sequel. We have $\beta_n - \alpha_n \to 0$ by (5) and in virtue of the uniform continuity of the function φ_{-1}. Hence, with regard to the uniform continuity of φ,

$$(7) \qquad \operatorname{diam} \varphi(\langle \alpha_n, \beta_n \rangle) \to 0 .$$

The theorem on θ-curves implies (for every n)

$$(8) \qquad \Omega - L_n = \Omega_{n1} \cup \Omega_{n2}$$

where Ω_{n1}, Ω_{n2} are disjoint Jordan regions with

$$(9) \qquad \partial\Omega_{n1} = L_n \cup \varphi(\langle \alpha_n, \beta_n \rangle) .$$

Since

$$\operatorname{diam} \bar{\Omega}_{n1} = \operatorname{diam} \partial\Omega_{n1} \leqq \operatorname{diam} L_n + \operatorname{diam} \varphi(\langle \alpha_n, \beta_n \rangle) ,$$

we conclude by (5) and (7) that

$$(10) \qquad \operatorname{diam} \bar{\Omega}_{n1} \to 0 .$$

Let Z_1, Z_2 be open half-planes determined by the straight line M. Let us denote by W_i $(i = 1, 2)$ the union of all squares $C_{m,n} \in \mathscr{S}$ which are contained in Z_i and intersect Ω. The sets W_i (being finite unions of compact sets $C_{m,n}$) are compact. If one of them is empty then our assertion is trivial as the empty family may be taken for L. Therefore, let $W_1 \neq \emptyset \neq W_2$; then

$$(11) \qquad \operatorname{dist}(\Omega \cap M, W_i) > 0 \quad \text{for} \quad i = 1, 2 .$$

Since $\bar{\Omega}_{n1} \cap (\Omega \cap M) \neq \emptyset$ for every n, by (10) and (11) there exists a positive integer p such that

$$(12) \qquad \bar{\Omega}_{n1} \cap (W_1 \cup W_2) = \emptyset \quad \text{for all} \quad n > p .$$

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[3]) The symbol φ_{-1} denotes, of course, the inverse mapping to φ.

Put $\mathscr{L} = \{L_1, \ldots, L_p\}$ and assume that there exists a component K of the set $\Omega - \bigcup_{L \in \mathscr{L}} L = \Omega - \bigcup_{n=1}^{p} L_n$ which intersects both W_1 and W_2. Let us choose points $z_i \in K \cap W_i$ $(i = 1, 2)$. As K is a region, there exists an arc N contained in K, with the end-points z_1, z_2. Further, there exists an $\eta > 0$ such that $\overline{U(N, \eta)} \subset K$. With regard to (10) there exists a $q > p$ such that diam $\bar{\Omega}_{n1} < \eta$ for all $n > q$. This implies

(13) $$\bar{\Omega}_{n1} \cap N = \emptyset \quad \text{for all} \quad n > q\,.$$

It is easy to see that the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n = (\mathbf{S} - \Omega) \cup \overline{\bigcup_{n>q} L_n}$$

is a continuum[4]). This continuum does not separate the points z_1, z_2 in $\mathbf{S}$, since its complement

$$\mathbf{S} - ((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) = \Omega - \bigcup_{n>q} L_n \supset \Omega - \bigcup_{n>q} \bar{\Omega}_{n1}$$

contains the connected set N. If the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n$ did not separate the points z_1, z_2 in $\mathbf{S}$, then the same would hold according to the Janiszewski theorem also for the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{\infty} L_n = ((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) \cup ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n)\,,$$

since the intersection

$$((\mathbf{S} - \Omega) \cup \bigcup_{n>q} L_n) \cap ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n) = \mathbf{S} - \Omega = \overline{\text{Ext}\,\partial\Omega}$$

is connected. However, this is not the case, since $M \subset (\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{\infty} L_n$ and M separates the points z_1, z_2 in $\mathbf{S}$ [5]). Consequently,

(14) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{q} L_n$ separates the points z_1, z_2 in $\mathbf{S}$.

On the other hand, we know that

(15) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{p} L_n$ does not separate the points z_1, z_2 in $\mathbf{S}$ (since the connected set $K \subset \Omega - \bigcup_{n=1}^{p} L_n = \mathbf{S} - ((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{p} L_n)$ contains these points).

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[4]) That is, a compact connected set.

[5]) Let us note that (12) implies that $z_i \in \Omega - \bigcup_{n>p} \bar{\Omega}_{n1}$, hence also $z_i \in \Omega - \bigcup_{n=1}^{\infty} L_n$ for $i = 1, 2$.

Let r be the least positive integer such that

(16′) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r} L_n$ separates the points z_1, z_2 in $\mathbf{S}$.

Then $p < r \leqq q$ and

(16″) the continuum $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n$ does not separate the points z_1, z_2 in $\mathbf{S}$.

The set $\mathbf{S} - \bar{\Omega}_{r1} = \operatorname{Ext} \partial\Omega_{r1}$ is connected and according to (12) contains both z_1 and z_2. Consequently, $\bar{\Omega}_{r1}$ does not separate these points in $\mathbf{S}$. Further, we have

$$((\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n) \cap \bar{\Omega}_{r1} = ((\mathbf{S} - \Omega) \cap \bar{\Omega}_{r1}) \cup (\bigcup_{n=1}^{r-1} L_n \cap \bar{\Omega}_{r1}) = \varphi(\langle \alpha_r, \beta_r \rangle) \cup \Lambda,$$

where Λ is the union of all segments L_n $(n = 1, \ldots, r - 1)$ satisfying $\tilde{L}_n \subset \Omega_{r1}$. The end-points of such segments belong to $(\mathbf{S} - \Omega) \cap \bar{\Omega}_{r1} = \varphi(\langle \alpha_r, \beta_r \rangle)$ and therefore the union $\varphi(\langle \alpha_r, \beta_r \rangle) \cup \Lambda$ is connected. The Janiszewski theorem implies that the set

$$(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r-1} L_n \cup \bar{\Omega}_{r1}$$

does not separate the points z_1, z_2 in $\mathbf{S}$, either. The less does so the smaller set $(\mathbf{S} - \Omega) \cup \bigcup_{n=1}^{r} L_n$; however, this contradicts (16′).

This completes the proof of Lemma 5.

Lemma 6. *Let Ω be a bounded Jordan region and let $\varepsilon > 0$. Then:*

1. *There exist segments $L_1, \ldots, L_q$ (with $q \geqq 0$) such that*

(17) $$\partial\Omega, L_1, \ldots, L_q \quad \text{is a net in} \quad \bar{\Omega};$$

(18) *every component of the set $\Omega - \bigcup_{n=1}^{q} L_n$ has a diameter less than ε.*

2. *If Ω is a polygon and the set Z is dense in $\partial\Omega$, then it is possible to choose the segments $L_1, \ldots, L_q$ so that, in adition to (17) and (18), the following two conditions hold:*

(19) *every point $z \in \partial\Omega$ belong to at most one set L_n $(1 \leqq n \leqq q)$;*

(20) $$L_n \cap \partial\Omega \subset Z \quad \text{for all} \quad n = 1, \ldots, q.$$

Proof. 1. Let Ω be a bounded Jordan region, $\varepsilon > 0$. Let us denote by $\mathscr{S}$ the family of all squares

$$C_{m,n} = \{z; \tfrac{1}{3}(m - 1)\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}m\varepsilon, \tfrac{1}{3}(n - 1)\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}n\varepsilon\}$$

where m, n are integers.

Let $\mathscr{M}_1$ be the family of all straight lines

(21) $$\{z; \operatorname{Im} z = \tfrac{1}{6}(2k - 1)\varepsilon\}$$

where k is an integer satisfying

(22) $$\{z \in \Omega;\ \operatorname{Im} z \geqq \tfrac{1}{3}k\varepsilon\} \neq \emptyset \neq \{z \in \Omega;\ \operatorname{Im} z \leqq \tfrac{1}{3}(k-1)\,\varepsilon\}\,,$$

and let $\mathcal{M}_2$ be the family of all straight lines

(23) $$\{z;\ \operatorname{Re} z = \tfrac{1}{6}(2j-1)\,\varepsilon\}$$

where j is an integer satisfying

(24) $$\{z \in \Omega;\ \operatorname{Re} z \geqq \tfrac{1}{3}j\varepsilon\} \neq \emptyset \neq \{z \in \Omega;\ \operatorname{Re} z \leqq \tfrac{1}{3}(j-1)\,\varepsilon\}\,.$$

Put $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. It follows from the boundedness of the set Ω that the family $\mathcal{M}$ is finite.

According to Lemma 5 we have to every straight line $M \in \mathcal{M}$ a finite system $\mathcal{L}(M)$ of segments satisfying

(25_1) $L \in \mathcal{L}(M) \Rightarrow L \subset M$;

(25_2) $L \in \mathcal{L}(M),\ L = u(a, b) \Rightarrow o(a, b) \subset \Omega,\ a, b \in \partial\Omega$;

(25_3) $L', L'' \in \mathcal{L}(M),\ L' \neq L'' \Rightarrow L' \cap \tilde{L}'' = \emptyset$;

(25_4) none of the components of the set $\Omega - \bigcup_{L \in \mathcal{L}(M)} L$ intersects simultaneously two squares $C', C'' \in \mathcal{S}$ which belong to different components of the set $\mathbf{E} - M$.

Let K be one of the components of the set

(26) $$\Omega - \bigcup_{M \in \mathcal{M}} \bigcup_{L \in \mathcal{L}(M)} L\,.$$

Then there exists a maximal integer k such that $\{z \in K;\ \operatorname{Im} z \leqq \tfrac{1}{3}(k-1)\,\varepsilon\} = \emptyset$. It follows from its definition that $K \cap C_{m_1,k} \neq \emptyset$ for a certain integer m_1. If it were $\{z \in K;\ \operatorname{Im} z \geqq \tfrac{1}{3}(k+1)\,\varepsilon\} \neq \emptyset$, then $K \cap C_{m_2,n} \neq \emptyset$ for some two numbers m_2 and $n \geqq k+2$. The straight line $M = \{z; \operatorname{Im} z = \tfrac{1}{6}(2k+1)\,\varepsilon\}$ would belong to $\mathcal{M}_1$. However, this is not possible, since K is part of a component K' of the set $\Omega - \bigcup_{L \in \mathcal{L}(M)} L$ and according to (25_4) K' does not intersect both squares $C_{m_1,k}$, $C_{m_2,n}$, as they belong (in virtue of the inequality $n \geqq k+2$) to different components of the set $\mathbf{E} - M$. Thus we have proved that

(27) $$K \subset \{z;\ \tfrac{1}{3}(k-1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}(k+1)\,\varepsilon\}$$

for k suitably chosen; we can prove similarly that

(27′) $$K \subset \{z;\ \tfrac{1}{3}(j-1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}(j+1)\,\varepsilon\}$$

for j suitably chosen. The last two inclusions imply immediately

$$\operatorname{diam} K \leqq \operatorname{diam} \{z; \tfrac{1}{3}(j - 1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}(j + 1)\,\varepsilon\,,$$

$$\tfrac{1}{3}(k - 1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}(k + 1)\,\varepsilon\} = \varepsilon\,\sqrt{\tfrac{8}{9}} < \varepsilon\,.$$

Consequently,

(28) if K is a component of the set (26), then $\operatorname{diam} K < \varepsilon$.

Let us now arrange all segments of the family $\bigcup_{M \in \mathcal{M}_1} \mathcal{L}(M)$ in a sequence $L_1, \ldots, L_p$ with mutually different terms (so that $p \geqq 0$). With regard to $(25_1)-(25_3)$ and to the fact that the straight lines from $\mathcal{M}_1$ are disjoint we obtain that

(29) the sequence $\partial\Omega, L_1, \ldots, L_p$ is a net in $\bar{\Omega}$.

On the basis of the families $\mathcal{L}(M)$, where $M \in \mathcal{M}_2$, we form new systems $\mathcal{L}^*(M)$ in the following way: For every $L \in \bigcup_{M \in \mathcal{M}_2} \mathcal{L}(M)$, let

$$L \cap (\partial\Omega \cup \bigcup_{n=1}^{p} L_n)\,^{6}) = \{a_0^L, \ldots, a_{r(L)}^L\} \tag{30}$$

where

$$\operatorname{Im} a_0^L < \operatorname{Im} a_1^L < \ldots < \operatorname{Im} a_{r(L)}^L\,. \tag{31}$$

Let the family $\mathcal{L}^*(M)$ contain exactly all the segments

$$u(a_0^L, a_1^L),\; u(a_1^L, a_2^L),\; \ldots,\; u(a_{r(L)-1}^L, a_{r(L)}^L)$$

where $L \in \mathcal{L}(M)$.

If all segments of the family $\bigcup_{M \in \mathcal{M}_2} \mathcal{L}^*(M)$ are arranged in a sequence $L_{p+1}, \ldots, L_q$ with mutually different terms, then $q \geqq p$ and it is evident that (17) holds. Further, obviously

$$\bigcup_{n=p+1}^{q} L_n = \bigcup_{M \in \mathcal{M}_2} \; \bigcup_{L \in \mathcal{L}(M)} L\,,$$

hence also

$$\bigcup_{n=1}^{q} L_n = \bigcup_{M \in \mathcal{M}} \; \bigcup_{L \in \mathcal{L}(M)} L\,. \tag{32}$$

In virtue of (28), (18) is proved.

2. Now let Ω be a polygon and Z a set dense in $\partial\Omega$. Without any loss of generality we may assume that

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[6]) This set is finite, since $L \cap \partial\Omega$ is a two-point set according to (25_2) and L is ortogonal to each segment $L_1, \ldots, L_p$.

(33) any vertex of the polygon Ω belongs neither to a line $\{z; \operatorname{Re} z = \frac{1}{6}l\varepsilon\}$ nor to a line $\{z; \operatorname{Im} z = \frac{1}{6}l\varepsilon\}$ with l odd,

(34) no point of the form $\frac{1}{6}(l_1 + il_2)\,\varepsilon$ with l_1, l_2 odd belongs to $\partial\Omega$,

since, if necessary, Ω may be suitably translated without affecting essentially our argument.[7])

Let the symbols $C_{m,n}$, $\mathscr{S}$, $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}$ have the same meaning as in Part 1 of the proof. According to (33) the set $M \cap \partial\Omega$ is finite for every $M \in \mathscr{M}$; let

(35) $$M \cap \partial\Omega = \{a_1^M, a_2^M, \ldots, a_{r(M)}^M\}$$

where

(36_1) $$\operatorname{Re} a_1^M < \operatorname{Re} a_2^M < \ldots < \operatorname{Re} a_{r(M)}^M \quad \text{if} \quad M \in \mathscr{M}_1\,,$$

(36_2) $$\operatorname{Im} a_1^M < \operatorname{Im} a_2^M < \ldots < \operatorname{Im} a_{r(M)}^M \quad \text{if} \quad M \in \mathscr{M}_2\,.$$

Further, let us denote by X_k^M the side of the polygon Ω which contains the point a_k^M; since the point a_k^M is not a vertex of the polygon Ω there is exactly one such side, it is not parallel to M, and the point a_k^M is its interior point. Let $P^+(M)$, $P^-(M)$ be the components of the set $\mathbf{E} - M$. Let $W^+(M)$ and $W^-(M)$ be the unions of all squares $C_{m,n} \in \mathscr{S}$ which intersect Ω and satisfy $C_{m,n} \subset P^+(M)$ and $C_{m,n} \subset P^-(M)$, respectively.

We shall prove that for every sufficiently small $\delta > 0$ the following five conditions hold:

(37_1) for any $M \in \mathscr{M}$, there is no vertex of the polygon Ω in the strip $\overline{U(M, \delta)}$;

(37_2) $\overline{U(M, \delta)} \cap (W^+(M) \cup W^-(M)) = \emptyset$ for each $M \in \mathscr{M}$;

(37_3) if $M_1 \neq M_2$ and either $M_1, M_2 \in \mathscr{M}_1$ or $M_1, M_2 \in \mathscr{M}_2$, then $U(M_1, \delta) \cap$ $\cap\, U(M_2, \delta) = \emptyset$;

(37_4) if $M_1 \in \mathscr{M}_1$, $M_2 \in \mathscr{M}_2$, then $U(M_1, \delta) \cap U(M_2, \delta) \cap \partial\Omega = \emptyset$;

(37_5) given arbitrary points $b_k^M \in X_k^M \cap U(M, \delta)$ (where $M \in \mathscr{M}$, $1 \leqq k \leqq r(M)$), then the angle between the segment $u(b_{k-1}^M, b_k^M)$ (where $1 < k \leqq r(M)$) and the line M is non-negative and less than $\frac{1}{4}\pi$.

With regard to (33) we find that (37_1) holds for all sufficiently small $\delta > 0$. As $W^+(M) \cup W^-(M)$ is a compact set disjoint with M, (37_2) holds for all sufficiently small $\delta > 0$ as well. If $0 < \delta < \frac{1}{6}\varepsilon$, then ($37_3$) holds, too. Further, (34) implies that

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[7]) It is sufficient to use a translation vector v which is parallel neither to the real and imaginary axes nor to any side of the polygon Ω, and whose magnitude is less than (i) the distance of every vertex of the polygon Ω from the union of all straight lines $\{z; \operatorname{Re} z = l\varepsilon/6\}$, $\{z; \operatorname{Im} z = l\varepsilon/6\}$ with l odd, not containing this vertex, (ii) the distance of every point $(l_1 + il_2)\,\varepsilon/6$ with l_1, l_2 odd from any side of the polygon Ω not containing the point in question.

also the condition (37_4) is fulfilled for all $\delta > 0$ sufficiently small. If the points b_k^M are sufficiently close to the points a_k^M, then all conditions concerning angles will be fulfilled as required in (37_5). Taking into account that the segments X_k^M are not parallel to the straight lines M we see that the points $b_k^M \in X_k^M \cap U(M, \delta)$ will be arbitrarily close to the points $a_k^M (\in X_k^M \cap M)$ provided $\delta > 0$ is sufficiently small.

Let us now fix a number $\delta > 0$ so that $(37_1)-(37_5)$ hold. Denote

$$(40_1) \qquad A_0^M = \{z \in M;\ \operatorname{Re} z < \operatorname{Re} a_1^M\}, \quad A_{r(M)}^M = \{z \in M;\ \operatorname{Re} z > \operatorname{Re} a_{r(M)}^M\}$$

for $M \in \mathcal{M}_1$,

$$(40_2) \qquad A_0^M = \{z \in M;\ \operatorname{Im} z < \operatorname{Im} a_1^M\}, \quad A_{r(M)}^M = \{z \in M;\ \operatorname{Im} z > \operatorname{Im} a_{r(M)}^M\}$$

for $M \in \mathcal{M}_2$, and let

$$(40_3) \qquad A_k^M = o(a_k^M, a_{k+1}^M)$$

for $M \in \mathcal{M}$ and $1 \leqq k < r(M)$.

Further, let G_k^M (where $M \in \mathcal{M}$, $0 \leqq k \leqq r(M)$) be the component of the set $U(M, \delta) - \bigcup_{k=1}^{r(M)} X_k^M$ containing the set A_k^M. Let us note that each set G_k^M is convex (being the intersection of three or four half-planes).

First, let us show that the number $r(M)$ is even for all $M \in \mathcal{M}$ and

$$(41) \qquad A_0^M \cup A_2^M \cup \ldots \cup A_{r(M)}^M \subset \mathbf{S} - \bar{\Omega}, \quad A_1^M \cup \ldots \cup A_{r(M)-1}^M \subset \Omega.$$

In virtue of its connectedness and of the condition $A_k^M \cap \partial\Omega = \emptyset$ each set A_k^M is contained either in $\mathbf{S} - \bar{\Omega}$ or in Ω. Since obviously $A_0^M \cup A_{r(M)}^M \subset \mathbf{S} - \bar{\Omega}$, (41) will be proved (together with the evenness of $r(M)$), if we show that

(42) for each $k = 1, \ldots, r(M)$ one of the sets A_{k-1}^M, A_k^M is contained in $\mathbf{S} - \bar{\Omega}$, the other one in Ω.

To prove (42), let us choose (with $k = 1, \ldots, r(M)$ fixed) a neighbourhood U of the point a_k^M so small that $U \cap \partial\Omega = U \cap X_k^M$. Then the set $U - \partial\Omega$ is the union of two open semicircles, one of them being contained in $\mathbf{S} - \bar{\Omega}$, the other one in Ω according to the Jordan theorem. At the same time it is apparent that one of the sets A_{k-1}^M, A_k^M intersects one of these semicircles while the other set intersects the other one. This implies immediately (42) in virtue of the fact that each set A_j^M is a subset either of $\mathbf{S} - \bar{\Omega}$ or of Ω.

If we show that

$$(43) \qquad G_k^M \cap \partial\Omega = \emptyset \quad \text{for every} \quad M \in \mathcal{M} \quad \text{and every} \quad k = 0, \ldots, r(M),$$

then (41) together with the connectedness of the sets G_k^M will imply

$$(44) \qquad G_0^M \cup G_2^M \cup \ldots \cup G_{r(M)}^M \subset \mathbf{S} - \bar{\Omega}, \quad G_1^M \cup \ldots \cup G_{r(M)-1}^M \subset \Omega.$$

Let us suppose that there is a point $z_0 \in G_k^M \cap \partial\Omega$. By (37_1), the point z_0 is an interior point of a certain side X of the polygon Ω, while the end-points of X do not belong to $\overline{G_k^M}$. Hence X intersects ∂G_k^M in two points $z_1 \neq z_2$. Since obviously $X \neq X_j^M$ for $j = 1, \ldots, r(M)$, we have $z_1, z_2 \notin \bigcup_{j=1}^{r(M)} X_j^M$ and, consequently, the points z_1, z_2 belong to the straight lines whose closures form $\partial U(M, \delta)$, and lie in different components of the $\mathbf{E} - M$. This implies that $X \cap M \neq \emptyset$; it is easy to see, with regard to the convexity of the sets G_k^M, that $o(z_1, z_2) \subset G_k^M$ so that the point of intersection of X and M belongs to $A_k^M = G_k^M \cap M$. However, this is a contradiction since $A_k^M \cap \partial\Omega = \emptyset$.

This completes the proof of (43), and thus also of (44).

Since the set Z is dense in $\partial\Omega$, there exist points

(45) $$b_k^M \in \dot{X}_k^M \cap Z \cap U(M, \delta) \,(M \in \mathcal{M},\ 1 \leqq k \leqq r(M)).$$

For $M \in \mathcal{M}_1$ let us put

(46_1) $$B_0^M = \{z;\ \operatorname{Re} z < \operatorname{Re} b_1^M,\ \operatorname{Im} z = \operatorname{Im} b_1^M\},$$

$$B_{r(M)}^M = \{z;\ \operatorname{Re} z > \operatorname{Re} b_{r(M)}^M,\ \operatorname{Im} z = \operatorname{Im} b_{r(M)}^M\},$$

for $M \in \mathcal{M}_2$ let

(46_2) $$B_0^M = \{z;\ \operatorname{Re} z = \operatorname{Re} b_1^M,\ \operatorname{Im} z < \operatorname{Im} b_1^M\},$$

$$B_{r(M)}^M = \{z;\ \operatorname{Re} z = \operatorname{Re} b_{r(M)}^M,\ \operatorname{Im} z > \operatorname{Im} b_{r(M)}^M\};$$

for every $M \in \mathcal{M}$ let

(46_3) $$B_k^M = o(b_k^M, b_{k+1}^M) \quad (1 \leqq k < r(M))$$

and

(46_4) $$B^M = \bigcup_{r=0}^{r(M)} \bar{B}_k^M.$$

As evidently $B_k^M \subset G_k^M$, we have, by (44),

(47) $$B_0^M \cup B_2^M \cup \ldots \cup B_{r(M)}^M \subset \mathbf{S} - \bar{\Omega}, \quad B_1^M \cup \ldots \cup B_{r(M)-1}^M \subset \Omega.$$

The set B^M is a topological circumference, hence the set $\mathbf{S} - B^M$ has exactly two components in virtue of the Jordan theorem. Since $B^M \subset \overline{U(M, \delta)}$, one of the components contains the half-plane $P^+(M) - \overline{U(M, \delta)}$, hence (by (37_2)) also the set $W^+(M)$, while the other one contains the half-plane $P^-(M) - \overline{U(M, \delta)}$, and hence also the set $W^-(M)$. This implies that every connected set intersecting both $W^+(M)$ and $W^-(M)$ intersects B^M as well, so that

(48) none of the components of the set $\Omega - B^M$ intersects both $W^+(M)$ and $W^-(M)$.

If for any $M \in \mathscr{M}$, the symbol $\mathscr{L}(M)$ stands for the system of all segments $\bar{B}_1^M$, $\bar{B}_3^M$, $\bar{B}_{r(M)-1}^M$, then the system is obviously disjoint and, by (47),

$$\Omega \cap B^M = \bigcup_{L \in \mathscr{L}(M)} \tilde{L}, \tag{49}$$

so that, by (48),

(50) none of the components of the set $\Omega - \bigcup_{L \in \mathscr{L}(M)} L$ intersects (for any $M \in \mathscr{M}$) both sets $W^+(M)$, $W^-(M)$.

Hence it follows similarly as in Part 1 of the proof that

(51) to every component K of the set $\Omega - \bigcup_{M \in \mathscr{M}} \bigcup_{L \in \mathscr{L}(M)} L$ there exist integers j, k such that
$$K \subset \{z; \tfrac{1}{3}(j-1)\,\varepsilon \leqq \operatorname{Re} z \leqq \tfrac{1}{3}(j+1)\,\varepsilon,\ \tfrac{1}{3}(k-1)\,\varepsilon \leqq \operatorname{Im} z \leqq \tfrac{1}{3}(k+1)\,\varepsilon\},$$

and consequently

(52) $\operatorname{diam} K < \varepsilon$ for every component K of the set $\Omega - \bigcup_{M \in \mathscr{M}} \bigcup_{L \in \mathscr{L}(M)} L$.

Now let us arrange all segments of the system $\bigcup_{M \in \mathscr{M}} \mathscr{L}(M)$ in a sequence $L_1, \ldots, L_p$ with mutually different terms. Similarly as in Part 1 of the proof

(53) the sequence $\partial\Omega$, $L_1, \ldots, L_p$ is a net in $\bar{\Omega}$.

If $L \in \mathscr{L}(M)$ with $M \in \mathscr{M}_2$, then the set

$$L \cap (\partial\Omega \cup \bigcup_{n=1}^{p} L_n) \tag{54}$$

is finite since $L \cap \partial\Omega$ is a two-point set and the segments L, L_n $(1 \leqq n \leqq p)$ are not parallel (as, by (37_5), the angle between the segment L and the imaginary axis as well as the angle between any one of the segments L_n and the real axis is less than $\frac{1}{4}\pi$).

Consequently, for every $M \in \mathscr{M}_2$ a system $\mathscr{L}^*(M)$ can be defined analogously as in Part 1 of the proof; we arrange the segments of the system $\bigcup_{M \in \mathscr{M}_2} \mathscr{L}^*(M)$ in a sequence $L_{p+1}, \ldots, L_q$ with mutually different terms as before.

Again, (17) and (18) hold by an analogous argument. Two different segments L', L'' belonging either both to $\bigcup_{M \in \mathscr{M}_1} \mathscr{L}(M)$ or both to $\bigcup_{M \in \mathscr{M}_2} \mathscr{L}(M)$ do not intersect (which is a consequence of the construction of the systems $\mathscr{L}(M)$ – cf. (37_3)). If $L' \in \mathscr{L}(M_1)$, $L'' \in \mathscr{L}(M_2)$ with $M_1 \in \mathscr{M}_1$, $M_2 \in \mathscr{M}_2$, then $L' \cap L''$ is part of $U(M_2, \delta) \cap \cap U(M_2, \delta)$ which is a set disjoint with $\partial\Omega$ according to (37_4).Hence it follows that the condition (19) is satisfied. The validity of the condition (20) follows from the fact that every point from $L_n \cap \partial\Omega$ with $1 \leqq n \leqq q$ is one of the points b_k^M.

This completes the proof of Part 2 of Lemma 6.

Definition. Let Ω and Ω^* be bounded Jordan regions. Suppose that

$$\partial\Omega = T_1, T_2, \ldots, T_n \quad \text{is a net in} \quad \bar{\Omega}, \tag{55}$$

$$\partial\Omega^* = T_1^*, T_2^*, \ldots, T_n^* \quad \text{is a net in} \quad \bar{\Omega}^*. \tag{55*}$$

A homeomorphic mapping h of the set $\bigcup_{n=1}^{q} T_n$ onto the set $\bigcup_{n=1}^{q} T_n^*$ is said to be *regular*[8]) if it is possible to number the components $\Omega_1, \ldots, \Omega_q$ of the set $\Omega - \bigcup_{n=1}^{q} T_n$ and the components $\Omega_1^*, \ldots, \Omega_q^*$ of the set $\Omega^* - \bigcup_{n=1}^{q} T_n^*$ in such a way that

$$h(\partial\Omega_n) = \partial\Omega_n^* \quad \text{for} \quad n = 1, \ldots, q. \tag{56}$$

Lemma 7. *Let Ω be a bounded Jordan region, Ω^* a polygon. Let f be a homeomorphic mapping of $\partial\Omega$ onto $\partial\Omega^*$. Then:*

1. *Under the assumption (55) there exist polygonal lines $T_2^*, \ldots, T_q^*$ such that (55*) holds and the mapping f can be extended to a regular homeomorphic mapping F of the set $\bigcup_{n=1}^{q} T_n$ onto the set $\bigcup_{n=1}^{q} T_n^*$.*

2. *Let Z denote the set of all points from $\partial\Omega$ which are linearly accessible from Ω; let $Z^* = f(Z)$, $f^* = f_{-1}$. Let us further assume that (55*) is satisfied,*

$$\bigcup_{n=2}^{q} T_n^* \cap \partial\Omega^* \subset Z^*, \tag{57}$$

and

(58) *there are no two arcs T_m^*, T_n^*, $2 \leqq m < n \leqq q$, with a common point in $\partial\Omega^*$.*

Then there exist polygonal lines $T_2, \ldots, T_q$ such that (55) is satisfied and that the mapping f^* can be extended to a regular homeomorphic F^* mapping of the set $\bigcup_{n=1}^{q} T_n^*$ onto the set $\bigcup_{n=1}^{q} T_n$.

Proof. Let the assumptions of Part 1 of Lemma 7 be satisfied; we shall proceed by induction with respect to q.

For $q = 1$ it is sufficient to notice that every homeomorphic mapping of $\partial\Omega$ onto $\partial\Omega^*$ is regular.

Let the assertion analogous to the above one hold for each $(q - 1)$-term net. If (55) is satisfied, then

$$\partial\Omega = T_1, \ldots, T_{q-1} \tag{59}$$

[8]) Cf. [1], p. 376.

is a $(q-1)$-term net so that there exist polygonal lines $T_2^*, \ldots, T_{q-1}^*$ such that $\partial\Omega^* = T_1^*, T_2^*, \ldots, T_{q-1}^*$ is a net in $\bar{\Omega}^*$ and the mapping f can be extended to a regular homeomorphic mapping g of the set $\bigcup_{n=1}^{q-1} T_n$ onto $\bigcup_{n=1}^{q-1} T_n^*$. This means that numbering suitably the components $G_1, \ldots, G_{q-1}$ of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ and the components $G_1^*, \ldots, G_{q-1}^*$ of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$ we have

$$g(\partial G_n) = \partial G_n^* \quad \text{for} \quad n = 1, \ldots, q-1\,. \tag{60}$$

According to the definition of a net, T_q is an arc contained in $\bar{\Omega}$ with its end-points a, b in $\bigcup_{n=1}^{q-1} T_n$ while $\mathring{T}_q \cap \bigcup_{n=1}^{q-1} T_n = \emptyset$. This implies that $\mathring{T}_q$ is part of a certain component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$. Without any loss of generality we may assume that this component is G_{q-1}. Then $a, b \in \partial G_{q-1}$. Since G_{q-1}^* is a polygon, the points $g(a), g(b) \in \partial G_{q-1}^*$ are linearly accessible from G_{q-1}^* (see Lemma 3); consequently, there exists an arc T_q^* with end-points $g(a), g(b)$ which is a polygonal line and satisfies $\mathring{T}_q^* \subset G_{q-1}^*$. The topological circumference ∂G_{q-1} is the union of two arcs M_1, M_2 with end-points a, b. Denoting $M_i^* = g(M_i)$ for $i = 1, 2$ we conclude that M_i^* are two arcs with end-points $g(a), g(b)$ whose union is ∂G_{q-1}^*. In virtue of the theorem on the θ-curves we have

$$G_{q-1} - T_q = \Omega_{q-1} \cup \Omega_q\,, \quad G_{q-1}^* - T_q^* = \Omega_{q-1}^* \cup \Omega_q^*\,, \tag{61}$$

where Ω_{q-1}, Ω_q as well as $\Omega_{q-1}^*, \Omega_q^*$ are disjoint Jordan regions with boundaries

$$\partial\Omega_{q-1} = T_q \cup M_1\,, \quad \partial\Omega_q = T_q \cup M_2\,, \quad \partial\Omega_{q-1}^* = T_q^* \cup M_1^*\,, \tag{62}$$
$$\partial\Omega_q^* = T_q^* \cup M_2^*\,,$$

respectively. Let h be a homeomorphic mapping of the arc T_q onto the arc T_q^* satisfying $h(a) = g(a)$, $h(b) = g(b)$. Then we may put

$$F = \begin{cases} g & \text{in} \quad \bigcup_{n=1}^{q-1} T_n\,, \\ h & \text{in} \quad T_q\,, \end{cases} \tag{63}$$

and F is evidently a homeomorphic mapping of $\bigcup_{n=1}^{q} T_n$ onto $\bigcup_{n=1}^{q} T_n^*$ which is an extension of f.

It is easy to see that the sets $\Omega - \bigcup_{n=1}^{q} T_n$ and $\Omega^* - \bigcup_{n=1}^{q} T_n^*$ have the components

$$\Omega_1 = G_1, \ldots, \Omega_{q-2} = G_{q-2}, \Omega_{q-1}, \Omega_q$$

and

$$\Omega_1^* = G_1^*, \ldots, \Omega_{q-2}^* = G_{q-2}^*, \Omega_{q-1}^*, \Omega_q^*\,,$$

respectively. With respect to (60), (62) and (63) we have evidently

$$F(\partial\Omega_n) = \partial\Omega_n^* \quad \text{for} \quad n = 1, \dots, q$$

so that the mapping F is regular.

2. Now let the assumptions of Part 2 of Lemma 7 be satisfied. Again we proceed by induction with respect to q. The assertion for $q = 1$ is evident; let the assertion analogous to the above one hold for any $(q-1)$-term net.

The induction hypothesis together with (55*), (57) and (58) implies that there exist polygonal lines $T_2, \dots, T_{q-1}$ such that $\partial\Omega = T_1, T_2, \dots, T_{q-1}$ is a net in $\bar{\Omega}$ and that the mapping f^* can be extended to a regular homeomorphic mapping g^* of the set $\bigcup_{n=1}^{q-1} T_n^*$ onto the set $\bigcup_{n=1}^{q-1} T_n$. This means that the components $G_1^*, \dots, G_{q-1}^*$ of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$ and the components $G_1, \dots, G_{q-1}$ of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ can be numbered in such a way that

$$g^*(\partial G_n^*) = \partial G_n \quad \text{for} \quad n = 1, \dots, q-1 \,. \tag{64}$$

The arc T_q^* let have end-points a^*, b^*. By (55*), $\mathring{T}_q^*$ is contained in a certain component of the set $\Omega^* - \bigcup_{n=1}^{q-1} T_n^*$; we may assume that the numbering is chosen so that $\mathring{T}_q^* \subset G_{q-1}^*$. Then also $a^*, b^* \in \partial G_{q-1}^*$. Let us denote further $a = g^*(a^*)$, $b = g^*(b^*)$; then $a, b \in \partial G_{q-1}$.

If $a^* \in \partial\Omega^*$ then $a^* \in Z^*$ according to (57) so that the point $a = g^*(a^*) = f_{-1}(a^*) \in Z$ is linearly accessible from Ω. According to (58) we have $a^* \notin \bigcup_{n=2}^{q-1} T_n^*$; consequently $a \notin \bigcup_{n=2}^{q-1} T_n$ and there exists a segment $u(a, a_1)$ satisfying $u(a, a_1) - \{a\} \subset \Omega - \bigcup_{n=1}^{q-1} T_n$. The connected set $u(a, a_1) - \{a\}$ is part of a certain component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$ and the point a belongs to its closure. However, in virtue of Lemma 1 the point $a \in \partial\Omega - \bigcup_{n=2}^{q-1} T_n$ belongs to the closure of only one component of the set $\Omega - \bigcup_{n=1}^{q-1} T_n$. Hence $u(a, a_1) - \{a\} \subset G_{q-1}$. If $a^* \in \bigcup_{n=1}^{q-1} T_n^* \cap \Omega^*$ then $a \in \bigcup_{n=1}^{q-1} T_n \cap \Omega$ and a similar segment $u(a, a_1)$ exists by Lemma 4. The existence of a point b_1 such that $u(b, b_1) - \{b\} \subset G_{q-1}$ is shown similarly.

Now it follows easily that there exists an arc T_q which is a polygonal line with end-points a, b and satisfying $\mathring{T}_q \subset G_{q-1}$.

If h^* is a homeomorphic mapping of the arc T_q^* onto T_q with $h^*(a^*) = a$, $h^*(b^*) = b$, then

$$F^* = \begin{cases} g^* & \text{in} \quad \bigcup_{n=1}^{q-1} T_n^* \,, \\ h^* & \text{in} \quad T_q^* \end{cases}$$

is the required regular homeomorphic extension of the mapping f^*. (The proof is quite analogous to that of a similar assertion for the mapping F in the proof of Part 1 of the lemma.)

Lemma 8. *Denote*

$$(65) \qquad Q = \{z;\ |\text{Re}\, z| < 2,\ |\text{Im}\, z| < 2\},\ Q_1 = \{z;\ |\text{Re}\, z| < 1,\ |\text{Im}\, z| < 1\}\,.$$

Let $-2 < a < b < 2$, *let* M_1, M_2 *be two arcs with end-points* a, b *and such that* $T = M_1 \cup M_2$ *is a topological circumference contained in* Q. *Further, let* $0 \in \text{Int}\, T$,

$$(66) \qquad i \max \{\text{Im}\, z;\ z \in T,\ \text{Re}\, z = 0\} \in M_1\,,$$

and

$$(67) \qquad (\tilde{M}_1 \cup \tilde{M}_2) \cap (\langle -2, a\rangle \cup \langle b, 2\rangle) = \emptyset\,.$$

Let f *be a homeomorphic mapping of the set* T *onto* ∂Q_1 *satisfying*

$$(68) \qquad f(a) = -1\,,\quad f(b) = 1\,,$$

$$(69) \qquad f(M_1) = \{z \in \partial Q_1;\ \text{Im}\, z \geqq 0\}\,,\quad f(M_2) = \{z \in \partial Q_1;\ \text{Im}\, z \leqq 0\}\,.$$

Then there exists a homeomorphic mapping F *of the set* $\mathbf{S}$ *onto itself satisfying*

$$(70) \qquad F \mid T = f\,,\quad F \mid (\mathbf{S} - Q) = \text{Id}\ ^{9)}\,.$$

Proof. Let us denote

$$(71_1) \qquad T_1 = T_1^* = \partial Q\,,$$

$$(71_2) \qquad T_2 = \langle -2, a\rangle \cup M_1 \cup \langle b, 2\rangle\,,$$

$$T_2^* = \langle -2, -1\rangle \cup \{z \in \partial Q_1;\ \text{Im}\, z \geqq 0\} \cup \langle 1, 2\rangle\,,$$

$$(71_3) \qquad T_3 = M_2\,,\quad T_3^* = \{z \in \partial Q_1;\ \text{Im}\, z \leqq 0\}\,.$$

It is easy to verify that the sequences T_1, T_2, T_3 and T_1^*, T_2^*, T_3^* are nets in $\bar{Q}$. Moreover, it is evident that the mapping f_1 defined in $\bigcup_{n=1}^{3} T_n$ by

$$(72) \qquad f_1(z) = \begin{cases} f(z) & \text{for} \quad z \in T\,, \\ z & \text{for} \quad z \in \partial Q\,, \\ [z - 2(a+1)]/(a+2) & \text{for} \quad z \in \langle -2, a\rangle\,, \\ [z + 2(1-b)]/(2-b) & \text{for} \quad z \in \langle b, 2\rangle \end{cases}$$

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[9]) Identical mapping; the symbol | is used for parcial mapping.

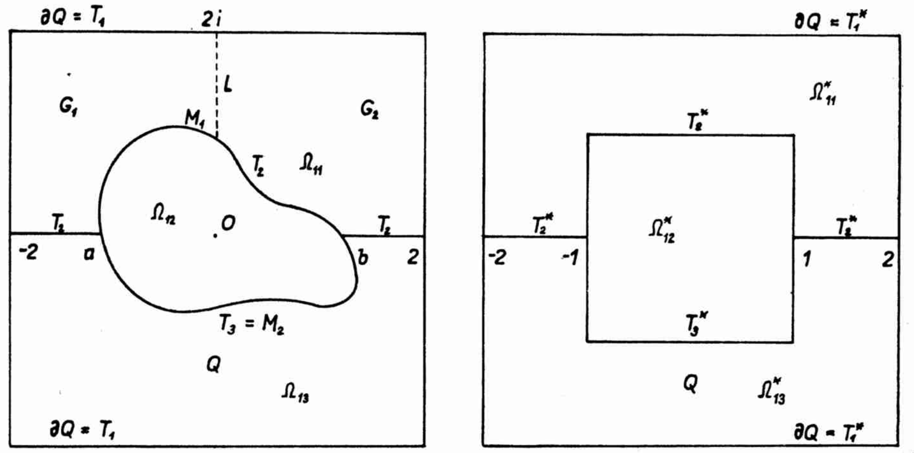
is a homeomorphic mapping of $\bigcup_{n=1}^{3} T_n$ onto $\bigcup_{n=1}^{3} T_n^*$ which is an extension of the mapping f.

Let us show that f_1 is regular. We have

$$Q - T_2 = \Omega_1 \cup \Omega_2 \tag{73}$$

in virtue of the theorem on the θ-curves, where Ω_1, Ω_2 are disjoint Jordan regions with boundaries

$$\partial\Omega_1 = T_2 \cup \{z \in \partial Q;\ \operatorname{Im} z \geqq 0\}\,, \quad \partial\Omega_2 = T_2 \cup \{z \in \partial Q;\ \operatorname{Im} z \leqq 0\}\,. \tag{74}$$



Further, we have $\tilde{T}_3 = \tilde{M}_2 \subset \Omega_1 \cup \Omega_2$ so that either $\tilde{M}_2 \subset \Omega_1$ or $\tilde{M}_2 \subset \Omega_2$. Let us denote by L the segment with end-points $2i$, $i \max\{\operatorname{Im} z;\ z \in T,\ \operatorname{Re} z = 0\}$. Then

$$\Omega_1 - L = G_1 \cup G_2 \tag{75}$$

where G_1, G_2 are disjoint Jordan regions satisfying

$$a \in \bar{G}_1 - \bar{G}_2\,, \quad b \in \bar{G}_2 - \bar{G}_1\,. \tag{76}$$

If it were $\tilde{M}_2 \subset \Omega_1$, then with regard to (75) and (76) necessarily

$$\tilde{M}_2 \cap G_1 \neq \emptyset \neq \tilde{M}_2 \cap G_2 \tag{77}$$

which would imply $\tilde{M}_2 \cap \partial G_1 \neq \emptyset$ as well. However, in virtue of the equality $\partial G_1 \cap \Omega_1 = \tilde{L}$ (which is an easy consequence of the theorem on the θ-curves) and the inclusion $\tilde{M}_2 \subset \Omega_1$, this would yield $\tilde{M}_2 \cap \tilde{L} \neq \emptyset$ which contradicts the assumption (66).

Consequently,

$$\tilde{T}_3 = \tilde{M}_2 \subset \Omega_2 \tag{78}$$

and the theorem on the θ-curves together with (73) yields

$$Q - \bigcup_{n=1}^{3} T_n = \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \tag{79}$$

where Ω_{1j} ($j = 1, 2, 3$) are the components of the sets on the left-hand side satisfying

$$\partial\Omega_{11} = T_2 \cup \{z \in \partial Q;\ \operatorname{Im} z \geqq 0\}\,, \tag{80_1}$$

$$\partial\Omega_{12} = T\,, \tag{80_2}$$

$$\partial\Omega_{13} = \langle -2, a\rangle \cup T_3 \cup \langle b, 2\rangle \cup \{z \in \partial Q;\ \operatorname{Im} z \leqq 0\}\,. \tag{80_3}$$

We obtain similarly

$$Q - \bigcup_{n=1}^{3} T_n^* = \Omega_{11}^* \cup \Omega_{12}^* \cup \Omega_{13}^* \tag{81}$$

where the sets on the right-hand side are the components of the set on the left-hand side, and

$$\partial\Omega_{11}^* = T_2^* \cup \{z \in \partial Q;\ \operatorname{Im} z \geqq 0\}\,, \tag{82_1}$$

$$\partial\Omega_{12}^* = \partial Q_1\,, \tag{82_2}$$

$$\partial\Omega_{13}^* = \langle -2, -1\rangle \cup T_3^* \cup \langle 1, 2\rangle \cup \{z \in \partial Q;\ \operatorname{Im} z \leqq 0\}\,. \tag{82_3}$$

The relations (79), (80_1)–(80_3), (81), (82_1)–(82_3) imply

$$f_1(\partial\Omega_{1j}) = \partial\Omega_{1j}^* \quad \text{for} \quad j = 1, 2, 3\,, \tag{83}$$

i.e., the regularity of the mapping f_1.

Let us note that diam $Q = \sqrt{32} < 6$ and the more so,

$$\operatorname{diam} \Omega_{1j} < 6\,, \quad \operatorname{diam} \Omega_{1j}^* < 6 \quad \text{for} \quad j = 1, 2, 3\,. \tag{84}$$

Put

$$q(1) = 3\,. \tag{85}$$

Let us assume that for a positive integer k we have defined nets $T_1, \ldots, T_{q(k)}$ and $T_1^*, \ldots, T_{q(k)}^*$ in $\bar{Q}$, that Ω_{kj} and Ω_{kj}^* with $j = 1, \ldots, q(k)$ are the components of the sets $Q - \bigcup_{n=1}^{q(k)} T_n$ and $Q - \bigcup_{n=1}^{q(k)} T_n^*$, respectively, that f_k is a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k)} T_n$ onto the set $\bigcup_{n=1}^{q(k)} T_n^*$ which is an extension of the mapping f_1, and

$$T_1^*, \ldots, T_{q(k)}^* \quad \text{are polygonal lines}\,, \tag{86}$$

$$\operatorname{diam} \Omega_{kj} < 6/k\,, \quad \operatorname{diam} \Omega_{kj}^* < 6/k \quad \text{for} \quad j = 1, \ldots, q(k)\,, \tag{87}$$

$$f_k(\partial\Omega_{kj}) = \partial\Omega_{kj}^* \quad \text{for} \quad j = 1, \ldots, q(k)\,. \tag{88}$$

We see immediately that the above conditions hold for $k = 1$.

In virtue of Lemma 6, Part 1, to every $j = 1, \ldots, q(k)$ there exist segments $L_{j1}, \ldots$ $\ldots, L_{jp(j)}$ such that

$$(89_1) \qquad \partial\Omega_{kj} = L_{j0}, L_{j1}, \ldots, L_{jp(j)} \quad \text{is a net in} \quad \bar{\Omega}_{kj}$$

and

(89_2) every component of the set $\Omega_{kj} - \bigcup_{i=1}^{p(j)} L_{ji}$ has a diameter less than $6/(k + 1)$.

In virtue of Lemma 7, Part 1, there exist polygonal lines $L^*_{j1}, \ldots, L^*_{jp(j)}$ such that

$$(90) \qquad \partial\Omega^*_{kj} = L^*_{j0}, L^*_{j1}, \ldots, L^*_{jp(j)} \quad \text{is a net in} \quad \bar{\Omega}^*_{kj},$$

and a regular homeomorphic mapping Φ_j of the set $\bigcup_{i=0}^{p(j)} L_{ji}$ onto the set $\bigcup_{i=0}^{p(j)} L^*_{ji}$ which is an extension of the mapping $f_k \mid \partial\Omega_{kj}$.

Let us put $r = q(k) + p(1) + \ldots + p(q(k))$, let

$$T_{q(k)+1} = L_{11}, \ldots, T_{q(k)+p(1)} = L_{1p(1)},$$

$$T_{q(k)+p(1)+1} = L_{21}, \ldots, T_{q(k)+p(1)+p(2)} = L_{2p(2)}, \ldots, T_r = L_{q(k)p(q(k))},$$

and similarly

$$T^*_{q(k)+1} = L^*_{11}, \ldots, T^*_r = L^*_{q(k)p(q(k))}.$$

Then it is evident that $T_1, \ldots, T_r$ and $T^*_1, \ldots, T^*_r$ are nets in $\bar{Q}$ and the mapping

$$\Phi = \Phi_j \quad \text{in} \quad \bigcup_{i=0}^{p(j)} L_{ji} \quad (j = 1, \ldots, q(k))$$

is a regular homeomorphic mapping of $\bigcup_{n=1}^{r} T_n$ onto $\bigcup_{n=1}^{r} T^*_n$ which is an extension of the mapping f_k (and thus also of f_1). Numbering suitably the components $X_1, \ldots, X_r$ and $X^*_1, \ldots, X^*_r$ of the sets $Q - \bigcup_{n=1}^{r} T_n$ and $Q - \bigcup_{n=1}^{r} T^*_n$, respectively, we have

$$(91) \qquad \Phi(\partial X_n) = \partial X^*_n \quad \text{for} \quad n = 1, \ldots, r;$$

besides, each of the sets X_n is a component of a certain set $\Omega_{kj} - \bigcup_{i=1}^{p(j)} L_{ji}$ so that

$$(92) \qquad \operatorname{diam} X_n < 6/(k + 1) \quad \text{for} \quad n = 1, \ldots, r.$$

Let us denote by Z_n the set of all points from ∂X_n which are linearly accessible from X_n, let $Z^*_n = \Phi(Z_n)$. Then Z^*_n is dense in ∂X^*_n and by Lemma 6, Part 1 there exist segments Λ^*_{ni}, $1 \leqq i \leqq s(n)$, such that

$$(93_1) \qquad \partial X^*_n = \Lambda^*_{n0}, \Lambda^*_{n1}, \ldots, \Lambda^*_{ns(n)} \quad \text{is a net in} \quad \bar{X}^*_n,$$

(93_2) every component of the set $X^*_n - \bigcup_{i=1}^{s(n)} \Lambda^*_{ni}$ has a diameter less than $6/(k + 1)$, (93₃) there are no two segments $\Lambda^*_{ni}, \Lambda^*_{nj}$, $1 \leqq i < j \leqq s(n)$, with a common point in ∂X^*_n,

$$(93_4) \qquad \Lambda^*_{ni} \cap \partial X^*_n \subset Z^*_n \quad \text{for} \quad i = 1, \ldots, s(n)\,.$$

In virtue of Lemma 7, Part 2 there exist polygonal lines Λ_{ni} such that

$$(94) \qquad \partial X_n = \Lambda_{n0}, \Lambda_{n1}, \ldots, \Lambda_{ns(n)} \quad \text{is a net in} \quad \overline{X}_n\,,$$

and a regular homeomorphic mapping Ψ_n of the set $\bigcup_{i=0}^{s(n)} \Lambda^*_{ni}$ onto $\bigcup_{i=0}^{s(n)} \Lambda_{ni}$ which is an extension of the mapping $\Phi_{-1} \mid \partial X^*_n$.

Let us denote $q(k+1) = r + s(1) + \ldots + s(r)$, let

$$T_{r+1} = \Lambda_{11}, \ldots, T_{r+s(1)} = \Lambda_{1s(1)}\,,$$

$$T_{r+s(1)+1} = \Lambda_{21}, \ldots, T_{r+s(1)+s(2)} = \Lambda_{2s(2)}, \ldots, T_{q(k+1)} = \Lambda_{rs(r)}\,,$$

and similarly

$$T^*_{r+1} = \Lambda^*_{11}, \ldots, T^*_{q(k+1)} = \Lambda^*_{rs(r)}\,.$$

The mapping

$$f^*_{k+1} = \Psi_n \quad \text{in} \quad \bigcup_{i=0}^{s(n)} \Lambda^*_{ni} \quad (n = 1, \ldots, r)$$

is then evidently a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k+1)} T^*_n$ onto $\bigcup_{n=1}^{q(k+1)} T_n$ which is an extension of the mapping Φ_{-1}, hence also of $(f_k)_{-1}$ and $(f_1)_{-1}$. The mapping $f_{k+1} = (f^*_{k+1})_{-1}$ is a regular homeomorphic mapping of the set $\bigcup_{n=1}^{q(k+1)} T_n$ onto $\bigcup_{n=1}^{q(k+1)} T^*_n$ which is an extension of the mapping Φ, hence also of f_k and f_1. Numbering suitably the components $\Omega_{k+1,j}$ and $\Omega^*_{k+1,j}$ $(j = 1, \ldots, q(k+1))$ of the sets $Q - \bigcup_{n=1}^{q(k+1)} T_n$ and $Q - \bigcup_{n=1}^{q(k+1)} T^*_n$, respectively, we obtain

$$(95) \qquad f_{k+1}(\partial\Omega_{k+1,j}) = \partial\Omega_{k+1,j} \quad \text{for} \quad j = 1, \ldots, q(k+1)\,.$$

It is easily seen from our construction that

$$(96) \qquad T^*_1, \ldots, T^*_{q(k+1)} \quad \text{are polygonal lines}\,.$$

Since every component $\Omega_{k+1,j}$ of the set $Q - \bigcup_{n=1}^{q(k+1)} T_n$ is part of one of the sets X_n and every component $\Omega^*_{k+1,j}$ of the set $Q - \bigcup_{n=1}^{q(k+1)} T^*_n$ is a component of a certain set $X^*_n - \bigcup_{i=1}^{s(n)} \Lambda^*_{ni}$, we have by (92) and (93₂)

$$(97) \qquad \operatorname{diam} \Omega_{k+1,j} < 6/(k+1)\,,$$

$$\operatorname{diam} \Omega^*_{k+1,j} < 6/(k+1) \quad \text{for} \quad j = 1, \ldots, q(k+1)\,.$$

This completes the induction: For every positive integer k we have constructed nets $T_1, \ldots, T_{q(k)}$ and $T_1^*, \ldots, T_{q(k)}^*$ in $\bar{Q}$, components Ω_{kj} and Ω_{kj}^* $(j = 1, \ldots, q(k))$ of the sets $Q - \bigcup_{n=1}^{q(k)} T_n$ and $Q - \bigcup_{n=1}^{q(k)} T_n^*$, respectively, and a regular homeomorphic mapping f_k of the set $\bigcup_{n=1}^{q(k)} T_n$ onto the set $\bigcup_{n=1}^{q(k)} T_n$ which is an extension of the mapping f_{k-1} (where $f_0 = f$) so that (86)–(88) hold.

Now we can show, similarly as in [1], pp. 378–379:

(98_1) the set $\bigcup_{n=1}^{\infty} T_n$, $\bigcup_{n=1}^{\infty} T_n^*$ are dense in $\bar{Q}$,

(98_2) the mapping $f_\omega = f_k$ in $\bigcup_{n=1}^{q(k)} T_n$ $(k = 1, 2, \ldots)$ is uniformly continuous in $\bigcup_{n=1}^{\infty} T_n$ and maps this set onto $\bigcup_{n=1}^{\infty} T_n^*$.

The mapping f_ω can be extended continuously onto the whole $\bar{Q}$ by a well-known theorem[10]). Denoting the resulting mapping by F we can show that F is one-to one[11]), and hence a homeomorphic mapping of $\bar{Q}$ onto itself. The mapping F is an extension of the mapping f_1, hence also of the mapping f, and since $f_1 \mid \partial Q = \mathrm{Id}$, we have $F \mid \partial Q = \mathrm{Id}$ as well. Putting now $F = \mathrm{Id}$ in $\boldsymbol{S} - Q$ we obtain the required homeomorphic mapping.

Theorem. *Every homeomorphic mapping of a topological circumference into* $\boldsymbol{S}$ *can be extended to a homeomorphic mapping of* $\boldsymbol{S}$ *onto itself.*

Proof. Let h be a homeomorphic mapping of a topological circumference $T \subset \boldsymbol{S}$ into $\boldsymbol{S}$. Let us choose points $A, B \in \boldsymbol{E}$ in different components of the set $\boldsymbol{S} - T$ and let $r \in (0, \infty)$ be so small that $\overline{U(A, r)} \cap T = \emptyset$. Putting

(99) $$\Phi(z) = \frac{r}{z - A} - \frac{r}{B - A} \quad (z \in \boldsymbol{S})$$

we have

$$|\Phi(z)| \leq \left|\frac{r}{z - A}\right| + \left|\frac{r}{B - A}\right| < 2 \quad \text{for} \quad z \in \boldsymbol{S} - \overline{U(A, r)},$$

so that

(100) $$\Phi(T) \subset Q \text{ }^{12)}.$$

Since Φ is a homeomorphic mapping $\boldsymbol{S}$ onto $\boldsymbol{S}$ and since the points A, B belong to

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[10]) See e.g. [2], p. 83.

[11]) The proof is not difficult. We refer the reader to [1], since here the present proof would bring nothing new.

[12]) Q means the same as in (65); similarly for Q_1.

different components of the set $\boldsymbol{S} - T$, the points $\infty = \Phi(A)$, $0 = \Phi(B)$ belong to different components of the set $\boldsymbol{S} - \Phi(T)$. Thus the point 0 belongs to Int $\Phi(T)$.

If we put

$$a = \min \{z \in \Phi(T);\ \operatorname{Im} z = 0\}\,, \quad b = \max \{z \in \Phi(T);\ \operatorname{Im} z = 0\}\ ^{13})\,,$$

we have $-2 < a < b < 2$. Denoting further by M_1, M_2 the arcs with endpoints a, b which satisfy $M_1 \cup M_2 = \Phi(T)$ we have

$$(\tilde{M}_1 \cup \tilde{M}_2) \cap (\langle -2, a\rangle \cup \langle b, 2\rangle) = \emptyset\,.$$

Moreover, let us choose the notation so that

$$i \max \{\operatorname{Im} z;\ z \in \Phi(T),\ \operatorname{Re} z = 0\} \in M_1\,.$$

Certainly there exists a homeomorphic mapping f of the set $\Phi(T)$ onto ∂Q_1 which satisfies (68) and (69). Therefore, by Lemma 8, there exists a homeomorphic mapping F of the set $\boldsymbol{S}$ onto itself, which is an extension of the mapping f.

The mapping $F \circ \Phi$ is a homeomorphic mapping of $\boldsymbol{S}$ onto $\boldsymbol{S}$ which maps the topological circumference T onto ∂Q_1. Similarly, to the topological circumference $h(T)$ there exists a homeomorphic mapping G of the set $\boldsymbol{S}$ onto itself with $G(h(T)) = \partial Q_1$. The mapping

$$\Psi = G \circ h \circ \Phi_{-1} \circ F_{-1}$$

maps ∂Q_1 homeomorphically onto itself. If it is extended by

$$\Psi(tz) = t\,\Psi(z) \quad \text{for} \quad z \in \partial Q_1\,, \quad t \in \langle 0, \infty)\,, \quad \Psi(\infty) = \infty$$

to the whole $\boldsymbol{S}$, it is seen immediately that the extended mapping Ψ is a homeomorphic mapping of $\boldsymbol{S}$ onto $\boldsymbol{S}$.

Hence it follows that

$$H = G_{-1} \circ \Psi\ ^{14}) \circ F \circ \Phi$$

is a homeomorphic mapping of $\boldsymbol{S}$ onto itself which is an extension of the mapping h.

References

[1] *Kuratowski K.:* Topologie II, 2nd edition, Warsaw 1952.
[2] *Černý I.:* Elements of analysis in the complex domain, Prague 1967 (in Czech).

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[13]) The right-hand extrema exist since $0 \in \operatorname{Int} \Phi(T)$ so that the intersection of the straight line $\{z; \operatorname{Im} z = 0\}$ with $\Phi(T)$ is a compact set containing at least one negative and one positive number.

[14]) Ψ stands here, of course, for the extended mapping.

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PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION

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INTRODUCTION

In this paper the existence of a solution to the equations

(0.1) $$u_{tt}(t, x) - u_{xx}(t, x) = \varepsilon F_\varepsilon(u)(t, x), \quad t \in R^+, \quad x \in R,$$

(0.2) $$u(t, x) = u(t, x + 2\pi) = -u(t, -x), \quad t \in R^+, \quad x \in R,$$

(0.3) $$u(t + 2\pi + \varepsilon\lambda, x) = u(t, x), \quad t \in R^+, \quad x \in R$$

is investigated for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. The number $\varepsilon_0 > 0$ is supposed to be sufficiently small and the number $\lambda > 0$ is supposed to be fixed. The operator F_ε has the form

(0.4) $$F_\varepsilon(u)(t, x) = f_\varepsilon(t, x, u(t, x), u_t(t, x), u_x(t, x)).$$

The function f_ε is assumed to satisfy the next two conditions:

(0.5) $$f_\varepsilon(t, x, y_0, y_1, y_2) = f_\varepsilon(t, x + 2\pi, y_0, y_1, y_2) =$$
$$= -f_\varepsilon(t, -x, -y_0, -y_1, y_2) = f_\varepsilon(t + 2\pi + \varepsilon\lambda, x, y_0, y_1, y_2)$$

for every $(t, x, y_0, y_1, y_2) \in R^+ \times R^4$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

(0.6) If the derivative

$$D \equiv D_x^\alpha D_{y_0}^{\beta_0} D_{y_1}^{\beta_1} D_{y_2}^{\beta_2}$$

satisfies $\alpha + \beta_0 + \beta_1 + \beta_2 \leqq 2$, $\alpha \leqq 1$, then the function Df_ε is continuous on $R^+ \times R^4$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$\lim_{\varepsilon \to 0} \sup \{|Df_\varepsilon(t, x, y_0, y_1, y_2) - Df_0(t, x, y_0, y_1, y_2)| ;$$
$$t \in [0, 2\pi + 1], \; x \in R, \; |y_0|, |y_1|, |y_2| \leqq \varrho\} = 0$$

for every $\varrho > 0$ and

$$\lim_{r \to 0+} \sup \{ |Df_\varepsilon(t, x, y_0, y_1, y_2) - Df_\varepsilon(t, x, \bar{y}_0, \bar{y}_1, \bar{y}_2)| ;$$
$$t \in [0, 2\pi + 1], \ x \in R, \ |y_i - \bar{y}_i| \leqq r, \ i = 0, 1, 2, \ \varepsilon \in [-\varepsilon_0, \varepsilon_0] \} = 0$$

for every $(y_0, y_1, y_2) \in R^3$.

The first section of this paper contains two assertions on the existence of periodic solutions of the problem described (Theorems 1.1 and 1.2) which are deduced under some additional assumptions on F_ε. This part is modelled by [1].

In the second section it is shown that a solution to (0.1)–(0.3) with F_ε given by

(0.7) $$F_\varepsilon(u)(t, x) = g(u, u_t, u_x) + h_\varepsilon(t, x)$$

exists for every ε with $|\varepsilon|$ sufficiently small provided

(0.8) the second derivatives of g are continuous on R^3,

(0.9) $$g(y_0, y_1, y_2) = -g(-y_0, -y_1, y_2) \quad \text{for} \quad (y_0, y_1, y_2) \in R^3 ,$$

(0.10) $$g_{y_1}(y_0, y_1, y_2) \geqq \gamma_1 , \quad |g_{y_0}(y_0, y_1, y_2)| \leqq \gamma_0 ,$$
$$|g_{y_2}(y_0, y_1, y_2)| \leqq \gamma_2 \quad \text{for} \quad (y_0, y_1, y_2) \in R^3 ,$$

(0.11) $$\gamma_1 - \gamma_2 - 2\gamma_0 > 0 ,$$

(0.12) $h_\varepsilon = h_\varepsilon(t, x) : R^+ \times R \to R$ and $(h_\varepsilon)_x$ are continuous for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$,

$$h_\varepsilon(t, x) = h_\varepsilon(t, x + 2\pi) = -h_\varepsilon(t, -x) = h_\varepsilon(t + 2\pi + \varepsilon\lambda, x) \quad \text{for} \quad (t, x) \in R^+ \times R$$

and

$$\lim_{\varepsilon \to 0} \sup \{ |D_x h_\varepsilon(t, x) - D_x h_0(t, x)| ; \ t \in [0, 2\pi + 1], \ x \in R \} = 0 .$$

These assumptions, from which (0.11) describes "some sort of monotonicity of F_ε", are similar to those in [3] where 2π-periodic solutions were investigated. Eventually, Section 2 contains a brief discussion of the existence of a $(2\pi + \varepsilon\lambda)$-periodic solution to

(0.13) $$u_{tt} - u_{xx} = \varepsilon(3u^2 u_t + h_\varepsilon(t, x))$$

for every ε from a neighbourhood of 0 provided (0.12) is satisfied and

$$\int_0^{2\pi} h_0(\vartheta, x - \vartheta) \, d\vartheta \neq 0 \text{ for some } x \in R.$$

Section 3 contains some auxiliary assertions.

The problem analogous to (0.1)–(0.3) was investigated by J. P. FINK and W. S. HALL in [1]. These authors developed a general theory for a system of first order equations and as a by-product they obtained the existence of periodic solutions for one special type of the wave equation (cf. (0.13)). In their paper the difficulties connected with the existence of periodic solutions whose periods depend on a parameter were also thoroughly discussed and therefore everybody who wants to be informed in detail is referred to [1].

The author is grateful to O. VEJVODA who attracted his attention to paper [1].

1. GENERAL THEOREMS

Let H_k be the space of all real valued 2π-periodic functions s which have generalized derivatives up to order k and satisfy

$$\int_0^{2\pi} s(\xi)\,\mathrm{d}\xi = 0 \quad\text{and}\quad \int_0^{2\pi} (s^{(k)}(\xi))^2\,\mathrm{d}\xi < +\infty\,.$$

The space H_k endowed with the inner product

$$(r, s)_k = \int_0^{2\pi} r^{(k)}(\xi)\, s^{(k)}(\xi)\,\mathrm{d}\xi$$

is a real Hilbert space. The norm in the space H_k will be denoted by $|\cdot|_k$. Putting

$$\mathscr{H}_k = \{s \in H_k;\ s(x) = -s(-x) \text{ for all } x \in R\}$$

and endowing $\mathscr{H}_k$ with the norm $|\cdot|_k$, we set

$$U_\infty = C^2([0, \infty); \mathscr{H}_0) \cap C^1([0, \infty); \mathscr{H}_1) \cap C^0([0, \infty); \mathscr{H}_2)$$

and

$$U_T = C^2([0, T]; \mathscr{H}_0) \cap C^1([0, T]; \mathscr{H}_1) \cap C^0([0, T]; \mathscr{H}_2)$$

for $0 < T < \infty$. The space U_T equipped with the norm

$$\|u\|_{U_T} = \sum_{i=0}^{2} \|u\|_{C^{2-i}([0,T];\mathscr{H}_i)}$$

is a Banach space. For the sake of simplicity we fix $T = 2\pi + 1$ and introduce an operator $Z : H_2 \to U_\infty$ by

$$Z\, s(t, x) = s(t + x) - s(t - x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

The space of all linear continuous mappings from X into Y will be denoted by $[X, Y]$. For $A \in [X, Y]$ we put

$$\|A\|_{[X,Y]} = \sup\{\|Ax\|_Y;\ x \in X,\ \|x\| \leqq 1\}\,.$$

Using Lemmas 3.1 and 3.2, we verify that a function $u \in U_\infty$ satisfying (0.1)–(0.3) for $\varepsilon \neq 0$ exists if and only if there is a pair of functions $(u, s) \in U_T \times H_2$ such that

$$(1.1)\qquad {}^{\varepsilon}G_1(u, s)(t, x) \equiv -u(t, x) + Z\,s(t, x) + \frac{\varepsilon}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta = 0\,, \quad t \in [0, T]\,, \quad x \in R\,,$$

$$(1.2)\qquad {}^{\varepsilon}G_2(u, s)(x) \equiv \frac{1}{\varepsilon}(s'(x) - s'(x - \varepsilon\lambda)) + \frac{1}{2}\int_0^{2\pi+\varepsilon\lambda} F_\varepsilon(u)(\vartheta, x - \vartheta)\, d\vartheta = 0\,, \quad x \in R\,.$$

Sufficient conditions under which a solution of (1.1) and (1.2) exists are described in the following two theorems.

Theorem 1.1. *Let $\lambda > 0$ and let a function f_ε satisfy (0.5) and (0.6). Let the following assumptions be satisfied:*

(i) *There exists $s_0 \in H_3$ such that $Ms_0 = 0$ where*

$$(1.3)\qquad Ms(x) = \lambda s''(x) + \frac{1}{2}\int_0^{2\pi} F_0(Zs)(\vartheta, x - \vartheta)\, d\vartheta = 0\,, \quad x \in R\,.$$

(ii) *There exists a constant m and a family of operators $Y^\varepsilon \in [H_1, H_2]$ such that*

$$(1.4)\qquad V^\varepsilon Y^\varepsilon = I_{H_1} \quad \text{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0\,,$$

$$(1.5)\qquad \|Y^\varepsilon\|_{[H_1, H_2]} \leqq m \quad \text{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0$$

where

$$(1.6)\quad V^\varepsilon \sigma(x) = |\varepsilon|^{-1}(\sigma'(x) - \sigma'(x - |\varepsilon|\,\lambda)) + \frac{1}{2}\int_0^{2\pi} F_0'(Zs_0)\, Z\sigma(\vartheta, x - \vartheta)\, d\vartheta\,, \ x \in R\,.$$

Then there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for every ε, $0 < |\varepsilon| \leqq \varepsilon_1$ there is $u \in U_\infty$ satisfying (0.1)–(0.3). Moreover, denoting this u by u^ε, we have

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Let us suppose that*

$$Y^\varepsilon V^\varepsilon = I_{H_2} \quad \textit{for} \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0]\,, \quad \varepsilon \neq 0\,.$$

Then there exist two numbers $r > 0$ and $\varepsilon_2 \in (0, \varepsilon_0]$ such that for every ε, $0 < |\varepsilon| \leqq \varepsilon_2$ there is a unique $u \in U_\infty$ satisfying (0.1)–(0.3) and $\|u - Zs_0\|_{U_T} \leqq r$.

Moreover, denoting this u by u^ε, we have

$$\lim_{\varepsilon\to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Proof of Theorem 1.1. Let us put $X = U_T \times H_2$, $Y = U_T \times H_1$ and ${}^\varepsilon G(u, s) = ({}^\varepsilon G_1(u, s), {}^\varepsilon G_2(u, s))$ where ${}^\varepsilon G_1$ and ${}^\varepsilon G_2$ are given by (1.1) and (1.2) respectively. Assuming $\varepsilon \in (0, \varepsilon_0]$, we shall prove that the mapping ${}^\varepsilon G$ satisfies the assumptions of Lemma 3.3. Routine but lengthy calculations show that the mapping ${}^\varepsilon G : X \to Y$ is continuous for every fixed $\varepsilon \in (0, \varepsilon_0]$. The derivative ${}^\varepsilon G'$ of ${}^\varepsilon G$ with respect to (u, s) is given by

$${}^\varepsilon G'(u, s) = ({}^\varepsilon G_1'(u, s), {}^\varepsilon G_2'(u, s))$$

where

$$({}^\varepsilon G_1'(u, s)(v, \sigma))(t, x) = -v(t, x) + Z\,\sigma(t, x) + \\ + \frac{\varepsilon}{2}\int_0^t\int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon'(u)\, v(\vartheta, \xi)\, d\xi\, d\vartheta\,, \quad t \in [0, T]\,, \quad x \in R\,,$$

$$({}^\varepsilon G_2'(u, s)(v, \sigma))(x) = \varepsilon^{-1}(\sigma'(x) - \sigma'(x - \varepsilon\lambda)) + \\ + \frac{1}{2}\int_0^{2\pi+\varepsilon\lambda} F_\varepsilon'(u)\, v(\vartheta, x - \vartheta)\, d\vartheta\,, \quad x \in R$$

for $(v, \sigma) \in X$. These relations imply that ${}^\varepsilon G'(u, s) \in [X, Y]$ for every $(u, s) \in X$ and $\varepsilon \in (0, \varepsilon_0]$. Denoting $u_0 = Zs_0$, we obtain

$$\limsup_{\varrho\to 0+} \{\|{}^\varepsilon G'(u, s) - {}^\varepsilon G'(u_0, s_0)\|_{[X,Y]}\,;\ \varepsilon \in (0, \varepsilon_0],\ \|(u, s) - (u_0, s_0)\|_X \leqq \varrho\} = 0\,.$$

The assumption (i) yields

$$\lim_{\varepsilon\to 0+} \|{}^\varepsilon G(u_0, s_0)\|_Y = 0\,.$$

We shall now define a pair of operators by

$$(A_1(v, \sigma))(t, x) = -v(t, x) + Z\,\sigma(t, x)\,, \quad t \in [0, T]\,, \quad x \in R\,,$$

$$({}^\varepsilon A_2(v, \sigma))(x) = \varepsilon^{-1}(\sigma'(x) - \sigma'(x - \varepsilon\lambda)) + \frac{1}{2}\int_0^{2\pi} F_0'(u_0)\, v(\vartheta, x - \vartheta)\, d\vartheta\,, \quad x \in R\,.$$

Putting ${}^\varepsilon A = (A_1, {}^\varepsilon A_2)$, we easily verify

$$\lim_{\varepsilon\to 0+} \|{}^\varepsilon G'(u_0, s_0) - {}^\varepsilon A\|_{[X,Y]} = 0\,. \tag{1.7}$$

We shall show that there exists a constant m_1 and a family of operators $B^\varepsilon \in [Y, X]$, $0 < \varepsilon \leqq \varepsilon_0$ satisfying

$${}^\varepsilon A B^\varepsilon = I_Y\,, \tag{1.8}$$

$$\|B^\varepsilon\|_{[Y,X]} \leqq m_1 \tag{1.9}$$

for every $\varepsilon \in (0, \varepsilon_0]$. For the sake of simplicity we put

$$P\,v(x) = \frac{1}{2}\int_0^{2\pi} F_0'(u_0)\,v(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta\,.$$

Then we set

$$B_2^\varepsilon(w, \eta) = Y^\varepsilon(\eta + Pw)\,, \quad B_1^\varepsilon(w, \eta) = -w + Z\,B_2^\varepsilon(w, \eta)$$

for $(w, \eta) \in Y$. The assumptions (1.4) and (1.5) show that the operator $B^\varepsilon = (B_1^\varepsilon, B_2^\varepsilon)$ satisfies (1.8) and (1.9). In virtue of (1.7) we can apply Lemma 3.5 to the operator ${}^\varepsilon G'(u_0, s_0)$. Hence there are $\bar{m} > 0$, $\bar{\varepsilon} \in (0, \varepsilon_0]$ and a family of operators T^ε, $0 < \varepsilon \leqq \bar{\varepsilon}$ such that ${}^\varepsilon G'(u_0, s_0)\,T^\varepsilon = I_Y$ and $\|T^\varepsilon\|_{[Y,X]} \leqq \bar{m}$. Thus all the assumptions of Lemma 3.3 are satisfied and therefore the theorem is proved for ε positive. The case $\varepsilon \in [-\varepsilon_0, 0)$ can be treated in the same way if Lemma 3.3 is applied to the pair of operators $({}^{-\varepsilon}G_1(u, s), {}^{-\varepsilon}G_3(u, s))$ where ${}^\varepsilon G_3(u, s)(x) = {}^\varepsilon G_2(u, s)(x + \varepsilon\lambda)$. This completes the proof.

Theorem 1.2 can be proved analogously to Theorem 1.1 if Lemma 3.4 is applied.

2. APPLICATIONS

We start by proving the following assertion:

Theorem 2.1. *Let two functions g and h_ε satisfy (0.8)–(0.12). Then there exist $\varepsilon_1 \in (0, \varepsilon_0]$, $r > 0$ and $s_0 \in H_3$ with the following property: For every ε, $0 < |\varepsilon| \leqq \varepsilon_1$ there is unique $u \in U_\infty$ satisfying $\|u - Zs_0\|_{U_T} \leqq r$ and (0.1)–(0.3) with F_ε given by (0.7). Moreover, denoting this u by u^ε, we have*

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - Zs_0\|_{U_T} = 0\,.$$

Proof. The theorem will follow from Theorem 1.2 if we prove:
(a) There is $s_0 \in H_3$ which satisfies

$$(2.1) \qquad s_0''(x) + (2\lambda)^{-1}\int_0^{2\pi} F_0(Zs_0)(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta = 0\,, \quad x \in R\,.$$

(b) There is $(V^\varepsilon)^{-1} \in [H_1, H_2]$ satisfying

$$\|(V^\varepsilon)^{-1}\|_{[H_1,H_2]} \leqq m$$

for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\varepsilon \neq 0$.

Here V^ε is given by (1.6). Firstly, we shall show that (a) is valid. Let us denote by K the linear operator from $[H_1, H_2]$ given by

$$Ks(x) = (2\lambda)^{-1}\left(\int_0^x s(\xi)\,\mathrm{d}\xi + (2\pi)^{-1}\int_0^{2\pi} \xi\, s(\xi)\,\mathrm{d}\xi\right), \quad x \in R$$

and by Φ the continuous and bounded operator from H_1 into itself given by

$$\Phi\,\sigma(x) = \int_0^{2\pi} F_0(2\,\lambda ZK\sigma)(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta =$$

$$= \int_0^{2\pi} g\left(\int_{-x+2\vartheta}^x \sigma(\xi)\,\mathrm{d}\xi,\ \sigma(x) - \sigma(-x + 2\vartheta),\ \sigma(x) + \sigma(-x + 2\vartheta)\right)\mathrm{d}\vartheta +$$

$$+ \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta, \quad x \in R.$$

The operator K is a linear compact mapping from H_1 into itself which satisfies

$$(Ks, s)_1 = (s, s')_0 = 0.$$

Denoting

$$g_j(x, \xi) = g_{y_j}\left(\int_\xi^x \sigma(\eta)\,\mathrm{d}\eta,\ \sigma(x) - \sigma(\xi),\ \sigma(x) + \sigma(\xi)\right), \quad j = 0, 1, 2,$$

$$\bar{h}(x) = \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\,\mathrm{d}\vartheta$$

we have

$$(\Phi\sigma)'(x) = \int_0^{2\pi} g_0(x, \xi)\,\sigma(x) + (g_1(x, \xi) + g_2(x, \xi))\,\sigma'(x)\,\mathrm{d}\xi + \bar{h}(x).$$

Thus

$$((\Phi\sigma)', \sigma')_0 = \int_0^{2\pi}\int_0^{2\pi} (g_1(x, \xi) + g_2(x, \xi))\,(\sigma'(x))^2 + g_0(x, \xi)\,\sigma(x)\,\sigma'(x)\,\mathrm{d}\xi\,\mathrm{d}x +$$

$$+ \int_0^{2\pi} \bar{h}'(\xi)\,\sigma'(\xi)\,\mathrm{d}\xi \geqq 2\pi(\gamma_1 - \gamma_2)\,|\sigma'|_0^2 - 2\pi\,\gamma_0|\sigma|_0\,|\sigma'|_0 - |\bar{h}'|_0\,|\sigma'|_0.$$

As $|\sigma|_0 \leqq |\sigma'|_0$, the preceding inequality yields

$$(\Phi\sigma, \sigma)_1 > 0$$

for all $\sigma \in H_1$, $|\sigma|_1 = R$ where $R = 1 + (2\pi(\gamma_1 - \gamma_2 - \gamma_0))^{-1}\,|\bar{h}'|_0$. Hence there do not exist $t \in [0, 1]$ and $\sigma \in H_1$, $|\sigma|_1 = R$ such that

$$\sigma + tK\Phi\sigma = 0.$$

Really, if there were such t and σ, then they should satisfy

$$0 = (\sigma + tK\Phi\sigma, \Phi\sigma)_1 = (\sigma, \Phi\sigma)_1 > 0 .$$

But this is a contradiction. Therefore the Leray-Schauder theorem implies that there is $\sigma_0 \in H_1$, $|\sigma_0|_1 < R$ satisfying

$$\sigma_0 + K\Phi\sigma_0 = 0 .$$

Let us set $s_0 = 2\lambda K\sigma_0$. Then $s_0 \in H_2$ and s_0 satisfies (2.1). In virtue of (0.8) and (0.12) we obtain $s_0 \in H_3$. Thus (a) is satisfied.

Secondly, we shall show that (b) is satisfied. Putting

$$\bar{g}_j(x, \xi) = g_{y_j}(s_0(x) - s_0(\xi), s_0'(x) - s_0'(\xi), s_0'(x) + s_0'(\xi)) ,$$

$j = 0, 1, 2$ we can write

$$\begin{aligned} V^\varepsilon \sigma(x) &= |\varepsilon|^{-1} (\sigma'(x) - \sigma'(x - |\varepsilon| \lambda)) + \\ &+ \frac{1}{2}\int_0^{2\pi} (\bar{g}_1(x, \xi) (\sigma'(x) - \sigma'(\xi)) + \bar{g}_2(x, \xi) (\sigma'(x) + \sigma'(\xi)) + \\ &\qquad + \bar{g}_0(x, \xi) (\sigma(x) - \sigma(\xi))) \,d\xi . \end{aligned}$$

Let us denote by $C_{2\pi}^\infty$ the space of infinitely differentiable 2π-periodic functions on R. Let $\eta \in C_{2\pi}^\infty \cap H_0$. Then

$$\begin{aligned} (V^\varepsilon\eta, -\eta''')_0 &= |\varepsilon|^{-1} \left(|\eta''|_0^2 - \int_0^{2\pi} \eta''(x)\, \eta''(x - |\varepsilon| \lambda) \,dx\right) + \\ &+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} ((\bar{g}_1(x, \xi) + \bar{g}_2(x, \xi)) (\eta''(x))^2 + \bar{g}_0(x, \xi)\, \eta'(x)\, \eta''(x) \,d\xi \,dx + \\ &+ \frac{1}{2}\int_0^{2\pi} \left(\int_0^{2\pi} (\bar{g}_{1x}(x, \xi) (\eta'(x) - \eta'(\xi)) + \bar{g}_{2x}(x, \xi) (\eta'(x) + \eta'(\xi)) + \right. \\ &\qquad \left. + \bar{g}_{0x}(x, \xi) (\eta(x) + \eta(\xi))) \,d\xi \right) \eta''(x) \,dx . \end{aligned}$$

As $|\eta|_0 \leqq |\eta'|_0 \leqq |\eta''|_0$ and

$$\int_0^{2\pi} \eta''(x)\, \eta''(x - |\varepsilon| \lambda) \,dx \leqq |\eta''|_0^2$$

we have

$$(2.2_1) \qquad \begin{aligned} (V^\varepsilon\eta, -\eta''')_0 &\geqq \pi(\gamma_1 - \gamma_2 - \gamma_0)\, |\eta''|_0^2 - c_1|\eta'|_0\, |\eta''|_0 \geqq \\ &\geqq 2^{-1}\pi(\gamma_1 - \gamma_2 - \gamma_0)\, |\eta''|_0^2 - c_1^2(2\pi(\gamma_1 - \gamma_2 - \gamma_0))^{-1}\, |\eta'|_0^2 . \end{aligned}$$

The constant c_1 does not depend on η. Similarly,

$$(V^\varepsilon\eta, \eta')_0 \geqq \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \gamma_1(\eta'(x) - \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi \; +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\bar{g}_1(x, \xi) - \gamma_1)\,(\eta'(x) - \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi \; +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \bar{g}_2(x, \xi)\,(\eta'(x) + \eta'(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi \; +$$

$$+ \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \bar{g}_0(x, \xi)\,(\eta(x) - \eta(\xi))\,\eta'(x)\,\mathrm{d}x\,\mathrm{d}\xi = I_1 + I_2 + I_3 + I_4\,.$$

Interchanging the variables x and ξ in I_2 and using the relations $\bar{g}_1(x, \xi) = \bar{g}_1(\xi, x)$ and $\bar{g}_1(x, \xi) \geqq \gamma_1$ we can write

$$2I_2 = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (\bar{g}_1(x, \xi) - \gamma_1)\,(\eta'(x) - \eta'(\xi))^2\,\mathrm{d}x\,\mathrm{d}\xi \geqq 0\,.$$

Thus simple estimations of I_3 and I_4 yield

$$(2.2_2) \qquad (V^\varepsilon\eta, \eta')_0 \geqq \pi\,\gamma|\eta'|_0^2$$

where $\gamma = \gamma_1 - \gamma_2 - 2\gamma_0$. Let Λ be an operator defined by

$$\Lambda\eta = -\eta''' + c_2\eta'\,, \quad c_2 = c_1^2(2\pi^2\,\gamma(\gamma_1 - \gamma_0 - \gamma_2))^{-1}\,.$$

By (2.2) Λ satisfies

$$(2.3) \qquad (V^\varepsilon\eta, \Lambda\eta)_0 \geqq \gamma_3|\eta''|_0^2 = \gamma_3|\eta|_2^2$$

with $\gamma_3 = 2^{-1}\,\pi(\gamma_1 - \gamma_2 - \gamma_0)$. Let

$$(V^\varepsilon)^*\,\varphi(x) = |\varepsilon|^{-1}\,(-1)\,(\varphi'(x) - \varphi'(x + |\varepsilon|\,\lambda)) \; -$$

$$- \frac{1}{2}\int_0^{2\pi} (\bar{g}_1(x, \xi)\,(\varphi(x) - \varphi(\xi)))_x\,\mathrm{d}\xi - \frac{1}{2}\int_0^{2\pi} (\bar{g}_2(x, \xi)\,(\varphi(x) - \varphi(\xi)))_x\,\mathrm{d}\xi \; +$$

$$+ \frac{1}{2}\int_0^{2\pi} \bar{g}_0(x, \xi)\,(\varphi(x) - \varphi(\xi))\,\mathrm{d}\xi\,.$$

Then

$$(2.4) \qquad (V^\varepsilon\eta, \varphi)_0 = (\eta, (V^\varepsilon)^*\,\varphi)_0$$

for every $\eta, \varphi \in C_{2\pi}^{\infty} \cap H_0$. Using the negative norms (cf. [4], p. 165–167), we complete the proof. The negative norm $|\cdot|_{-k}$, k positive integer is defined by

$$|v|_{-k} = \sup \{|(v, w)_0| \, |w|_k^{-1};\ 0 \neq w \in H_k\} .$$

The completion of H_0 with respect to the norm $|\cdot|_{-k}$ will be denoted by H_{-k}. Applying Fourier series, we easily show that for every $\varphi \in C_{2\pi}^{\infty} \cap H_0$ there exists a unique $\eta \in C_{2\pi}^{\infty} \cap H_0$ such that $\Lambda\eta = \varphi$. By (2.3) and (2.4),

$$|\eta|_2 \, |(V^\varepsilon)^* \varphi|_{-2} \geqq (\eta, (V^\varepsilon)^* \varphi)_0 = (V^\varepsilon\eta, \Lambda\eta)_0 \geqq \gamma_3 |\eta|_2^2 .$$

Hence

$$(2.5) \qquad |(V^\varepsilon)^* \varphi|_{-2} \geqq \gamma_3 |\eta|_2 .$$

By definition,

$$|\varphi|_{-1} = \sup \{|(\varphi, w)_0| \, |w|_1^{-1};\ 0 \neq w \in H_1\} =$$
$$= \sup \{|(-\eta''' + c_2\eta', w)_0| \, |w|_1^{-1};\ 0 \neq w \in H_1\} \leqq (1 + c_2) \, |\eta|_2 .$$

This inequality together with (2.5) yields

$$(2.6) \qquad |(V^\varepsilon)^* \varphi|_{-2} \geqq \gamma_3 (1 + c_2)^{-1} \, |\varphi|_{-1}$$

for every $\varphi \in C_{2\pi}^{\infty} \cap H_0$. Finally, let $g \in H_1$. Let us put $Q = (V^\varepsilon)^* (C_{2\pi}^{\infty} \cap H_0)$. To every $\psi \in Q$ we assign the value

$$l(\psi) = (\varphi, g)_0$$

where $\psi = (V^\varepsilon)^* \varphi$. This is possible because by (2.6) the function φ is uniquely determined for every ψ. Using (2.6), we conclude

$$|l(\psi)| \leqq |\varphi|_{-1} \, |g|_1 \leqq (\gamma_3^{-1}(1 + c_2) \, |g|_1) \, |\psi|_{-2} .$$

Hence l is a linear functional on $Q \subset H_{-2}$. According to the Hahn-Banach theorem, there is a linear functional l' on H_{-2} such that l' is an extension of l and the norm of l' equals that of l. By Lax's theorem ([4], p. 167) there exists a unique $v \in H_2$ such that

$$l'(\psi) = (\psi, v)_0$$

and

$$(2.7) \qquad |v|_2 \leqq \gamma_3^{-1}(1 + c_2) \, |g|_1 .$$

Putting $\psi = (V^\varepsilon)^* \varphi$ for $\varphi \in C_{2\pi}^{\infty} \cap H_0$, we have

$$l'(\psi) = (\varphi, g)_0 = ((V^\varepsilon)^* \varphi, v)_0 = (\varphi, V^\varepsilon v)_0 ,$$

i.e. $(\varphi, g - V^\varepsilon v)_0 = 0$. As $g, V^\varepsilon v \in H_0$, the last equality yields $V^\varepsilon v = g$. This implies that $(V^\varepsilon)^{-1} \in [H_1, H_2]$ exists. By (2.7),

$$\|(V^\varepsilon)^{-1}\|_{[H_1,H_2]} \leqq \gamma_3^{-1}(1 + c_2).$$

Hence the condition (b) is satisfied. This completes the proof.

In the second part of this section we show that for every ε from a neighbourhood of 0 there is a solution $u \in U_\infty$ to the equation

$$u_{tt}(t, x) - u_{xx}(t, x) = \varepsilon(3u^2 u_t + h_\varepsilon(t, x)), \quad t \in R^+, \quad x \in R \tag{2.8}$$

satisfying the conditions (0.2) and (0.3). We shall suppose that the function h_ε fulfils (0.12) and that the function

$$\bar{h}(x) = \int_0^{2\pi} h_0(\vartheta, x - \vartheta)\, d\vartheta$$

does not vanish identically. The existence of solutions follows from Theorem 1.2 if the next two conditions are satisfied.

(c) There is a function $s \in H_3$, $s \neq 0$ such that

$$s''(x) + (2\lambda)^{-1} \int_0^{2\pi} 3(s(x) - s(\xi))^2 (s'(x) - s'(\xi))\, d\xi + \bar{h}(x) = 0, \quad x \in R. \tag{2.9}$$

(d) The operator $V^\varepsilon \in [H_2, H_1]$ given by

$$\begin{aligned} V^\varepsilon \sigma(x) &= |\varepsilon|^{-1} (\sigma'(x) - \sigma'(x - |\varepsilon| \lambda)) + \\ &+ \frac{3}{2} \int_0^{2\pi} (s(x) - s(\xi))^2 (\sigma'(x) - \sigma'(\xi))\, d\xi + \\ &+ 3 \int_0^{2\pi} (s(x) - s(\xi)) (s'(x) - s'(\xi)) (\sigma(x) - \sigma(\xi))\, d\xi, \quad x \in R \end{aligned}$$

has an inverse $(V^\varepsilon)^{-1} \in [H_1, H_2]$ whose norm is bounded by a constant independent of ε.

The existence of solutions to (2.8), (0.2) and (0.3) was proved in [1] under the assumption that h_ε is a function π-antiperiodic in the variable x. The authors obtained this result as a by-product when investigating a system of two first order equations. The same theorems as in [1] have to be applied to complete the proofs of (c) and (d) which are indicated below. They can however be applied after simpler calculations and without the assumption of π-antiperiodicity of the function h_ε.

Firstly, we shall treat (c). Let L_p be the space of all 2π-periodic real functions s satisfying

$$\int_0^{2\pi} s(\xi)\, d\xi = 0 \quad \text{and} \quad \int_0^{2\pi} s^p(x)\, dx < \infty.$$

Let us denote by K the linear compact operator from $L_{4/3}$ into L_4 given by

$$K\, s(x) = \int_0^x s(\xi)\, d\xi + (2\pi)^{-1} \int_0^{2\pi} \xi\, s(\xi)\, d\xi\,, \quad x \in R\,.$$

As

$$\int_0^{2\pi} \left\{ \int_0^{2\pi} (s(x) - s(\xi))^3\, d\xi \right\} s(x)\, dx \geqq 2\pi \int_0^{2\pi} s^4(x)\, dx$$

we can use the theorem which was applied in the corresponding step in [1]. Thus there is $s \in L_4$ such that

$$s + (2\lambda)^{-1} K\left(\int_0^{2\pi} (s(\cdot) - s(\xi))^3\, d\xi \right) + K^2 \bar{h} = 0\,.$$

Differentiating this equation, we can show that $s \in H_3$. Clearly $s \neq 0$ and (2.9) is satisfied.

In the end we shall show how to treat (d). Let $g \in H_1$. Let us denote $W^\varepsilon = K V^\varepsilon$. Then

$$W^\varepsilon\, \sigma(x) = |\varepsilon|^{-1} (\sigma(x) - \sigma(x - |\varepsilon|\, \lambda)) +$$
$$+ \frac{3}{2} \int_0^{2\pi} (s(x) - s(\xi))^2 (\sigma(x) - \sigma(\xi))\, d\xi$$

and the equation $W^\varepsilon \sigma = Kg$ is equivalent to $V^\varepsilon \sigma = g$. Let us put $I = \frac{3}{2} \int_0^{2\pi} s^2(\xi)\, d\xi$. Then we immediately verify

$$(W^\varepsilon \sigma, \sigma)_0 \geqq I |\sigma|_0^2\,,$$
$$((W^\varepsilon \sigma)', \sigma')_0 \geqq I |\sigma'|_0^2 - M_1 |\sigma|_0\, |\sigma'|_0\,,$$
$$((W^\varepsilon \sigma)'', \sigma'')_0 \geqq I |\sigma''|_0^2 - M_2 |\sigma'|_0\, |\sigma''|_0$$

for every $\sigma \in H_2$ with M_1 and M_2 independent of σ and ε. Using the Lax-Milgram theorem in the same way as in [1], we see that (d) is satisfied.

3. AUXILIARY ASSERTIONS

Lemma 3.1. *Let $\varepsilon \neq 0$ satisfy $0 < 2\pi + \varepsilon\lambda < T$. Let $u \in U_\infty$ and $s \in H_2$ satisfy*

(3.1) $$u(t, x) = Z\, s(t, x) + \frac{\varepsilon}{2} \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta\,, \quad t \in R^+\,, \quad x \in R$$

and

(3.2) $$u(t, x) = u(t + 2\pi + \varepsilon\lambda, x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

Then the pair of functions consisting of the restriction of the function u to $[0, T] \times R$ *and the function s satisfies* (1.1) *and* (1.2).

Proof. (3.1) implies that (1.1) holds. Thus only (1.2) has to be shown. Let us put $\omega = 2\pi + \varepsilon\lambda$. Inserting u from (3.1) into (3.2) and making use of the obvious relations

$$\int_{-x+t+\omega-\vartheta}^{x-t-\omega+\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi = 0\,,$$

$$\int_0^{t+\omega} \int_{x-t-\omega+\vartheta}^{x+t+\omega-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta = \int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta + $$
$$+ \int_0^{\omega} \int_{x-t-\omega+\vartheta}^{x+t+\omega-\vartheta} F_\varepsilon(u)(\vartheta, \xi)\, d\xi\, d\vartheta\,,$$

we obtain

$$s(t + x + \omega) - s(t + x) + \frac{\varepsilon}{2}\int_0^{\omega} \int_0^{t+x} F_\varepsilon(u)(\vartheta, \xi + \omega - \vartheta)\, d\xi\, d\vartheta =$$
$$= s(t - x + \omega) - s(t - x) + \frac{\varepsilon}{2}\int_0^{\omega} \int_0^{t-x} F_\varepsilon(u)(\vartheta, \xi + \omega - \vartheta)\, d\xi\, d\vartheta$$

for every $t \in R^+$ and $x \in R$. From here (1.2) follows immediately.

Lemma 3.2. *Let* $\varepsilon \neq 0$ *satisfy* $0 < 2\pi + \varepsilon\lambda < T$. *Let* $u \in U_T$ *and* $s \in H_2$ *satisfy* (1.1) *and* (1.2). *Let us denote by* $\bar{u}$ *the function satisfying*

(3.3) $$\bar{u}(t, x) = u(t, x)\,, \quad t \in [0, 2\pi + \varepsilon\lambda)\,, \quad x \in R$$

and

(3.4) $$\bar{u}(t + 2\pi + \varepsilon\lambda, x) = \bar{u}(t, x)\,, \quad t \in R^+\,, \quad x \in R\,.$$

Then $\bar{u} \in U_\infty$ *and*

(3.5) $$\bar{u}(t, x) = Z\, s(t, x) + \frac{\varepsilon}{2}\int_0^t \int_{x-t+\vartheta}^{x+t-\vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi)\, d\xi\, d\vartheta$$

for every $t \in R^+$ *and* $x \in R$.

Proof. From (1.1) it follows that

$$u_t(t, x) = s'(t + x) - s'(t - x) + \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x + t - \vartheta)\, d\vartheta +$$
$$+ \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x - t + \vartheta)\, d\vartheta\,,$$

$$u_x(t, x) = s'(t + x) + s'(t - x) + \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x + t - \vartheta)\, d\vartheta -$$
$$- \frac{\varepsilon}{2}\int_0^t F_\varepsilon(u)(\vartheta, x - t + \vartheta)\, d\vartheta$$

for $t \in [0, T)$ and $x \in R$. Let $\omega = 2\pi + \varepsilon\lambda$. Using (1.2) we obtain

$$u_t(t + \omega, x) - u_t(t, x) = \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x + t - \vartheta) - F_\varepsilon(u)(\vartheta, x + t - \vartheta)\} \,d\vartheta$$

$$+ \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x - t + \vartheta) - F_\varepsilon(u)(\vartheta, x - t + \vartheta)\} \,d\vartheta,$$

$$u_x(t + \omega, x) - u_x(t, x) = \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x + t - \vartheta) - F_\varepsilon(u)(\vartheta, x + t - \vartheta)\} \,d\vartheta -$$

$$- \frac{\varepsilon}{2}\int_0^t \{F_\varepsilon(u)(\vartheta + \omega, x - t + \vartheta) - F_\varepsilon(u)(\vartheta, x - t + \vartheta)\} \,d\vartheta$$

for $t \in [0, T - \omega)$ and $x \in R$. In virtue of (0.2) we have

$$|u(t, x)| \leqq \int_0^{|x|} |u_x(t, \xi)| \,d\xi .$$

By Gronwall's lemma we deduce from the last three relations:

$$u(t, x) = u(t + \omega, x)$$

for $t \in [0, T - \omega)$ and $x \in R$. This shows that there is a function $\bar{u} \in U_\infty$ satisfying (3.3) and (3.4). Induction will be used to prove (3.5). Let $n \geqq 1$ be an integer such that (3.5) holds for $t \in [0, n\omega]$. Let $\tau \in (n\omega, (n + 1)\omega]$. Then we have

$$\bar{u}(\tau, x) = \bar{u}(\tau - \omega, x) = Z\, s(\tau - \omega, x) + \frac{\varepsilon}{2}\int_0^{\tau-\omega}\int_{x-\tau+\omega+\vartheta}^{x+\tau-\omega-\vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi) \,d\xi \,d\vartheta =$$

$$Z\, s(\tau, x) + \frac{\varepsilon}{2}\int_0^{\tau}\int_{x-\tau+\vartheta}^{x+\tau-\vartheta} F_\varepsilon(\bar{u})(\vartheta, \xi) \,d\xi \,d\vartheta + \Xi(\tau, x)$$

where

$$\Xi(\tau, x) = Z\, s(\tau - \omega, x) - Z\, s(\tau, x) - \frac{\varepsilon}{2}\int_0^{\omega}\int_{x-\tau+\vartheta}^{x+\tau-\vartheta} F_\varepsilon(u)(\vartheta, \xi) \,d\xi \,d\vartheta .$$

By (1.2), $\Xi(\tau, x) = 0$. Thus (3.5) holds for $t \in [0, (n + 1)\omega]$. This completes the proof.

The next two lemmas are modifications of the implicit function theorem and are closely related to Theorems 2.3 and 2.4 in [1].

Lemma 3.3. *Let X, Y be Banach spaces, $\bar{m}$, $\bar{\varepsilon}$ positive numbers and $x_0 \in X$. Let a family of mappings ${}^\varepsilon G : X \to Y$, $\varepsilon \in (0, \bar{\varepsilon}]$ satisfy the following assumptions:*

(i) *The mapping ${}^\varepsilon G : X \to Y$ is continuous and its derivative ${}^\varepsilon G' : X \to [X, Y]$ exists for every $\varepsilon \in (0, \bar{\varepsilon}]$.*

(ii) $\lim_{\varrho \to 0+} \sup \{\|{}^\varepsilon G'(x) - {}^\varepsilon G'(x_0)\|_{[X,Y]};\ \varepsilon \in (0, \bar{\varepsilon}],\ \|x - x_0\|_X < \varrho\} < 1/\bar{m}.$

(iii) $\lim_{\varepsilon \to 0+} \|{}^{\varepsilon}G(x_0)\|_Y = 0.$

(iv) *For every* $\varepsilon \in (0, \bar{\varepsilon}]$ *there exists* $T^{\varepsilon} \in [Y, X]$ *satisfying* ${}^{\varepsilon}G'(x_0)\, T^{\varepsilon} = I_Y$, $\|T^{\varepsilon}\|_{[Y,X]} \leqq \bar{m}$.

Then there exists $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon}_1]$ *there is* $x^{\varepsilon} \in X$ *satisfying* ${}^{\varepsilon}G(x^{\varepsilon}) = 0$. *Moreover*, $\lim_{\varepsilon \to 0} x^{\varepsilon} = x_0$.

Proof. Let us choose $\alpha \in (0, 1)$ and $\varrho > 0$ such that

$$\sup \{\|{}^{\varepsilon}G'(x) - {}^{\varepsilon}G'(x_0)\|_{[X,Y]};\ x \in B(x_0, \varrho),\ \varepsilon \in (0, \bar{\varepsilon}]\} < \alpha/\bar{m}\,.$$

Let $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ be such that $\varepsilon \in (0, \bar{\varepsilon}_1]$ implies

$$\|{}^{\varepsilon}G(x_0)\|_Y \leqq (1 - \alpha)\, \varrho/\bar{m}\,.$$

Let us put $x_0^{\varepsilon} = x_0$ and $x_{n+1}^{\varepsilon} = x_n^{\varepsilon} - T^{\varepsilon}\, {}^{\varepsilon}G(x_n^{\varepsilon})$ for $\varepsilon \in (0, \bar{\varepsilon}_1]$, $n = 0, 1, \ldots$. We easily obtain

$$\|x_{k+1}^{\varepsilon} - x_k^{\varepsilon}\|_X \leqq \bar{m}\|{}^{\varepsilon}G(x_k^{\varepsilon})\|_Y \tag{3.6}$$

for $k = 0, 1, \ldots$. If for an integer $n \geqq 1$ we have $x_k^{\varepsilon} \in B(x_0, \varrho)$, $k = 1, 2, \ldots, n$, then by [2] (relation 8.6.2),

$$\begin{aligned} \|{}^{\varepsilon}G(x_k^{\varepsilon})\|_Y &= \|{}^{\varepsilon}G(x_k^{\varepsilon}) - {}^{\varepsilon}G(x_{k-1}^{\varepsilon}) - {}^{\varepsilon}G'(x_0)\,(x_k^{\varepsilon} - x_{k-1}^{\varepsilon})\|_Y \leqq \\ &\leqq \|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X \sup \{\|{}^{\varepsilon}G'(x) - {}^{\varepsilon}G'(x_0)\|_{[X,Y]};\ \varepsilon \in (0, \bar{\varepsilon}_1],\ x \in B(x_0, \varrho)\} \leqq \\ &\leqq \alpha\|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X/\bar{m}\,. \end{aligned} \tag{3.7}$$

This estimate together with (3.6) implies

$$\|x_{k+1}^{\varepsilon} - x_k^{\varepsilon}\|_X \leqq \alpha\|x_k^{\varepsilon} - x_{k-1}^{\varepsilon}\|_X \tag{3.8}$$

for $k = 1, 2, \ldots, n$. Using (3.6) for $k = 0$ and (3.8), we obtain

$$\|x_{k+1}^{\varepsilon} - x_0\|_X \leqq \bar{m}\|{}^{\varepsilon}G(x_0)\|_Y/(1 - \alpha)\,. \tag{3.9}$$

Thus $x_n^{\varepsilon} \in B(x_0, \varrho)$ for all $\varepsilon \in (0, \bar{\varepsilon}_1]$ and all positive integers n. By (3.8) we can put $x^{\varepsilon} = \lim_{n \to \infty} x_n^{\varepsilon}$. ${}^{\varepsilon}G(x^{\varepsilon}) = 0$ and $\lim_{\varepsilon \to 0} x^{\varepsilon} = x_0$ are consequences of (3.7) and (3.9) respectively.

Lemma 3.4. *Let all the assumptions of Lemma* 3.3 *be satisfied. Let*

$$T^{\varepsilon}\, {}^{\varepsilon}G'(x_0) = I_X \quad \textit{for} \quad \varepsilon \in (0, \bar{\varepsilon}]\,.$$

Then there exist two numbers $\bar{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ and $\varrho > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon}_1]$ there is a unique $x^\varepsilon \in B(x_0, \varrho)$ satisfying ${}^\varepsilon G(x^\varepsilon) = 0$. Moreover, $\lim_{\varepsilon \to 0} x^\varepsilon = x_0$.

Proof. Let $\bar{\varepsilon}_1$, α and ϱ be the numbers chosen in the proof of Lemma 3.3. Let $x_1^\varepsilon, x_2^\varepsilon \in B(x_0, \varrho)$, $0 < \varepsilon \leqq \bar{\varepsilon}_1$ satisfy ${}^\varepsilon G(x_1^\varepsilon) = {}^\varepsilon G(x_2^\varepsilon) = 0$. Then we can write

$$x_1^\varepsilon - x_2^\varepsilon = -T^\varepsilon({}^\varepsilon G(x_1^\varepsilon) - {}^\varepsilon G(x_2^\varepsilon) - {}^\varepsilon G'(x_0)(x_1^\varepsilon - x_2^\varepsilon)) .$$

Using [2] (relation 8.6.2), we obtain

$$\|x_1^\varepsilon - x_2^\varepsilon\|_X \leqq \bar{m}\|x_1^\varepsilon - x_2^\varepsilon\|_X \,.$$

$$. \sup \{\|{}^\varepsilon G'(x) - {}^\varepsilon G'(x_0)\|_{[X,Y]};\ x \in B(x_0, \varrho),\ \varepsilon \in (0, \bar{\varepsilon}_1]\} \leqq \alpha \|x_1^\varepsilon - x_2^\varepsilon\|_X \,.$$

As $\alpha < 1$, we have $x_1^\varepsilon = x_2^\varepsilon$. This completes the proof.

Lemma 3.5. *Let X and Y be Banach spaces. Let $A \in [X, Y]$ and $B \in [Y, X]$ satisfy $AB = I_Y$. Then for every $\Delta \in [X, Y]$, $\|\Delta\|_{[X,Y]} \leqq (2\|B\|_{[Y,X]})^{-1}$ there exists $B_\Delta \in \in [Y, X]$ such that*

$$(A + \Delta) B_\Delta = I_Y \,, \tag{3.10}$$

$$\|B_\Delta\|_{[Y,X]} \leqq 2\|B\|_{[Y,X]} \,. \tag{3.11}$$

If in addition the operators A and B fulfil $BA = I_X$, then B_Δ satisfies (3.10), (3.11) *and $B_\Delta(A + \Delta) = I_X$.*

Proof is easy.

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KLEINE DESARGUES-BEDINGUNG IN GEWEBEN

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Bekanntlich existiert eine enge Beziehung zwischen 3-Geweben und Loops. In diesem Zusammenhang ergeben sich als wichtig solche 3-Gewebe, welche den Loops mit Moufang-Identitäten, bzw. mit assoziativem Gesetz, bzw. mit kommutativem Gesetzt entsprechen und welche durch die Gültigkeit der Bolschen Schließungsbedingungen, bzw. der Reidemeister-, bzw. Thomsen-Bedingung gekennzeichnet sind. Deswegen werden üblicherweise diese Schließungsbedingungen als gewiße Grundlage für die Untersuchungen der 3-Gewebe bezeichnet.

Auf der anderen Seite kann man einige 3-Gewebe in affine Ebenen einbetten (ohne Zunahme der eigentlichen Punkte) und für solche 3-Gewebe kann man dann die Schließungsbedingungen bezüglich der Ebene verengen, vor allem die Schließungsbedingungen vom Desarguesschen Typ. Dagegen ist es aber möglich die Schließungsbedingungen vom Desarguesschen Typ natürlicherweise in Geweben des Grades $\geqq 4$ unabhängig auf der Einbettbarkeit in affine Ebenen definieren. Insbesondere die sog. kleine Desargues-Bedingung zeigt sich als sehr geeignet für die Eintrittsbetrachtungen. Es entsteht die Frage über die algebraische Charakterisierung der Gewebe des Grades $\geqq 4$, in denen die affine kleine Desargues-Bedingung universal gilt, also über eine direkte Verallgemeinerung der Translationsebenen. Eine solche Charakterisierung möchte ich im weiteren herleiten. Dabei gebrauche ich die „zulässigen" Algebren, welche ich schon bei einer anderen Gelegenheit eingeführt habe (vgl. [4], § 2); der Bequemheit halber werde ich die Definition von solchen Algebren kurz wiedergeben. Diese Algebren haben den Vorteil, daß ihre binären Operationen sämtlich Loopoperationen sind und daß sie beim Übergang vom 3-Gewebe zur affinen Ebene unmittelbar in planare Ternärringe übergehen. Bei den Beweisen werde ich die Eigenschaften der Gewebeautomorphismen in expliziter Form nicht verwenden. Eine Ausdehnung des Baerschen Begriffs der (P, g)-Transitivität auf Gewebe habe ich in der Note [6] durchgeführt. Eine Anregung für den vorliegenden Artikel gaben die kürzlich erschienenen Resultate von V. D. Belousov und G. B. Beljavskaja aus dem ersten Teil von [3]; diese Resultate kommen als Sonderfälle in unseren Ergebnissen vor.

Erstens werden wir einige Begriffe über Gewebe einführen, sowohl wie den Begriff der zulässigen Algebra. Dann werden wir über die gegenseitige Beziehung zwischen

Geweben und zulässigen Algebren kurz berichten, wonach schon die Untersuchung der Desargues-Bedingungen in Geweben beginnen kann.

Unter einem *Gewebe* werden wir ein Tripel $(\mathscr{P}, \mathscr{G}, (V_\iota)_{\iota\in I})$ verstehen, wobei $\mathscr{P}$ eine nichtleere Menge, $\mathscr{G}$ eine Menge von gewissen wenigstens zweielementigen Untermengen von $\mathscr{P}$, I eine nichtleere Indexmenge und $\iota \mapsto V_\iota$ eine injektive Abbildung der Menge I in die Menge $\mathscr{P}$ ist, so daß die folgenden Axiome erfüllt sind:

(i) $$v := \{V_\iota \mid \iota \in I\} \in \mathscr{G},$$

(ii) $$\forall P \in \mathscr{P} \setminus v \quad \forall \iota \in I \quad \exists!\, g \in \mathscr{G} \quad P, V_\iota \in g,$$

(iii) $$\forall g \in \mathscr{G} \setminus \{v\} \quad \exists!\, \iota \in I \quad V_\iota \in g,$$

(iv) $$\forall a, b \in \mathscr{G} \setminus \{v\}\,;\; a \neq b \quad \#(a \cap b) = 1.$$

Die Elemente von $\mathscr{P}$ heißen *Punkte*, die Elemente von $\mathscr{G}$ heißen *Geraden*, die Punkte von v heißen *singulär* oder *uneigentlich*, während die übrigen als *gewöhnliche* oder *eigentliche* Punkte bezeichnet werden; die Gerade v heißt *singulär* oder *uneigentlich*, die übrigen heißen dann *gewöhnliche* oder *eigentliche* Geraden; eigentliche Geraden durch V_ι $(\iota \in I)$ werden *ι-Geraden* genannt und $\# I$ heißt der *Grad* des Gewebes. Die Bezeichnung $\overline{A_1, \ldots, A_n}$ soll bedeuten, daß die Punkte $A_1, \ldots, A_n$ auf derselben Geraden liegen. Sind A, B verschiedene Punkte, dann ist enweder $\{g \in \mathscr{G} \mid A, B \in g\} = \emptyset$ oder $\#\{g \in \mathscr{G} \mid A, B \in g\} = 1$; im letzten Fall nennen wir die Gerade durch A, B die *Verbindungsgerade* von A, B und bezeichnen sie mit AB. Sind a, b verschiedene Geraden, dann gibt es genau einen Punkt, der auf beiden Geraden liegt; dieser heißt der *Durchschnittspunkt* von a, b und wird mit $a \sqcap b$ bezeichnet.

Im weiteren beschränken wir uns auf Gewebe des Grades $\geqq 3$. Ist $\mathbf{G}$ (bzw. $\mathbf{G}$ mit irgendeinem Index rechts oben) ein Gewebe, dann werden wir stets $\mathbf{G} =: (\mathscr{P}, \mathscr{G}, (V_\iota)_{\iota\in I})$ (bzw. dasselbe mit entsprechenden Indexen rechts oben bei allen Symbolen) setzen und auch das Symbol v (bzw. v mit entsprechendem Index rechts oben) für die uneigentliche Gerade benützen.

Ist $\mathbf{G}$ ein Gewebe, dann ist $\#(g \setminus v)$ konstant für alle eigentlichen Geraden $g \in \mathscr{G}$ und diese Kardinalzahl heißt die *Ordnung* des Gewebes. Im weiteren sollen nur Gewebe der Ordnung >1 untersucht werden.

Ist $\mathbf{G}$ ein Gewebe des Grades $\geqq n$, dann bezeichnen wir für jedes n-Tupel voneinander verschiedener Indexe $(\iota_1, \ldots, \iota_n)$ mit $\mathbf{G}_{\iota_1,\ldots,\iota_n}$ das Gewebe $((\mathscr{P} \setminus v) \cup \{V_{\iota_1}, \ldots, V_{\iota_n}\},$ $\{g \in \mathscr{G} \setminus \{v\} \mid \exists i \in \{1, \ldots, n\}\; V_i \in g\} \cup \{\{V_{\iota_1}, \ldots, V_{\iota_n}\}\}, (V_{\iota_1}, \ldots, V_{\iota_n}))$.

Eine *zulässige Algebra* ist definiert als eine Menge M, ausgestattet mit einem ausgezeichneten Element $O \in M$, mit einem indexierten System $(\sigma_\iota)_{\iota\in J}$ von Permutationen der Menge M, wobei J eine Indexmenge, in welcher ein Index θ eine besondere Rolle spielt, und mit einem indexierten System $(+_\iota)_{\iota\in J}$ von Loopoperationen auf der Menge M, so daß die folgenden Bedingungen erfüllt sind:

(i) $$\forall \iota \in J \quad O^{\sigma_\iota} = O,$$

(ii) $$\sigma_\theta = \mathrm{id}_M,$$

(iii) $$\forall \xi, \eta \in J\,;\; \xi \neq \eta \quad \forall b, c \in M \quad \exists!\, a \in M \quad a^{\sigma_\xi} +_\xi b = a^{\sigma_\eta} +_\eta c.$$

Ist G ein Gewebe, dann erklären wir für ihr *Bezugssystem* jedes Quadrupel $(O, \alpha, \beta, \gamma)$, wo $O \in \mathscr{P} \setminus v$ und $\alpha, \beta, \gamma \in I$ mit $\alpha \neq \beta \neq \gamma \neq \alpha$ ist.

Es sei G ein Gewebe und $(O, \alpha, \beta, \gamma)$ ihr Bezugssystem. Dann ist die zulässige Algebra $(M, O, (\sigma_\iota)_{\iota\in J}, (+_\iota)_{\iota\in J})$ eindeutig bestimmt, so daß $M := OV_\alpha \setminus \{V_\alpha\}$, $J := I \setminus \{\alpha, \beta\}$; das ausgezeichnete Element aus J sei γ, für jedes $\iota \in J$ sei $\sigma_\iota : M \to M$; $x \mapsto ((xV_\beta \sqcap OV_\gamma)\, V_\alpha \sqcap OV_\iota)\, V_\beta \sqcap OV_\alpha$ und für alle $a, b \in M$ sei $a^{\sigma_\iota} +_\iota b = ((aV_\beta \sqcap OV_\gamma)\, V_\alpha \sqcap bV_\iota)\, V_\beta \sqcap OV_\alpha$ (Abb. 1–2). Das Erfülltsein sämtlicher Bedingungen aus der Definition der zulässigen Algebra kann ohne Mühe verifiziert werden. Diese Algebra nennen wir die *Koordinatenalgebra* von G (bezüglich $(O, \alpha, \beta, \gamma)$) und die Abbildung $\mu : M \times M \to \mathscr{P} \setminus v$; $(a, b) \mapsto (aV_\beta \sqcap OV_\gamma)\, V_\alpha \sqcap bV_\beta$ die *Koordinatenabbildung* (bezüglich $(O, \alpha, \beta, \gamma)$).

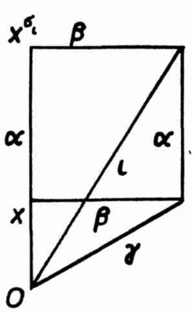

Abb. 1.

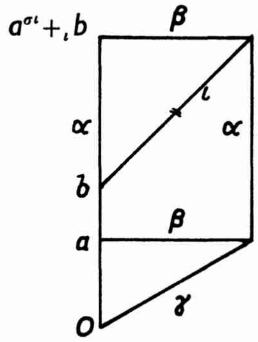

Abb. 2.

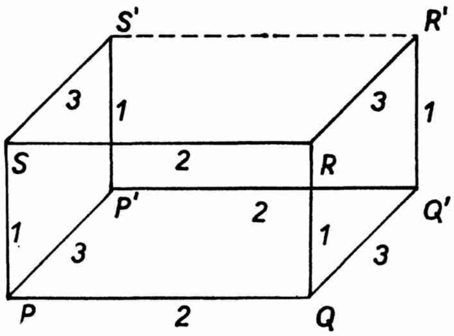

Abb. 3.

Es sei $\mathbf{A} = (M, O, (\sigma_\iota)_{\iota\in J}, (+_\iota)_{\iota\in J})$, $\#M > 1$, eine zulässige Algebra mit ausgezeichnetem Index θ. Dann bestimmen wir ein Gewebe G folgendermaßen: $I := J \cup \{\omega_1, \omega_2\}$, wo $\{\omega_1, \omega_2\}$ eine willkürliche zu J fremde zweielementige Menge, $\mathscr{P} := (M \times M) \cup I$, $v := I$, $\mathscr{G} := \{\{(x, y) \mid x = a\} \cup \{\omega_2\} \mid a \in M\} \cup \{\{(x, y) \mid y = b\} \cup \{\omega_1\} \mid b \in M\} \cup \{\{(x, y) \mid y = x^{\sigma_\iota} +_\iota c\} \cup \{\iota\} \mid c \in M,\ \iota \in J\}$.

Es ist wieder nur Routinensache zu überprüfen, daß es sich wirklich um ein Gewebe handelt. Wir nennen dieses G das Gewebe *über* $\mathbf{A}$.

Es sei G ein Gewebe des Grades 3, wo $I = \{1, 2, 3\}$. Unter der Reidemeister-Bedingung in G verstehen wir die Implikation (Abb. 3):

$$(\forall P, Q, R, S, P', Q', R', S' \in \mathscr{P} \setminus v)\,(\overline{P, Q, V_2} \,\&\, \overline{Q, R, V_1} \,\&$$
$$\&\, \overline{R, S, V_2} \,\&\, \overline{P, S, V_1} \,\&\, \overline{P', Q', V_2} \,\&\, \overline{Q', R', V_1} \,\&$$
$$\&\, \overline{P, P', V_3} \,\&\, \overline{Q, Q', V_3} \,\&\, \overline{R, R', V_3} \,\&\, \overline{S, S', V_3} \Rightarrow \overline{R', S', V_2})\,.$$

Lehrsatz 1. *Es sei G ein Gewebe und $(O, \alpha, \beta, \gamma)$ irgendeines seiner Bezugssysteme. Dann ist $+_\iota$ für $\iota \in I \setminus \{\alpha, \beta\}$ genau dann assoziativ, wenn die Reidemeister-Bedingung in $\mathsf{G}_{\alpha,\beta,\iota}$ erfüllt ist.*

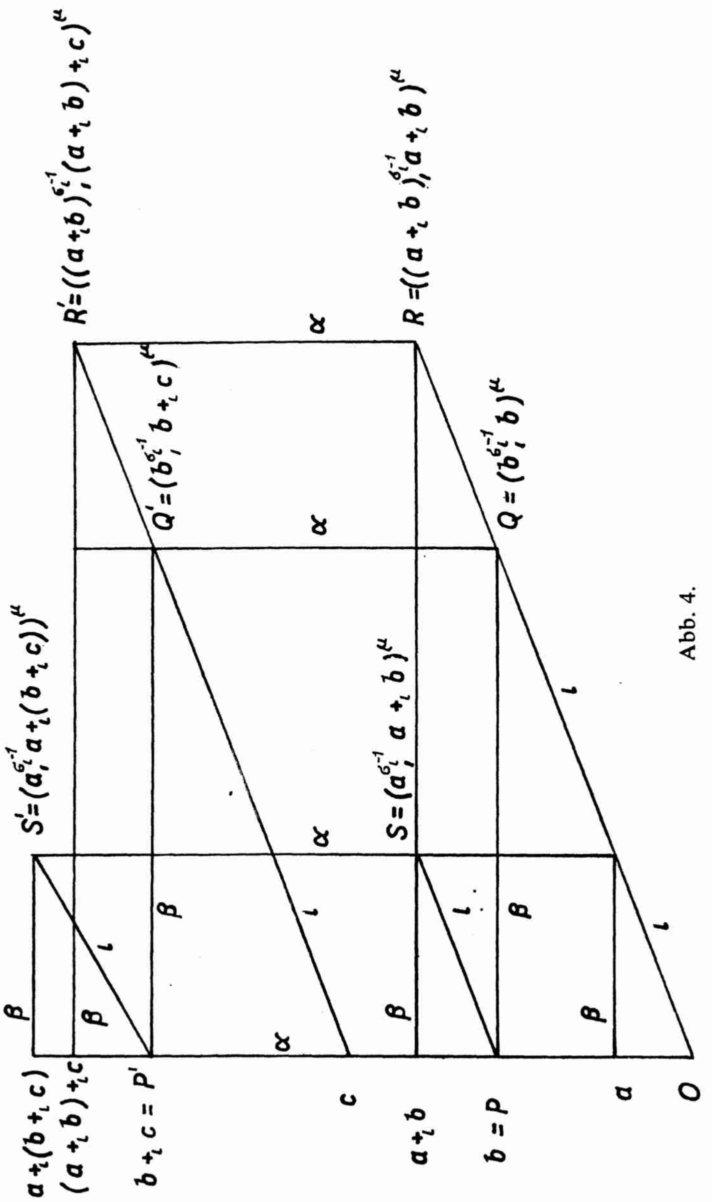

Abb. 4.

Beweis (Abb. 4). Für alle $a, b, c \in M := OV_\gamma \setminus v$ setzen wir $P := b$, $Q := (b^{\sigma_\iota^{-1}}, b)^\mu$, $R := ((a +_\iota b)^{\sigma_\iota^{-1}}, a +_\iota b)^\mu$, $S := (a^{\sigma_\iota^{-1}}, a +_\iota b)^\mu$, $R' := ((a +_\iota b)^{\sigma_\iota^{-1}}, (a +_\iota b) +_\iota c)^\mu$, $S' := (a^{\sigma_\iota^{-1}}, a +_\iota (b +_\iota c))^\mu$. Es gilt also $(a +_\iota b) +_\iota c = a +_\iota (b +_\iota c)$, genau wenn $\overline{R', S', V_\beta}$ erfüllt ist. Daraus folgt, daß $+_\iota$ genau dann assoziativ ist, wenn die Reidemeister-Bedingung in $\mathbf{G}_{\alpha,\beta,\iota}$ mit der Begrenzung $QV_\iota \sqcap PV_\gamma = O$ gilt. Es ist aber gut bekannt (siehe z. B. [7], S. 52–53), daß die Begrenzung $QV_\iota \sqcap PV_\gamma = O$ weggelassen werden kann und die Behauptung behält ihre Gültigkeit. ∎

Es sei **G** ein Gewebe des Grades $\geqq 4$ und $\alpha, \beta, \gamma, \delta$ von einander verschiedene Indexe (aus I). Unter der *Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ in **G** verstehen wir die Implikation (Abb. 5)

$$(\forall A, B, C, A', B', C' \in \mathscr{P} \setminus v)\,(\overline{A, A', V_\delta} \,\&\, \overline{B, B', V_\delta} \,\&\, \overline{C, C', V_\delta} \,\&$$
$$\&\, \overline{A, B, V_\gamma} \,\&\, \overline{A', B', V_\gamma} \,\&\, \overline{A, C, V_\beta} \,\&\, \overline{A', C', V_\beta} \,\&\, \overline{B, C, V_\alpha} \Rightarrow \overline{B', C', V_\alpha}).$$

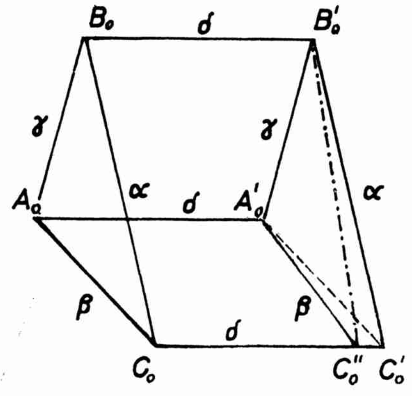

Abb. 5.

Gilt in **G** die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ für einen festen Index δ und für jede drei voneinander verschiedene Indexe $\alpha, \beta, \gamma \neq \delta$, dann sagen wir, daß in **G** die *Desargues-Bedingung des Typs* (δ) gilt.

Gilt in **G** die Desargues-Bedingung des Typs (δ) für jeden Index δ, dann sagen wir, daß in **G** die Desargues-Bedingung *universal* gilt.

Lehrsatz 2. *Es sei **G** ein Gewebe des Grades $\geqq 4$ und $\alpha, \beta, \gamma, \delta$ voneinander verschiedene Indexe.* a) *Gilt in **G** die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$, dann gilt in **G** jede Implikation, welche aus der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ durch die Vertauschung des Schlusses $\overline{B', C', V_\alpha}$ mit irgendwelcher der Voraussetzungen $\overline{A, A', V_\delta}$, $\overline{B, B', V_\delta}$, $\overline{C, C', V_\delta}$, $\overline{A, B, V_\gamma}$, $\overline{A', B', V_\gamma}$, $\overline{A, C, V_\beta}$, $\overline{A', C', V_\beta}$, $\overline{B, C, V_\alpha}$.* b) *Es gilt die zu* a) *umgekehrte Behauptung.*

Beweis des Teiles a) für den Fall der Vertauschung des Schlusses gegen die Voraussetzung $\overline{A', C', V_\beta}$ kann folgendermaßen durchgeführt werden (ähnlicherweise schreitet man auch bei Vertauschung des Schlusses gegen irgendwelche der Voraussetzungen $\overline{A, B, V_\gamma}$, $\overline{A', B', V_\gamma}$, $\overline{A, C, V_\beta}$, $\overline{B, C, V_\alpha}$ fort). Es seien $A_0, B_0, C_0, A'_0, B'_0, C'_0$ die Punkte, welche die neuen Voraussetzungen erfüllen und es gelte $\overline{A'_0, C'_0, V_\delta}$ nicht

(Abb. 6a). Die β-Gerade durch A_0' schneidet die Gerade C_0V_δ im Punkt $C_0'' \neq C_0'$ und nach der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$, die auf die Punkte A_0, B_0, C_0, A_0', B_0', C_0'' angewendet ist, bekommen wir $\overline{B_0', C_0'', V_\alpha}$, im Widerspruch mit $\overline{B_0', C_0', V_\alpha}$. Der Fall der Vertauschung des Schlusses $\overline{B', C', V_\alpha}$ gegen $\overline{C, C', V_\delta}$ (und ähnlicherweise schließt man bei der Vertauschung des Schlusses gegen irgendwelche der Voraussetzungen $\overline{A, A', V_\delta}$, $\overline{B, B', V_\gamma}$) behandelt man wie folgt (siehe Abb. 6b): Es seien A_0, B_0, C_0, A_0', B_0', C_0' Punkte, welche die neuen Vorsetzungen erfüllen,

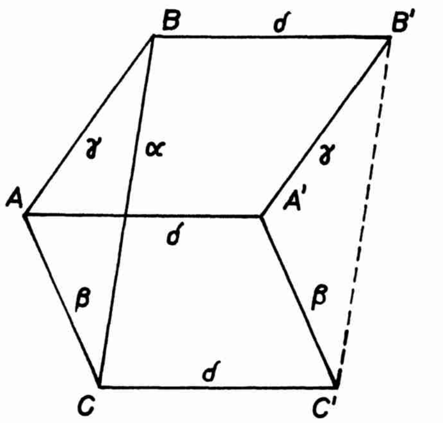

Abb. 6a.

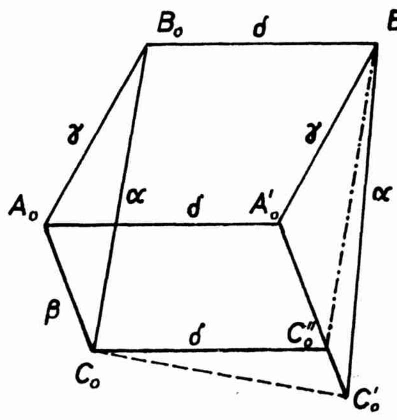

Abb. 6b.

wobei C_0, C_0', V_γ nicht gelte. Die δ-Gerade durch C_0 schneidet die Gerade $A_0'V_\beta$ im Punkt $C_0'' \neq C_0'$. Nach der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$, welche auf Punkte A_0, B_0, C_0, A_0', B_0', C_0'' angewendet ist, bekommen wir $\overline{B_0', C_0'', V_\alpha}$, im Widerspruch mit $\overline{B_0', C_0', V_\alpha}$. Der Beweis für den Teil b) ist analog. ∎

Folgerung. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$ *und* $\alpha, \beta, \gamma, \delta$ *voneinander verschiedene Indexe. Gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$, *dann gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha^\pi, \beta^\pi, \gamma^\pi, \delta^\pi)$ *für alle Permutationen* π *der Menge* $\{\alpha, \beta, \gamma, \delta\}$.

Lehrsatz 3. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$, $\alpha, \beta, \gamma, \delta$ *einander verschiedene Indexe und* O *ein eigentlicher Punkt.* a) *Gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit Begrenzung* $A = O$, *bzw.* $B = O$, *bzw.* $C = O$, *dann gilt in* **G** *jede Implikation, welche aus der Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit Begrenzung* $A = O$, *bzw.* $B = O$, *bzw.* $C = O$ *so entsteht, daß der Schluß* $\overline{B', C', V_\gamma}$ *gegen irgendwelche der Voraussetzungen* $\overline{A, A', V_\delta}$, $\overline{B, B', V_\delta}$, $\overline{C, C', V_\delta}$, $\overline{A, B, V_\gamma}$, $\overline{A', B', V_\gamma}$, $\overline{A, C, V_\beta}$, $\overline{A', C', V_\beta}$, $\overline{B, C, V_\alpha}$ *vertauscht wird.* b) *Es gilt die zu* a) *umgekehrte Behauptung.*

Beweis. Zuerst untersuchen wir die Behauptung a). Der Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen irgendwelche der Voraussetzungen $\overline{B, B', V_\gamma}$, $\overline{C, C', V_\delta}$, $\overline{A', C', V_\beta}$, $\overline{A', B', V_\gamma}$, $\overline{B, C, V_\alpha}$ kann ähnlich wie im Beweis des Lehrsatzes 2 behandelt werden, weil die Begrenzung $A = O$ keine Verletzung bedeutet.

Also widmen wir uns dem Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen $\overline{A, A', V_\delta}$ (Abb. 7). Es seien $A_0, B_0, C_0, A_0', B_0', C_0'$ Punkte, welche alle neuen Voraussetzungen erfüllen und es gelte nicht $\overline{A_0, A_0', A_\delta}$. Setzen wir $A_0'' := OV_\delta \sqcap B'V_\gamma \mp A_0$, $C_0'' := A_0''V_\beta \sqcap C_0V_\gamma \mp C_0'$. Dann erfüllen die Punkte $A_0, B_0, C_0, A_0'', B_0'', C_0''$ die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit Begrenzung $A_0 = O$, so daß $\overline{B_0', C_0'', V_\alpha}$ folgt, im Widerspruch mit $\overline{B_0', C_0', V_\alpha}$.

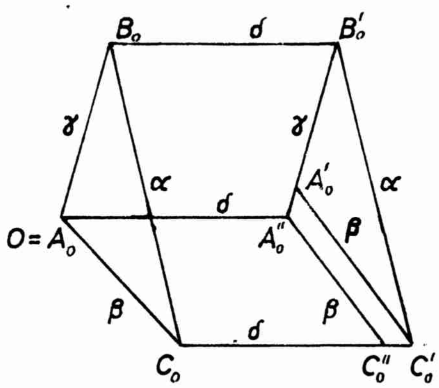

Abb. 7.

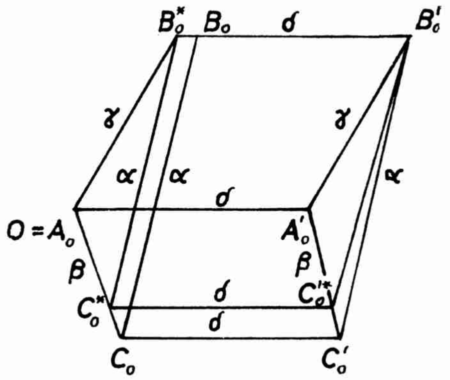

Abb. 8.

Weiter untersuchen wir den Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen $\overline{A, B, V_\gamma}$ (Abb. 8). Es seien also $A_0, B_0, C_0, A_0', B_0', C_0'$ Punkte, welche die neuen Voraussetzungen erfüllen, wobei $\overline{A_0, B_0, V_\gamma}$ nicht gelte. Setzen wir $B_0^* := OV_\gamma \sqcap B'V_\delta \mp B_0$, $C_0^* := B_0^*V_\alpha \sqcap OV_\beta \mp C_0$, $C_0'^* := C_0^*V_\delta \sqcap A_0'V_\beta \mp C_0'$, dann erfüllen die Punkte $A_0, B_0, C_0, A_0', B_0', C_0'^*$ die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit Begrenzung $A = O$, so daß $\overline{B_0', C_0'^*, V_\alpha}$ gilt, im Widerspruch mit $\overline{B', C', V_\alpha}$. Der Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen $\overline{A, C, V_\beta}$ kann ähnlicherweise behandelt werden.

Auch die Behauptung b) kann schon nur durch Modifikationen von vorangehenden Betrachtungen bewiesen werden. ■

Lehrsatz 4. *Es sei* $\mathbf{G}$ *ein Gewebe des Grades* $\geqq 4$, $\alpha, \beta, \gamma, \delta$ *einander verschiedene Indexe und* g_0 *eine* δ*-Gerade.* a) *Gilt in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit Begrenzung* $A \in g_0$, *bzw.* $B \in g_0$, *bzw.* $C \in g_0$, *dann gilt in* $\mathbf{G}$ *jede Implikation, welche aus der Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *durch Vertauschung des Schlusses* $\overline{B', C', V_\alpha}$ *gegen irgendwelche der Voraussetzungen*

$\overline{A, A', V_\delta}$, $\overline{B, B', V_\delta}$, $\overline{C, C', V_\delta}$, $\overline{A, B, V_\gamma}$, $\overline{A', B', V_\gamma}$, $\overline{A, C, V_\beta}$, $\overline{A', C', V_\beta}$, $\overline{B, C, V_\alpha}$ *mit Begrenzung* $A \in g_0$, *bzw.* $B \in g_0$, *bzw.* $C \in g_0$. b) *Es gilt die zu* a) *umgekehrte Behauptung.*

Beweis. Erstens betrachten wir den Teil a), und zwar den Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen $\overline{A', B', V_\gamma}$ bei Begrenzung $A_\gamma \in g_0$ (Abb. 9). Es seien also $A_0, B_0, C_0, A_0', B_0', C_0'$ Punkte, welche die neuen Voraussetzungen erfüllen, wobei $\overline{A_0', B_0', V_\gamma}$. Die γ-Gerade durch A_0' schneide die Gerade B_0V_δ im Punkt $B_0'' \neq B_0'$. Nun verwenden wir die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit Begrenzung $A \in g_0$ auf die Punkte $A_0, B_0, C_0, A_0', B_0'', C_0'$. Wir bekommen $\overline{B_0'', C_0', V_\gamma}$, im Widerspruch mit $\overline{B_0', C_0', V_\gamma}$. Ähnlich schließt man bei Vertauschung von $\overline{B', C', V_\alpha}$ gegen $\overline{A, B, V_\gamma}$, bzw. $\overline{A', C', V_\beta}$, bzw. $\overline{A, C, V_\beta}$ mit der Begrenzung $A \in g_0$.

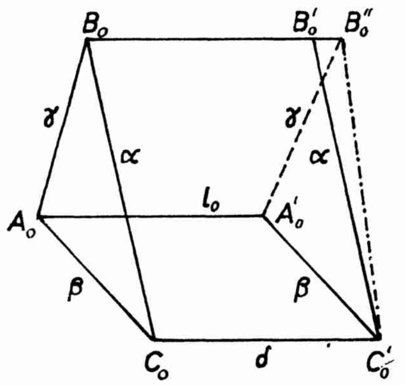

Abb. 9.

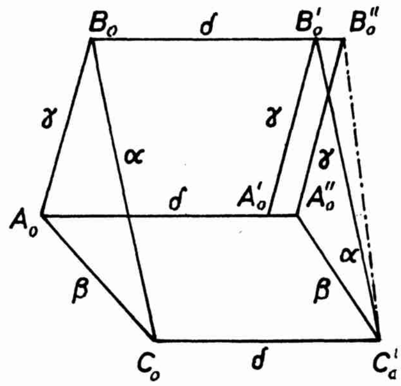

Abb. 10.

Den Fall der Vertauschung des Schlusses $\overline{B', C', V_\alpha}$ gegen $\overline{B, B', V_\delta}$ behandelt man sofort, wenn man im Auge hat, daß die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ äquivalent als die Implikation $(\forall A, B, C, A', B', C' \in \mathscr{P} \setminus v)$ $(\overline{A, B, V_\gamma} \,\&\, \overline{A, C, V_\beta} \,\&\, \overline{B, C, V_\alpha} \,\&\, \overline{A, A', V_\delta} \,\&\, \overline{C, C', V_\delta} \Rightarrow B'V_\delta, A'V_\gamma, C'V_\alpha$ gehen durch denselben Punkt) formuliert werden kann.

Nun übergehen wir zum Fall der Vertauschung des Schlusses B', C', V_α gegen $\overline{A', C', V_\beta}$ bei Begrenzung $B \in g_0$ (Abb. 10). Es seien also für die Punkte $A_0, B_0, C_0, A_0', B_0', C_0'$ die neuen Voraussetzungen erfüllt, wobei A_0', C_0', V_β nicht gelte. Weiter sei A_0'' der Schnittpunkt der β-Geraden durch C_0' und der Geraden A_0V_δ. Nun verwenden wir die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit Begrenzung $B \in g_0$ auf die Punkte $A_0, B_0, C_0, A_0'', B_0'', C_0'$. Dann folgt $\overline{B_0'', C_0', V_\alpha}$, aber dies widerspricht mit $\overline{B_0', C_0', V_\alpha}$.

Der Fall der Vertauschung von $\overline{B', C', V_\alpha}$ gegen irgendeine der übrigbleibenden Voraussetzungen bei der Begrenzung $B \in g_0$ und der Fall der Vertauschung von

B', C', V_α gegen irgendwelche der Voraussetzungen bei der Begrenzung $C \in g_0$ kann man immer auf irgendeinen vorher behandelten Fall überführen. Die Behauptung b) beweist man ähnlicherweise. ■

Folgerung. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$, $\alpha, \beta, \gamma, \delta$ *voneinander verschiedene Indexe und* g_0 *eine* δ*-Gerade. Gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Einschränkung* $A \in g_0$, *bzw.* $B \in g_0$, *bzw.* $C \in g_0$, *dann gilt die Desargues-Bedingung des Typs* $(\alpha^\pi, \beta^\pi, \gamma^\pi, \delta^\pi)$ *in* **G** *mit der Einschränkung, daß irgendwelcher der Punkte* A, B, C *auf* g_0 *liegt, und zwar für jede Permutation* π *der Menge* $\{\alpha, \beta, \gamma, \delta\}$.

Den Sonderfall der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ aus der vorherigen Folgerung werden wir als *Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta, g_0)$ bezeichnen. Gilt in **G** die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta, g_0)$ für einen festen Index δ, eine feste δ-Gerade g und alle Indexe α, β, γ, so daß $\alpha, \beta, \gamma, \delta$ voneinander verschieden ist, so sagen wir, daß in **G** die *Desargues-Bedingung des Typs* (δ, g) gilt.

Lehrsatz 5. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$, $\alpha, \beta, \gamma, \delta$ *voneinander verschiedene Indexe und* O *ein eigentlicher Punkt. Gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Einschränkung* $A = O$, *bzw.* $B = O$, *bzw.* $C = O$, *dann gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta, OV_\delta)$.

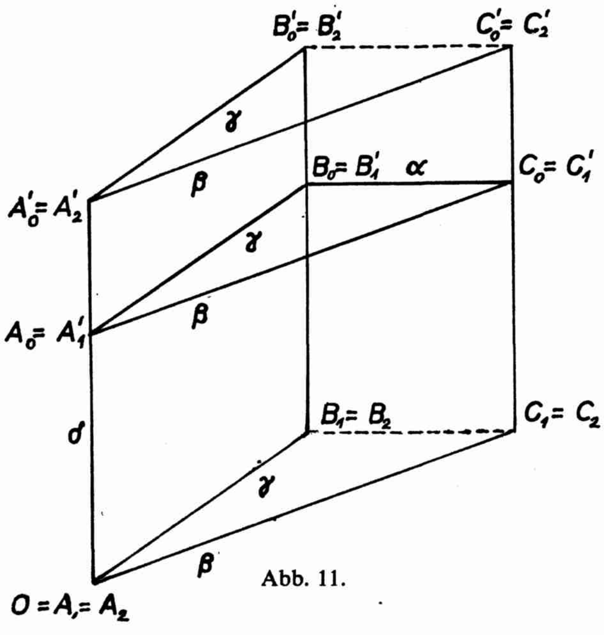

Abb. 11.

B e w e i s. Es seien also $A_0, B_0, C_0, A'_0, B'_0, C'_0$ die Punkte, welche die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ bei der Einschränkung $A \in OV_\gamma$ erfüllen (Abb. 11). Die Punkte $A_1 := O$, $B_1 := OV_\gamma \sqcap B_0V_\delta$, $C_1 := OV_\beta \sqcap C_0V_\delta$,

$A_1' := A_0$, $B_1' := B_0$, $C_1' := C_0$ erfüllen die Voraussetzungen der Implikation, welche aus der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit der Begrenzung $A = O$ durch die Verwechslung des Schlusses $\overline{B', C', V_\alpha}$ mit der Voraussetzung $\overline{B, C, V_\alpha}$ entsteht. Auf Grund des Lehrsatzes 3 bekommen wir also $\overline{B_1, C_1, V_\alpha}$. Die Punkte $A_2 := O$, $B_2 := B_1$, $C_2 := C_1$, $A_2' := A_0$, $B_2' := B_0'$, $C_2' := C_0'$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit der Einschränkung $A = O$, so daß $\overline{B_2', C_2', V_\alpha}$, d. h. $\overline{B_0', C_0', V_\alpha}$ folgt. Der Rest folgt schon nach dem Lehrsatz 3. Ähnlich schließt man bei der Einschränkung $B = O$, bzw. $C = O$. ■

Bemerkung. Es sei **G** ein Gewebe, $\alpha, \beta, \gamma, \delta$ voneinander verschiedene Indexe und O ein eigentlicher Punkt. Gilt in **G** die Implikation, welche aus der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ mit der Einschränkung $A = O$, bzw. $B = O$, bz$_2$. $C = O$ durch die Verwechslung des Schlusses $\overline{B', C', V_\alpha}$ mit irgendwelcher der Voraussetzungen $\overline{A, A', V_\delta}$, $\overline{B, B', V_\delta}$, $\overline{C, C', V_\delta}$, $\overline{A, B, V_\gamma}$, $\overline{A', B', V_\gamma}$, $\overline{A, C, V_\beta}$, $\overline{A', C', V_\beta}$, $\overline{B, C, V_\gamma}$ entsteht, dann gilt in **G** dieselbe Implikation mit schwacher Einschränkung $A \in OV_\delta$, bzw. $B \in OV_\delta$, bzw. $C \in OV_\delta$.

Der Beweis ist ähnlich durchführbar wie beim Lehrsatz 5. ■

Satz 1. *Es sei **G** ein Gewebe des Grades $\geqq 4$, β, γ, δ voneinander verschiedene Indexe und g eine δ-Gerade. Gilt in **G** die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta, g)$ für alle Indexe $\alpha \in I \setminus \{\beta, \gamma, \delta\}$, dann gilt in **G** die Desargues-Bedingung des Typs (δ, g).*

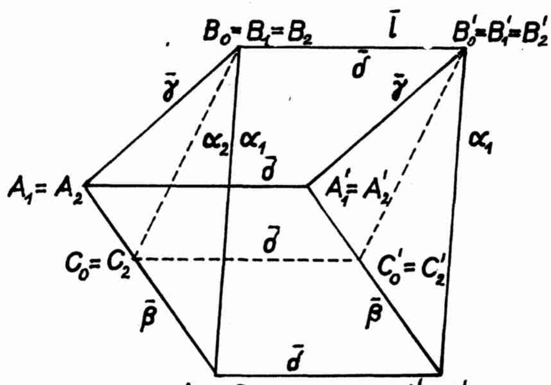

Abb. 12.

Beweis. Es gelte also in **G** die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta, g)$ für jeden Index $\alpha \in I \setminus \{\beta, \gamma, \delta\}$. Ohne Einschränkung der Allgemeinheit sei $B \in g$. Nehmen wir willkürliche Indexe $\alpha_1, \alpha_2 \in I \setminus \{\beta, \gamma, \delta\}$; $\alpha_1 \neq \alpha_2$ und Punkte $A_1, B_1, C_1, A_1', B_1', C_1'$, welche die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha_1, \beta, \gamma, \delta)$ mit Begrenzung $B \in g$ erfüllen (Abb. 12). Es folgt $\overline{B_1', C_1', V_{\alpha_1}}$. Weiter setzen wir

$A_2 := A_1$, $B_2 := B_1$, $C_2 := B_1 V_{\alpha_2} \sqcap AV_\beta$, $A'_2 := A'_1$, $B'_2 := B'_1$, $C'_2 := C_2 V_\delta \sqcap \sqcap A'_1 V_\beta$. Diese Punkte erfüllen die Desargues-Bedingung des Typs $(\alpha_2, \beta, \gamma, \delta)$ mit Begrenzung $B \in g$. Die Punkte $A_0 := C_1$, $B_0 := B_1$, $C_0 := C_2$, $A'_0 := C'_1$, $B'_0 := B'_1$, $C'_0 := C'_2$ erfüllen dann die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha_2, \beta, \alpha_1, \delta)$ mit der Einschränkung $B \in g$, so daß B'_2, C'_2, V_{α_2} folgt, was auch als $\overline{B'_0, C'_0, V_{\alpha_2}}$ geschrieben werden kann. Es folgt also die Gültigkeit der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta, g)$ in $\mathbf{G}$ für alle Indexe α, γ, welche voneinander und auch von β, γ verschieden sind. Nach Folgerung des Lehrsatzes 4 ist die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ in $\mathbf{G}$ der Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta)$ in $\mathbf{G}$ unter der Begrenzung $C \in g$ äquivalent. Wählen wir also willkürliche

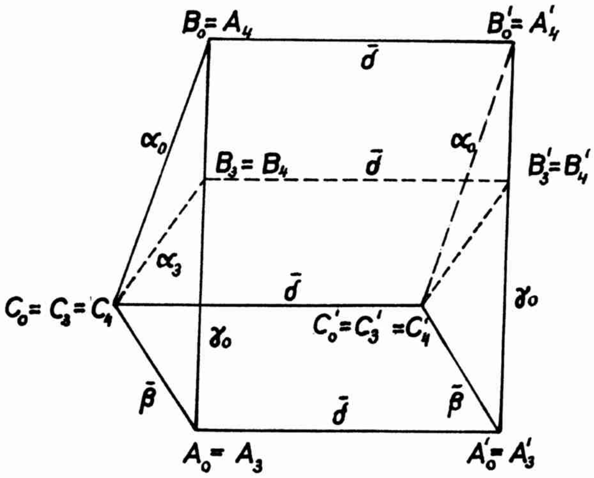

Abb. 13.

Indexe α_0, γ_0 in solcher Weise, daß $\alpha_0, \beta, \gamma_0, \delta$ voneinander verschieden sind und nehmen wir Punkte $A_0, B_0, C_0, A'_0, B'_0, C'_0$, welche die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha_0, \beta, \gamma_0, \delta)$ in $\mathbf{G}$ mit der Begrenzung $C \in g$ erfüllen (Abb. 13). Dann folgt $\overline{B'_0, C'_0, V_{\alpha_0}}$. Weiter sei $\alpha_3 \in I \setminus \{\beta, \gamma_0, \delta\}$. Die Punkte $A_3 := A_0$, $B_3 := C_0 V_{\alpha_3} \sqcap A_0 V_{\gamma_0}$, $C_3 := C_0$, $A'_3 := A'_0$, $B'_3 := B_3 V_\delta \sqcap A'_0 V_{\gamma_0}$, $C'_3 := C'_0$ erfüllen dann die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha_3, \beta, \gamma_0, \delta)$ in $\mathbf{G}$ mit Begrenzung $C \in g$, so daß sich $\overline{B'_3, C'_3, V_{\alpha_3}}$ ergibt. Andererseits erfüllen die Punkte $A_4 := B_0$, $B_4 := B_3$, $C_4 := C_3$, $A'_4 := B'_0$, $B'_4 := B_3$, $C'_4 := C'_3$ die Voraussetzungen der Desargues-Bedingung des Typs $(\alpha_3, \alpha_0, \gamma_0, \delta)$ bei der Einschränkung $C \in g$. Nachdem $\overline{B'_3, C'_3, V_{\alpha_3}}$ auch als $\overline{B'_4, C'_4, V_{\alpha_3}}$ geschrieben werden kann, gilt in $\mathbf{G}$ die Desargues-Bedingung des Typs $(\alpha, \beta, \gamma, \delta, g)$ für alle $\alpha, \beta, \gamma \in I \setminus \{\delta\}$, für welche $\alpha, \beta, \gamma, \delta$ voneinander verschieden sind, d. h. es gilt in $\mathbf{G}$ die Desargues-Bedingung des Typs (δ, g). ■

Folgerung.*) *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$ *und* β, γ, δ *voneinander verschiedene Indexe. Gilt in* **G** *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *für jedes* $\alpha \in I \setminus \{\beta, \gamma, \delta\}$, *dann gilt in* **G** *die Desargues-Bedingung des Typs* (δ).

Lehrsatz 6. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$ *und* δ *irgendein Index. Gilt in* **G** *die Desargues-Bedingung des Typs* (δ), *dann gilt die Reidemeister-Bedingung in* $\mathbf{G}_{\delta,\eta,\zeta}$ *für alle* $\eta, \zeta \in I \setminus \{\delta\}$; $\eta \neq \zeta$.

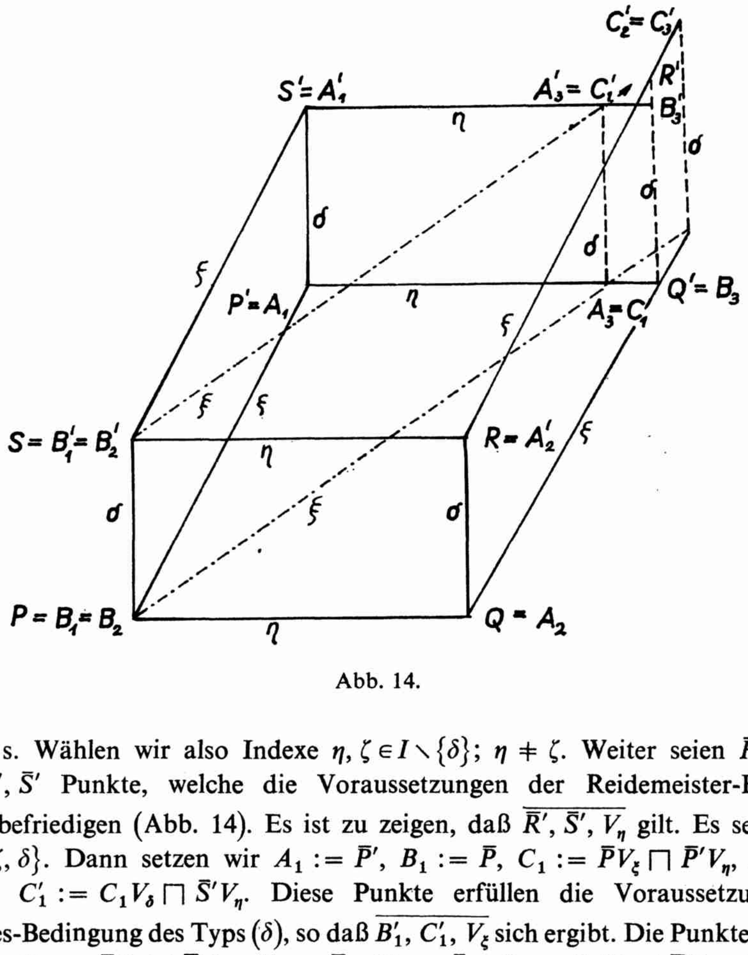

Abb. 14.

Beweis. Wählen wir also Indexe $\eta, \zeta \in I \setminus \{\delta\}$; $\eta \neq \zeta$. Weiter seien $\bar{P}, \bar{Q}, \bar{R}, \bar{S}, \bar{P}', \bar{Q}', \bar{R}', \bar{S}'$ Punkte, welche die Voraussetzungen der Reidemeister-Bedingung in $\mathbf{G}_{\delta,\eta,\zeta}$ befriedigen (Abb. 14). Es ist zu zeigen, daß $\overline{\bar{R}', \bar{S}', V_\eta}$ gilt. Es sei also $\xi \in I \setminus \{\eta, \zeta, \delta\}$. Dann setzen wir $A_1 := \bar{P}'$, $B_1 := \bar{P}$, $C_1 := \bar{P}V_\xi \sqcap \bar{P}'V_\eta$, $A_1' := \bar{S}'$, $B_1' := \bar{S}$, $C_1' := C_1 V_\delta \sqcap \bar{S}'V_\eta$. Diese Punkte erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (δ), so daß $\overline{B_1', C_1', V_\xi}$ sich ergibt. Die Punkte $A_2 := \bar{Q}$, $B_2 := \bar{P}$, $C_2 := \bar{P}V_\xi \sqcap \bar{Q}V_\zeta$, $A_2' := \bar{Q}$, $B_2' := \bar{S}$, $C_2' := C_2 V_\delta \sqcap \bar{R}V_\zeta$ erfüllen die Voraussetzungen der Desargues-Bedingungen des Typs (δ), so daß sich $\overline{B_2', C_2', V_\xi}$ ergibt. Auch die Punkte $A_3 := C_1$, $B_3 := \bar{Q}'$, $C_3 := C_2$, $A_3' := C_1'$, $B_3' := \bar{Q}'$, $C_3' := C_2'$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (δ), so daß $\overline{B_3', C_3', V_\zeta}$ hervorgeht. Also gilt auch $\bar{R}' = B_3'$ und endlich $\overline{\bar{R}', \bar{S}', V_\eta}$. ■

*) Wurde schon in [3], S. 42—43, bewiesen.

Lehrsatz 7. *Es sei* $\mathbf{G}$ *ein Gewebe mit* $I = \{1, 2, 3, 4\}$ *und* g *eine 1-Gerade. Weiter gelte in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(1, g)$ *und in* $\mathbf{G}_{1,2,3}$, $\mathbf{G}_{1,2,4}$ *die Reidemeister-Bedingung. Dann gilt in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* (1).

Beweis (Abb. 15). Es seien $A_0, B_0, C_0, A'_0, B'_0, C'_0$ Punkte, welche die Voraussetzungen der Desargues-Bedingung des Typs (1) in $\mathbf{G}$ befriedigen. Dann erfüllen die Punkte $A_1 := A_0V_2 \sqcap g$, $B_1 := B_0V_2 \sqcap A_1V_3$, $C_1 := C_0V_2 \sqcap A_1V_3$, $A'_1 := A'_0V_2 \sqcap g$, $B'_1 := B_0V_2 \sqcap A'_1V_3$, $C'_1 := C'_0V_2 \sqcap A'_1V_3$ die Voraussetzungen der Desargues-Bedingung des Typs (1) mit der Einschränkung $A \in g$, so daß $\overline{B'_1, C'_1, V_2}$

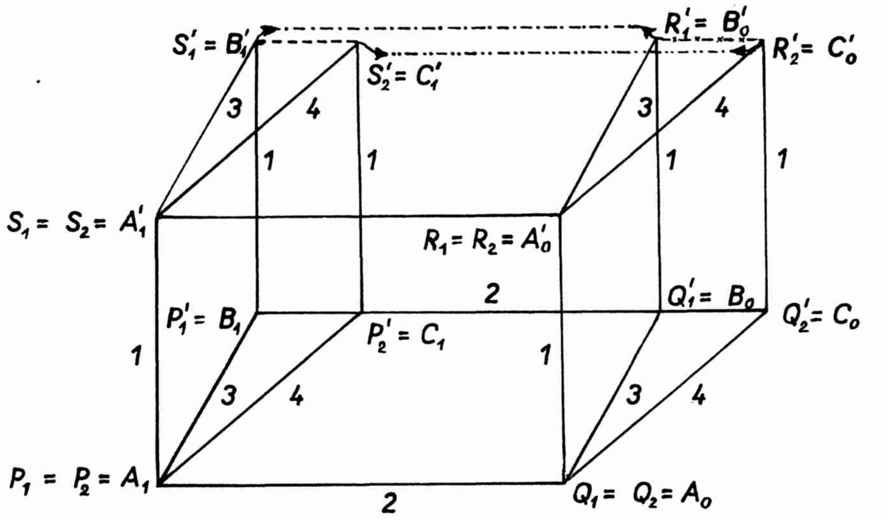

Abb. 15.

hervorgeht. Die Punkte $P_1 := A_1$, $Q_1 := A_0$, $R_1 := A'_0$, $S_1 := A'_1$, $P'_1 := B_1$, $Q'_1 := B_0$, $R'_1 := B'_0$, $S'_1 := B'_1$ befriedigen die Voraussetzungen der Reidemeister-Bedingung in $\mathbf{G}_{1,2,3}$, so daß $\overline{R'_1, S'_1, V_2}$ folgt. Also liegen die Punkte B'_1, C'_1, B_0 auf derselben 2-Geraden. Die Punkte $P_2 := A_1$, $Q_2 := A_0$, $R_2 := A'_0$, $S_2 := A'_1$, $P'_2 := C_1$, $Q'_2 := C_0$, $R'_2 := C'_0$, $S'_2 := C'_1$ erfüllen die Voraussetzungen der Reidemeister-Bedingung in $\mathbf{G}_{1,2,4}$, so daß sich $\overline{R'_2, S'_2, V_2}$ ergibt. Folglich liegt auch der Punkt C'_0 mit den Punkten B'_1, C'_1, B'_0 auf derselben 2-Geraden, so daß auch $\overline{B'_0, C'_0, V_2}$ gilt. Daraus ergibt sich schon die Gültigkeit der Desargues-Bedingung des Typs (1) in $\mathbf{G}$. ∎

Lehrsatz 8. *Es sei* $\mathbf{G}$ *ein Gewebe mit* $I := \{1, 2, 3, 4\}$ *und* $(O, 1, 2, 3)$ *eines seiner Bezugssysteme. Dann gilt in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(1, OV_1)$ *genau dann, wenn die beiden Operationen* $+_3$, $+_4$ *der Koordinatenalgebra bezüglich* $(O, 1, 2, 3)$ *zusammenfallen.*

Beweis (Abb. 16). Für willkürliche Elemente $a, b \in S := OV_1 \setminus \{V_1\}$ setzen wir $A = O$, $B = (a^{\sigma_3^{-1}}, a)^\mu$, $C = (a^{\sigma_4^{-1}}, a)^\mu$, $A' = b$, $B' = (a^{\sigma_3^{-1}}, a +_3 b)^\mu$, $C' = (a^{\sigma_4^{-1}}, a +_4 b)^\mu$. Die Punkte A, B, C, A', B', C' erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (1) in **G** mit der Einschränkung $A = O$ und $a +_3 b = a +_4 b$ gilt genau dann, wenn $\overline{B', C', V_1}$ gilt. Daraus ergibt sich die Äquivalenz zwischen $+_3 = +_4$ und der Desargues-Bedingung des Typs (1) in **G** mit der Einschränkung $A = O$. Nach Lehrsatz 2 ist in **G** die Desargues-Bedingung des Typs (1) mit der Einschränkung $A = O$ mit der Desargues-Bedingung des Typs (1) mit der Einschränkung $A \in OV_1$ äquivalent und nach der Folgerung des Lehrsatzes 4 ist in **G** die Desargues-Bedingung des Typs (1) mit der Einschränkung $A \in OV_1$ und sogar mit der Desargues-Bedingung des Typs $(1, OV_1)$ äquivalent. ■

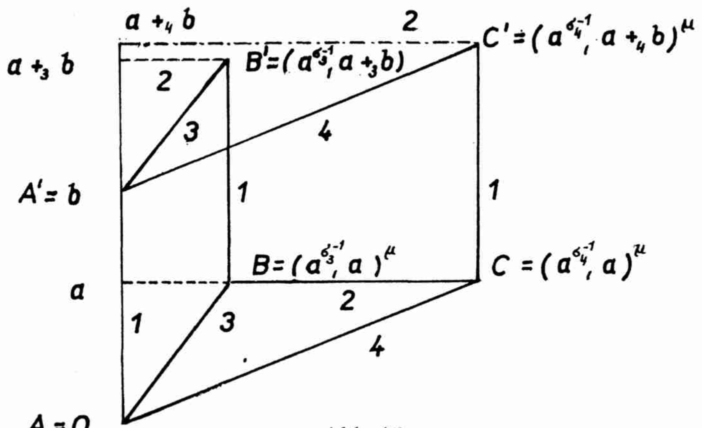

Abb. 16.,

Lehrsatz 9. *Es sei* **G** *ein Gewebe mit* $I = \{1, 2, 3, 4\}$, $(O, 1, 2, 3)$ *eines seiner Bezugssysteme und* $+_3$, $+_4$ *die beiden binären Operationen der Koordinatenalgebra bezüglich* $(O, 1, 2, 3)$. *Dann gilt in* **G** *die Desargues-Bedingung des Typs* (1) *genau dann, wenn* $+_3$ *eine Gruppenoperation ist, die mit* $+_4$ *zusammenfällt.*

Beweis. Nach dem Lehrsatz 8 ist $+_3 = +_4$ mit der Geltung der Desargues-Bedingung des Typs $(1, OV_1)$ in **G** äquivalent. Nach dem Lehrsatz 1 ist die Operation $+_3$, bzw. $+_4$ genau dann eine Gruppenoperation, wenn in $\mathbf{G}_{1,2,3}$, bzw. in $\mathbf{G}_{1,2,4}$ die Reidemeister-Bedingung erfüllt ist. Nach den Lehrsätzen 6–7 gilt in **G** die Desargues-Bedingung des Typs $(1, OV_1)$ zusammen mit der Reidemeister-Bedingung in $\mathbf{G}_{1,2,3}$ und $\mathbf{G}_{1,2,4}$, genau wenn in **G** die Desargues-Bedingung des Typs (1) gilt. Daraus ist die weitere Beweisführung klar. ■

Satz 2. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$ *und* $(O, \alpha, \beta, \gamma)$ *eines seiner Bezugssysteme. Dann gilt in* **G** *die Desargues-Bedingung des Typs* (α), *genau wenn die* *binäre Operation* $+_\gamma$ *der Koordinatenalgebra bezüglich* $(O, \alpha, \beta, \gamma)$ *eine Gruppenoperation ist und mit* $+_\iota$ *für jedes* $\iota \in I \setminus \{\alpha, \beta\}$ *zusammenfällt.*

Beweis. Wir wenden den Lehrsatz 9 auf jedes Gewebe $\mathbf{G}_{1,2,\iota}$, $\iota \in I \setminus \{\alpha, \beta\}$, an und gebrauchen noch die Folgerung des Satzes 1. ■

Lehrsatz 10. *Es sei* $\mathbf{G}$ *ein Gewebe mit* $I = \{1, 2, 3, 4\}$ *und* $(O, 1, 2, 3)$ *eines seiner Bezugssysteme. Weiter gelte in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(1, OV_1)$. *Dann gilt in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(2, OV_2)$ *genau dann, wenn die unäre Operation* σ_4 *der Koordinatenalgebra bezüglich* $(O, 1, 2, 3)$ *ein Automorphismus von* $(M, +)$ *ist, wobei* $M := OV_1 \setminus \{V_1\}$, $+ := +_3 = +_4$.

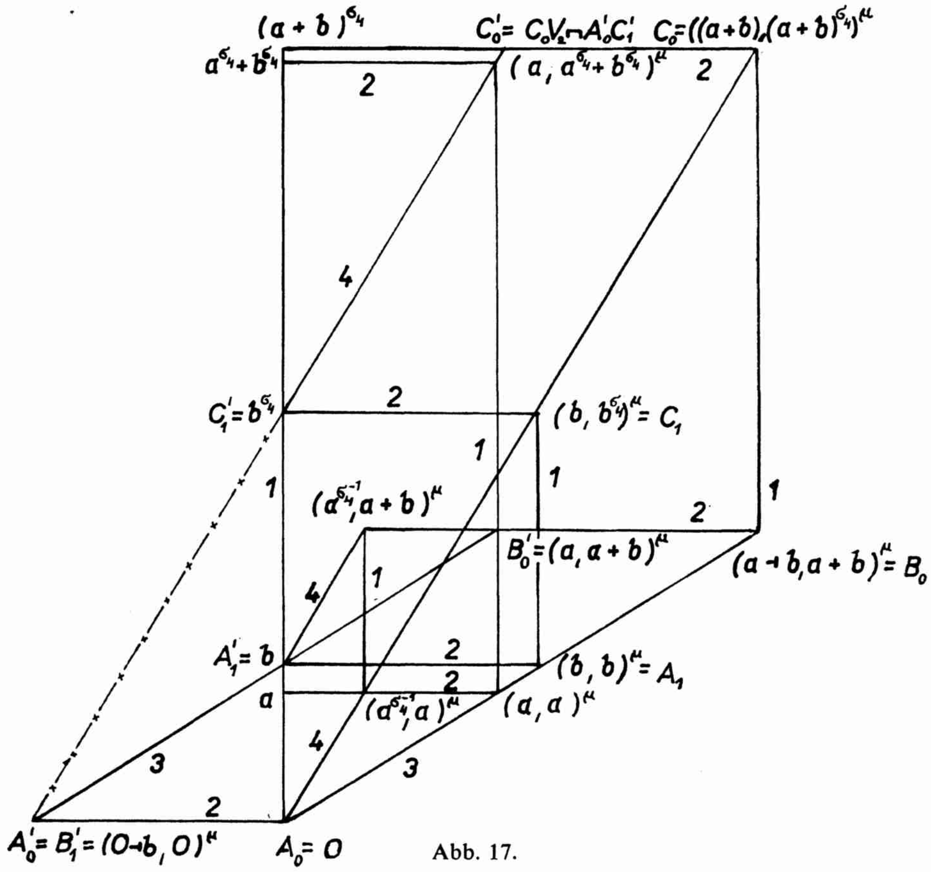

Abb. 17.

Beweis (Abb. 17).*) Es sei also $\mathbf{G}$ das gegebene Gewebe, das außer den Voraussetzungen des Satzes noch die Desargues-Bedingung des Typs $(2, OV_2)$ erfüllt.

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*) Im Beweis bedeutet das Symbol $\dashv$ das folgende: $x + y = z \Leftrightarrow x =: z \dashv y$ bezüglich des Loops $(M, +)$. Ähnlicherweise setzen wir $x + y = z \Leftrightarrow y := x \vdash z$.

Weiter seien a, b beliebige Elemente aus M. Die Punkte $A_1 := (b, b)^\mu$, $B_1 := O$, $C_1 := (b, b^{\sigma_4})^\mu$, $A'_1 := b$, $B'_1 := (O \dashv b, O)^\mu$, $C'_1 := b^{\sigma_4}$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs $(2, OV_2)$ in **G**, so daß $\overline{B'_1, C'_1, V_4}$ folgt. Die Punkte $A_0 := O$, $B_0 := (a + b, \ a + b)^\mu$, $C_0 := (a + b, \ (a + b)^{\sigma_4})^\mu$, $A'_0 := (O \dashv b, O)^\mu$, $B'_0 := (a, a + b)^\mu$, $C'_0 := (a, a^{\sigma_4} + b^{\sigma_4})^\mu$ erfüllen die Voraussetzungen der Implikation, welche aus der Desargues-Bedingung des Typs $(2, OV_2)$ entsteht, indem man den Schluß $\overline{B', C', V_1}$ mit $\overline{C, C', V_4}$ umtauscht. Also folgt nach dem Lehrsatz 3 $\overline{C_0, C'_0, V_2}$, d. h. $C'_0 = (a, a^{\sigma_4} + b^{\sigma_4})^\mu$, $(a + b)^{\sigma_4} = a^{\sigma_4} + b^{\sigma_4}$. Folglich ist σ_4 der geforderte Automorphismus.

Umgekehrt sei σ_4 ein Automorphismus des Loops $(M, +)$. Dann gilt für jedes $a \in M$ auch $(O \dashv a)^{\sigma_4} = O \dashv a^{\sigma_4}$, so daß sich für jedes $b \in M$ die Geraden bV_3, $b^{\sigma_4}V_4$ im Punkt $(O \dashv b, O)^\mu$ durchschneiden. Setzen wir also $A_0 := O$, $B_0 := (a + b, a + b)^\mu$, $C_0 := (a, (a + b)^{\sigma_4})^\mu$, $A'_0 := (O \dashv b, O)^\mu$, $B'_0 := (a, a + b)^\mu$, $C'_0 := (a, (a + b)^{\sigma_4})^\mu$, dann erfüllen diese Punkte die Voraussetzungen der Umkehrung der Desargues-Bedingung des Typs (2) in **G** mit Begrenzung $A = O$, die durch Umtauschen von $\overline{B', C', V_1}$ gegen $\overline{C, C', V_4}$ entsteht. Aus $(a + b)^{\sigma_4} = a^{\sigma_4} + b^{\sigma_4}$ folgt dann sofort $\overline{C_0, C'_0, V_4}$. Nach Lehrsatz 3 gilt also in **G** auch die Desargues-Bedingung des Typs (2) mit Begrenzung $A = O$ und nach Lehrsatz 5 auch die Desargues-Bedingung des Typs $(2, OV_2)$ in **G**. ■

Lehrsatz 11. *Es sei **G** ein Gewebe mit $I = \{1, 2, 3, 4\}$ und $(O, 1, 2, 3)$ eines seiner Bezugssysteme. Weiter gelte in **G** die Desargues-Bedingung des Typs* (1). a) *Dann gilt in **G** sogar die Desargues-Bedingung des Typs* (2). b) *Die unäre Operation σ_4 der Koordinatenalgebra bezüglich $(O, 1, 2, 3)$ ist ein Automorphismus der Gruppe $(M, +)$, wo $M := OV_1 \setminus \{V_1\}$, $+ := +_3$, genau dann, wenn in **G** die Desargues-Bedingung des Typs* (2) *gilt.*

Beweis. Es sei also **G** das Gewebe, welches die Voraussetzungen des Lehrsatzes befriedigt. a) Überdies gelte in **G** die Desargues-Bedingung des Typs (1). Dann gilt nach Lehrsatz 6 in $\mathbf{G}_{1,2,3}$ und $\mathbf{G}_{1,2,4}$ die Reidemeister-Bedingung. Aus der Geltung der Reidemeister-Bedingung in $\mathbf{G}_{1,2,3}$ und $\mathbf{G}_{1,2,4}$ und der Desargues-Bedingung des Typs $(2, OV_2)$ in **G** folgt dann nach Lehrsatz 7 auch die Gültigkeit der Desargues-Bedingung des Typs (2) in **G**. b) Es gelte in **G** außer den Voraussetzungen des Lehrsatzes noch die Desargues-Bedingung des Typs (1). Wegen Lehrsätzen 9–10 ist dann die binäre Operation $+ := +_3$ der Koordinatenalgebra bezüglich $(O, 1, 2, 3)$ eine Gruppenoperation, die mit $+_4$ zusammenfällt und σ_4 ein Automorphismus der Gruppe $(M, +)$, genau dann, wenn in **G** die Desargues-Bedingung des Typs $(2, OV_2)$ gilt, was aber nach Lehrsatz 11a) sogar mit der Geltung der Desargues-Bedingung des Typs (2) in **G** äquivalent ist.

Satz 3. *Es sei **G** ein Gewebe des Grades ≥ 4 und $(O, \alpha, \beta, \gamma)$ eines seiner Bezugssysteme. In **G** gelte weiter die Desargues-Bedingung des Typs (α), so daß also in* *der Koordinatenalgebra bezüglich $(O, \alpha, \beta, \gamma)$ eine Gruppenoperation ist, welche für jedes $\iota \in I \setminus \{\alpha, \beta\}$ mit $+_\iota$ zusammenfällt. Dann gilt in* **G** *die Desargues-Bedingung des Typs (β) genau dann, wenn für jedes $\iota \in I \setminus \{\alpha, \beta\}$ die Operation σ_ι ein Automorphismus der Gruppe $(OV_\alpha \setminus \{V_\alpha\}, +_\gamma)$ ist.*

Beweis. Folgt nach Verwendung des Lehrsatzes 11 auf jedes $\mathbf{G}_{\alpha,\beta,\iota}$, $\iota \in I \setminus \{\alpha, \beta\}$ und der Folgerung des Satzes 1. ■

Lehrsatz 12. *Es sei* **G** *ein Gewebe mit $I = \{1, 2, 3, 4\}$ und $(O, 1, 2, 3)$ eines seiner Bezugssysteme. Weiter gelte in* **G** *die Desargues-Bedingung des Typs (1) und (2). Dann gilt in* **G** *die Desargues-Bedingung des Typs $(3, OV_3)$, genau wenn die Gruppe $(M, +)$ abelsch ist, wobei sich die Bezeichnung $M := OV_1 \setminus \{V_1\}$, $+ := +_3$ auf die Koordinatenalgebra bezüglich $(O, 1, 2, 3)$ bezieht.*

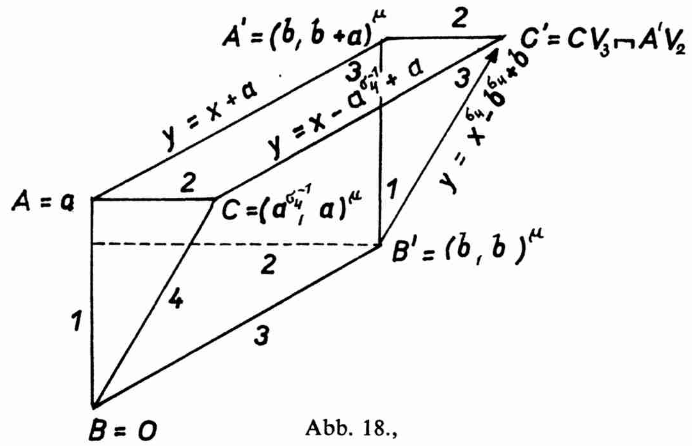

Abb. 18.,

Beweis (Abb. 18). Bei gegebenen Voraussetzungen ist $+$ eine Gruppenoperation und σ_4 ist ein Automorphismus der Gruppe $(M, +)$. Untersuchen wir also willkürliche Elemente $a, b \in M$. Die Punkte $A := a$, $B := O$, $C := (a^{\sigma_4^{-1}}, a)^\mu$, $A' := (b, b + a)^\mu$, $B' := (b, b)^\mu$, $C' := C_1V_3 \sqcap A'_1V_2 = (c, b + a)^\mu$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (3) in **G** mit der Einschränkung $B = O$ und $\overline{B', C', V}$ gilt genau dann, wenn $b + a = c^{\sigma_4} - b^{\sigma_4} + b$, was mit $b + a = c - a^{\sigma_4^{-1}} + a$ und weiter mit $c = b + a^{\sigma_4^{-1}}$ äquivalent ist. Mit Verwendung der Eigenschaft, daß σ_4 ein Automorphismus der Gruppe $(M, +)$ ist, kann man die letzte Gleichung als $c^{\sigma_4} = b^{\sigma_4} + a$ schreiben. Nach Einsetzen in die erste Gleichung bekommt man $b + a = b^{\sigma_4} + a + (-b^{\sigma_4} + b)$ und weiter $a = (-b + b^{\sigma_4}) + a + (-b^{\sigma_4} + b)$. Umgekehrt folgen aus dieser Gleichung mit $c := b + a^{\sigma_4^{-1}}$ die Gleichungen $b + a = c^{\sigma_4} - b^{\sigma_4} + b = c - a^{\sigma_4^{-1}} + a$. Nun setzen wir $d := -b + b^{\sigma_4}$, d. h. $b + d = b^{\sigma_4}$; umgekehrt gibt es zu jedem $d \in M$ genau ein $b \in M$ mit

$b + d = b^{\sigma_4}$ nach der Eigenschaft (ii) aus der Definition der zulässigen Algebra. Also kann man $a = (-b + b^{\sigma_4}) + a + (-b^{\sigma_4} + b)$ $\forall a, b \in M$ äquivalent als $a = d + a - d$ $\forall a, d \in M$ schreiben. Da nach dem Lehrsatz 5 in $\boldsymbol{G}$ die Desargues-Bedingung des Typs (3) mit Begrenzung der $B = O$ mit der Desargues-Bedingung des Typs $(3, OV_3)$ äquivalent ist, ist der Beweis beendet. ∎

Bemerkung. Der Lehrsatz 7 kann auch folgendermaßen modifiziert werden: Es sei $\boldsymbol{G}$ ein Gewebe mit $I = \{1, 2, 3, 4\}$, welches die Desargues-Bedingung des Typs (1) und (2) erfüllt. Gilt in $\boldsymbol{G}$ darüber hinaus noch die Desargues-Bedingung des Typs $(3, g_0)$ für eine 3-Gerade g_0, dann gilt in $\boldsymbol{G}$ die Desargues-Bedingung des Typs (3). Der Lehrsatz 12 gilt also auch, indem man die Desargues-Bedingung des Typs $(3, OV_3)$ durch die Desargues-Bedingung des Typs (3) ersetzt.

Satz 4. *Es sei $\boldsymbol{G}$ ein Gewebe des Grades $\geqq 4$, $(O, \alpha, \beta, \gamma)$ eines seiner Bezugssysteme und es gelte in $\boldsymbol{G}$ die Desargues-Bedingung des Typs (α) und (β), so daß also die binäre Operation $+_\gamma$ der Koordinatenalgebra bezüglich $(O, \alpha, \beta, \gamma)$ eine Gruppenoperation ist, die für jedes $\iota \in I \setminus \{\alpha, \beta\}$ mit $+_\iota$ zusammenfällt. Die Operation $+_\gamma$ ist überdies genau dann kommutativ, wenn in $\boldsymbol{G}$ die Desargues-Bedingung des Typs $(\alpha, \beta, \delta, \gamma, OV_3)$ für wenigstens ein $\delta \in I \setminus \{\alpha, \beta, \gamma\}$ gilt, bzw. wenn in $\boldsymbol{G}$ die Desargues-Bedingung des Typs (γ) gilt.*

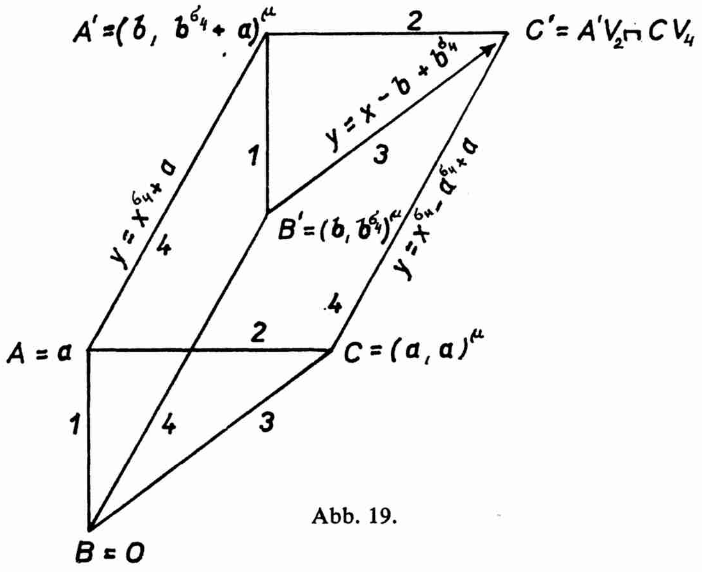

Abb. 19.

Beweis. Wir verwenden den Lehrsatz 12 und die Bemerkung hinter dem Lehrsatz 12 auf jedes $\boldsymbol{G}_{\alpha,\beta,\gamma,\iota}$, $\iota \in I \setminus \{\alpha, \beta, \gamma\}$ mit Hilfe der Folgerung des Satzes 1 und der Lehrsätze 6–7. ∎

Bemerkung. Einen Teil des Satzes 4 bildet die folgende Behauptung, welche wir wegen ihrer rein geometrischen Gestalt getrennt aussprechen: Es sei **G** ein Gewebe des Grades $\geqq 4$ und α, β, γ voneinander verschiedene Indexe. Gilt in **G** die Desargues-Bedingung des Typs (α), des Typs (β) und des Typs $(\alpha, \beta, \delta, \gamma)$ für ein $\delta \in I \setminus \{\alpha, \beta, \gamma\}$, dann gilt in **G** die Desargues-Bedingung des Typs (γ).

Lehrsatz 13. *Es sei* **G** *ein Gewebe mit* $I = \{1, 2, 3, 4\}$ *und* $(O, 1, 2, 3)$ *eines seiner Bezugssysteme. Weiter gelte in* **G** *die Desargues-Bedingung des Typs* (1) *und* $(3, OV_3)$. *Dann gilt in* **G** *genau dann die Desargues-Bedingung des Typs* $(4, OV_4)$, *wenn für die Operationen* $+ := +_3$ *und* σ_4 *der Koordinatenalgebra bezüglich* $(O, 1, 2, 3)$ *die Gleichung* $b^{\sigma_4} + a = b + a - b + b^{\sigma_4}$ $\forall a, b \in M$ *gilt.*

Beweis (Abb. 19). Untersuchen wir beliebige Elemente $a, b \in M := OV_1 \setminus \{V_1\}$. Die Punkte $A := a$, $B := O$, $C := (a, a)^\mu$, $A' := (b, b^{\sigma_4} + a)^\mu$, $B' := (b, b^{\sigma_4})^\mu$, $C' := (c, b^{\sigma_4} + a)^\mu = A'V_2 \sqcap CV_4$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (4) in **G** mit der Begrenzung $B = O$. Der Schluß $\overline{B', C', V_2}$ dieser Desargues-Bedingung ist mit $b^{\sigma_4} + a = c - b + b^{\sigma_4} = c^{\sigma_4} - a^{\sigma_4} + a$ äquivalent. Die Gleichung $b^{\sigma_4} + a = c^{\sigma_4} - a^{\sigma_4} + a$ kann zu $b^{\sigma_4} = c^{\sigma_4} - a^{\sigma_4}$ und weiter zu $b^{\sigma_4} + a^{\sigma_4} = c^{\sigma_4}$, $b + a = c$ vereinfacht werden. Aus $b^{\sigma_4} + a = b + a + b^{\sigma_4}$ folgt umgekehrt $b^{\sigma_4} + a = c^{\sigma_4} - a^{\sigma_4} + a$, wobei $c := b + a$ gesetzt ist. Hieraus und vom Lehrsatz 5 ist die weitere Beweisführung klar. ■

Folgerung. a) *Ist in den Voraussetzungen des Lehrsatzes* 13 *die Operation* $+$ *sogar kommutativ, dann gilt* $b^{\sigma_4} + a = b + a - b + b^{\sigma_4}$ $\forall a, b \in M$. b) *Wenn neben den Voraussetzungen des Lehrsatzes* 13 *noch die Desargues-Bedingung des Typs* $(3, OV_3)$ *in* **G** *gilt, dann gilt in* **G** *die Desargues-Bedingung des Typs* $(4, OV_3)$. c) *Ist* **G** *ein Gewebe mit* $I = \{1, 2, 3, 4\}$, *in welchem die Desargues-Bedingung des Typs* (1), (2) *und* (3) *gilt, dann gilt in* **G** *die Desargues-Bedingung des Typs* (4).

Beweis einfach.

Lehrsatz 14. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$, $(O, \alpha, \beta, \gamma)$ *eines seiner Bezugssysteme und* ϱ, σ, τ *Indexe, die voneinander und von* α, β *verschieden sind. Weiter gelte in* **G** *die Desargues-Bedingung des Typs* (α) *und* (β). *Dann gilt in* **G** *genau dann die Desargues-Bedingung des Typs* $(\alpha, \beta, \varrho, \tau, OV_\tau)$, *wenn in der Koordinatenalgebra bezüglich* $(O, \alpha, \beta, \gamma)$ *für die unären Operationen* $\sigma_\varrho, \sigma_\tau$ *und für die binäre Operation* $+ := +_\gamma$ *die Gleichung* $b^{\sigma_\tau} + a = b^{\sigma_\varrho} + a - b^{\sigma_\varrho} + b^{\sigma_\tau}$ $\forall a, b \in M := OV_\alpha \setminus \{V_\alpha\}$ *gilt.*

Beweis (Abb. 20). Untersuchen wir beliebige Punkte $a, b \in M$. Die Punkte $A := a$, $B := O$, $C := (a^{\sigma_\varrho^{-1}}, a)^\mu$, $A' := (b, b^{\sigma_\tau} + a)^\mu$, $B' := (b, b^{\sigma_\tau})^\mu$, $C' := (c, b^{\sigma_\tau} + a)^\mu$ erfüllen die Voraussetzungen der Desargues-Bedingung des Typs (τ)

in **G** mit Einschränkung $B = O$. Der Schluß $\overline{B', C', V_\varrho}$ dieser Desargues-Bedingung gilt genau dann, wenn $b^{\sigma_\tau} + a = c^{\sigma_\varrho} - b^{\sigma_\varrho} + b^{\sigma_\tau} = b^{\sigma_\tau} - a^{\sigma_\varrho^{-1}\sigma_\tau} + a$. Wie beim Beweis des Lehrsatzes 13 überführen wir die Gleichung $b^{\sigma_\tau} + a = c^{\sigma_\tau} - a^{\sigma_\varrho^{-1}\sigma_\tau} + a$ schrittweise auf die Gestalt $b^{\sigma_\tau} = c^{\sigma_\tau} - a^{\sigma_\varrho^{-1}\sigma_\tau}$, $b^{\sigma_\tau} + a^{\sigma_\varrho^{-1}\sigma_\tau} = c^{\sigma_\tau}$, $b + a^{\sigma_\varrho^{-1}} = c$. Nach Einsetzen in die Gleichung $b^{\sigma_\varrho} + a = c^{\sigma_\varrho} - b^{\sigma_\varrho} + b^{\sigma_\tau}$ bekommen wir $b^{\sigma_\tau} + a = b^{\sigma_\varrho} + a - b^{\sigma_\varrho} + b^{\sigma_\tau}$. Umgekehrt folgt hieraus mit $c := b + a^{\sigma_\varrho^{-1}}$ die ursprüngliche Gleichung $b^{\sigma_\tau} + a = c^{\sigma_\varrho} - b^{\sigma_\varrho} + b^{\sigma_\tau}$. Mit Rücksicht auf den Lehrsatz 5 ist die weitere Beweisführung klar. ■

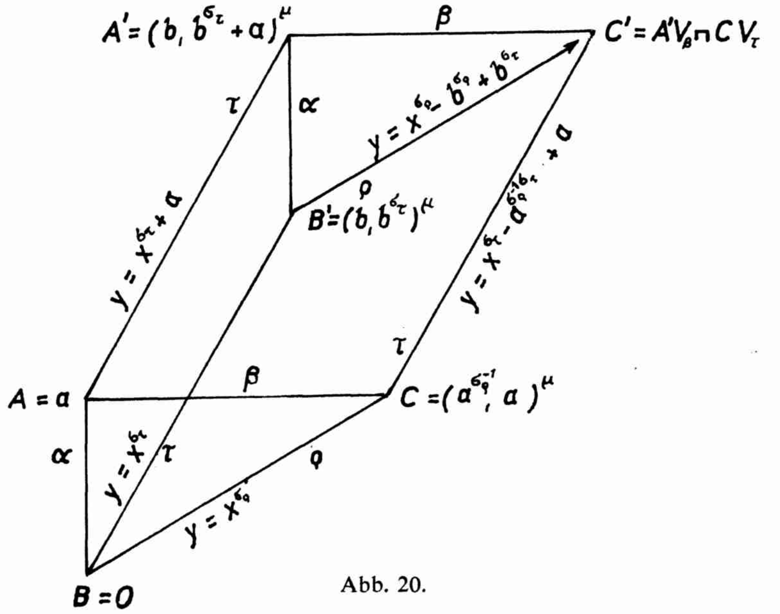

Abb. 20.

Folgerung. a) *Ist in den Voraussetzungen des Lehrsatzes* 14 *die Operation* $+$ *kommutativ, dann gilt* $b^{\sigma_\tau} + a = b^{\sigma_\varrho} + a - b^{\sigma_\varrho} + b^{\sigma_\tau}$ $\forall a, b \in M$. b) *Es gelten die Voraussetzungen des Lehrsatzes* 14 *und darüber hinaus noch die Desargues-Bedingung des Typs* $(\alpha, \beta, \bar{\delta}, \gamma)$ *in* **G** *für ein* $\bar{\delta} \in I \setminus \{\alpha, \beta, \gamma\}$. *Dann gilt in* **G** *auch die Desargues-Bedingung des Typs* (δ) *für jedes* $\delta \in I \setminus \{\alpha, \beta, \gamma\}$.

Beweis. Für a) klar, für b) folgt aus der Bemerkung nach dem Satz 4, aus dem Lehrsatz 14 und aus der Folgerung des Satzes 1. ■

In den Schlußbetrachtungen führen wir noch einige Ergänzungen durch, in denen die sog. Diagonalbedingung auftritt (Abb. 21): Es sei **G** ein Gewebe des Grades ≥ 4 und es seien $\alpha, \beta, \gamma, \delta$ Indexe, so daß α, β, γ voneinander verschieden sind und $\delta \in I \setminus \{\alpha, \beta\}$. Unter der *Diagonalbedingung* in **G** verstehen wir dann die Implikation der Gestalt $(\forall A, B, C, P \in \mathscr{P} \setminus v)$ $(\overline{A, B, V_\beta} \,\&\, \overline{C, D, V_\beta} \,\&\, \overline{A, D, V_\alpha} \,\&\, \overline{B, C, V_\alpha} \,\&\, \overline{A, C, V_\gamma} \Rightarrow \overline{B, D, V_\delta})$.

Lehrsatz 15. *Es sei* **G** *ein Gewebe des Grades* $\geqq 4$ *und* $(O, \alpha, \beta, \gamma)$ *eines seiner Bezugssysteme. Weiter gelte für die Koordinatenalgebra bezüglich* $(O, \alpha, \beta, \gamma)$, *daß* $+ := +_\gamma$ *eine Gruppenoperation ist, welche mit sämtlichen* $+_\iota$, $\iota \in I \setminus \{\alpha, \beta\}$, *zusammenfällt. Dann gibt es ein* $\delta \in I \setminus \{\alpha, \beta\}$ *mit* $x^{\sigma\delta} + x = O$ $\forall x \in M := := OV_\alpha \setminus \{V_\alpha\}$ *genau dann, wenn in* **G** *die Diagonalbedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Begrenzung* $A = O$ *gilt.*

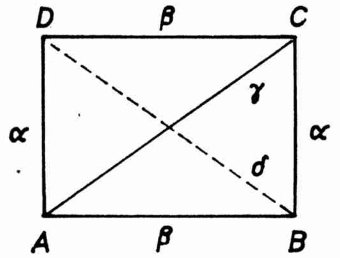

Abb. 21.

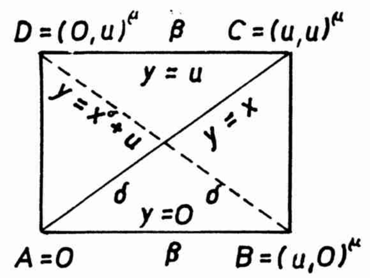

Abb. 22.

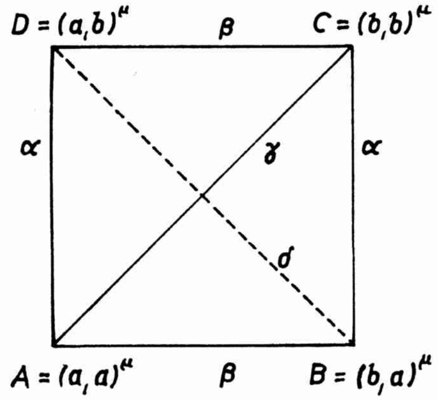

Abb. 23.

Beweis (Abb. 22). Für jedes $u \in M$ setzen wir

$$A := O, \quad B := (u, O)^\mu, \quad C := (u, u)^\mu, \quad D := (O, u)^\mu.$$

Diese Punkte erfüllen die Voraussetzungen der Diagonalbedingung des Typs $(\alpha, \beta, \gamma, \delta)$ in **G** mit Einschränkung $A = O$ und ihr Schluß $\overline{B, D, V_\delta}$ gilt genau dann, wenn $O = u^{\sigma\delta} + u$ gilt. Daraus folgt schon der Rest des Beweises. ■

Lehrsatz 16. *Es sei* **G** *ein Gewebe, welches dieselbe Voraussetzungen erfüllt wie im Lehrsatz 15. Dann gibt es ein* $\delta \in I \setminus \{\alpha, \beta\}$ *mit* $x^{\sigma\delta} + x = O$ $\forall x \in M := OV_\alpha \setminus \setminus \{V_\alpha\}$ *und* $+$ *ist kommutativ genau dann, wenn in* **G** *die Diagonalbedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Begrenzung* $A \in OV_\gamma$ *erfüllt ist.*

Beweis (Abb. 23). Untersuchen wir beliebige Elemente $a, b \in M$. Die Punkte $A := (a, a)^{\mu}$, $B := (b, a)^{\mu}$, $C := (b, b)^{\mu}$, $D := (a, b)^{\mu}$ erfüllen die Voraussetzungen der Diagonalbedingung des Typs $(\alpha, \beta, \gamma, \delta)$ in $\mathbf{G}$ mit der Begrenzung $A \in OV_{\gamma}$. Ihr Schluß $\overline{B, D, V_{\delta}}$ gilt genau dann, wenn $b = a^{\sigma\delta} + u$, $a = b^{\sigma\delta} + u$ für ein $u \in M$ gilt. Für $a = O$ bekommt man $b^{\sigma\delta} + b = O$ und beide Gleichungen lauten dann $a + b = u = b + a$, woraus schon der Rest des Beweises folgt. ∎

Folgerung. *Es sei* $\mathbf{G}$ *ein Gewebe des Grades* $\geqq 4$, $(O, \alpha, \beta, \gamma)$ *eines seiner Bezugssysteme und* $\delta \in I \setminus \{\alpha, \beta\}$. *Weiter gelte in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* (α) *und* (β). *Dann gilt in* $\mathbf{G}$ *die Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Einschränkung* $A \in OV_{\gamma}$ *genau dann, wenn in* $\mathbf{G}$ *die Diagonalbedingung des Typs* $(\alpha, \beta, \gamma, \delta)$ *mit der Einschränkung* $A = O$ *zusammen mit der Desargues-Bedingung des Typs* $(\alpha, \beta, \gamma, \delta, OV_{\gamma})$ *gilt.*

Beweis. Folgt aus den Lehrsätzen 16, 15 und Satz 4. ∎

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ON INVERSION OF LAPLACE TRANSFORM (I)

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The aim of this note is to show how a complex inversion theorem may be deduced from the general Post-Widder inversion theorem.

1. We denote by R and C respectively the real and complex number fields and by R^+ the set of all positive numbers. Further, if M_1, M_2 are two arbitrary sets, then $M_1 \to M_2$ will denote the set of all mappings of the set M_1 into the set M_2.

2. Lemma. *For every* $\alpha \geqq 0$ *and* $r \in \{1, 2, \ldots\}$ *such that* $r > \alpha$, *we have*

$$\left(\frac{r}{r-\alpha}\right)^r \leqq e^{(r/r-\alpha)\alpha} = e^{\alpha} e^{\alpha^2/(r-\alpha)}.$$

Proof. Under our assumptions we have

$$\log\left(\frac{r}{r-\alpha}\right)^r = \log\left(\frac{1}{1-\frac{\alpha}{r}}\right)^r = -r\log\left(1-\frac{\alpha}{r}\right) = r\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\alpha}{r}\right)k \leqq$$

$$\leqq r\sum_{k=1}^{\infty}\left(\frac{\alpha}{r}\right)^k = r\frac{\alpha}{r}\frac{1}{1-\frac{\alpha}{r}} = \alpha\frac{r}{r-\alpha} = \alpha + \frac{\alpha^2}{r-\alpha}$$

and our result follows.

3. Lemma. *For every* $z \in C$, $(1 + z/q)^q \to e^z (q \to \infty)$.

Proof. According to the binomial theorem, we can write

$$(1)\quad \left(1+\frac{z}{q}\right)^q=\sum_{k=0}^{q}\binom{q}{k}\frac{z^k}{q^k}=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{k!}\frac{z^k}{q^k}=$$

$$=1+z+\sum_{k=2}^{q}\frac{q(q-1)\dots(q-k+1)}{q^k}\frac{z^k}{k!}=$$

$$=1+z+\sum_{k=2}^{q}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z^k}{k!}$$

for every $q \in \{2, 3, \dots\}$.

Let now $z \in C$ and $\varepsilon > 0$. Then there exists a $k_0 \in \{2, 3, \dots\}$ such that

$$(2)\quad \sum_{k=k_0+1}^{\infty}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}.$$

It follows from (2) that

$$(3)\quad \left|e^z-\sum_{k=0}^{k_0}\frac{z^k}{k!}\right|\leqq\frac{\varepsilon}{3}.$$

Further by (1) and (2),

$$(4)\quad \left|\left(1+\frac{z}{q}\right)^q-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|=$$

$$=\left|\sum_{k=k_0+1}^{q}\left(1-\frac{1}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right|\leqq\sum_{k=k_0+1}^{k}\frac{|z|^k}{k!}\leqq\frac{\varepsilon}{3}$$

for every $q \geqq k_0 + 1$.

Finally it is easy to see that there exists a $q_0 \in \{k_0 + 1, k_0 + 2, \dots\}$ such that

$$(5)\quad \left|\sum_{k=0}^{k_0}\frac{z^k}{k!}-\left[1+z+\sum_{k=2}^{k_0}\left(1-\frac{1}{q}\right)\left(1-\frac{2}{q}\right)\dots\left(1-\frac{k-1}{q}\right)\frac{z}{k!}\right]\right|\leqq\frac{\varepsilon}{3}$$

for every $q \geqq q_0$.

Now we have immediately from (3), (4) and (5)

$$\left|e^z-\left(1-\frac{z}{q}\right)^q\right|\leqq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

for every $q \geqq q_0$ and this gives the assertion.

4. Theorem (Post-Widder). *Let $f \in R^+ \to E$ and let M, ω be two nonnegative constants. If* (α) *the function f is measurable over* R^+,

(β) $|f(t)| \leqq Me^{\omega t}$ *for almost all* $t \in R^+$, *then*

(a) $$\int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \text{ exists for every } t \in R^+ \text{ and } p + 1 > \omega t,$$

(b) $$\left|\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\int_0^\infty e^{-(p+1)/\tau}\tau^p f(\tau)\,\mathrm{d}\tau\right| \leqq Me^{\omega t}e^{\omega^2 t^2/(p+1-\omega t)}$$

for every $t \in R^+$ *and* $p + 1 > \omega t$,

(c) $$\frac{1}{p!}\left(\frac{p+1}{t}\right)^{p+1}\int_0^\infty e^{-((p+1)/t)\tau}\tau^p f(\tau)\,\mathrm{d}\tau \to f(t) \quad (p \to \infty,\ p+1 > \omega t)$$

for almost all $t \in R^+$.

5. Lemma. *Let* α *be a nonnegative constant and* $J \in \{z : \mathrm{Re}\, z \geqq \alpha\} \to C$. *If*

(α) J *is continuous on* $\{z : \mathrm{Re}\, z \geqq \alpha\}$,

(β) J *is analytic on* $\{z : \mathrm{Re}\, z > \alpha\}$,

(γ) $J(\lambda) \to 0$ $(\lambda \geqq \alpha,\ \lambda \to \infty)$,

(δ) *there exist a constant* K *and a number* $k \in \{0, 1, \ldots\}$ *so that for every* $\mathrm{Re}\, z \geqq \alpha$, *we have* $|J(z)| \leqq K(1 + |z|)^k$,

(ε) $$\int_{-\infty}^{\infty} \frac{|J(\alpha + i\beta)|}{1 + |\beta|}\,\mathrm{d}\beta < \infty,$$

then for every $\lambda > \alpha$ *and* $p \in \{0, 1, \ldots\}$,

$$\frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} J(\lambda) = (-1)^p \frac{p!}{2\pi i}\int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\,\mathrm{d}\beta.$$

Proof. Let us first fix a $\lambda > \alpha$.

Moreover, we choose fixed numbers K, k so that the assumption (δ) holds.

By virtue of Cauchy's integral theorem, we obtain from (α), (β) that

(1) $$\begin{aligned}\frac{2\pi}{p!} J^{(p)}(\lambda) = &-\int_{-N}^{N} \frac{1}{(\alpha + i\beta - \lambda)^{p+1}} J(\alpha + i\beta)\,\mathrm{d}\beta + \\ &+ \int_{-N}^{N} \frac{1}{(\alpha + 2N + i\beta - \lambda)^{p+1}} J(\alpha + 2N + i\beta)\,\mathrm{d}\beta - \\ &- i\int_0^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta + iN)\,\mathrm{d}\eta + \\ &+ i\int_0^{2N} \frac{1}{(\alpha + \eta - iN - \lambda)^{p+1}} J(\alpha + \eta - iN)\,\mathrm{d}\eta\end{aligned}$$

for every $p \in \{0, 1, \ldots\}$ and $N > \frac{1}{2}\lambda$.

Using (δ), we obtain

$$(2)\qquad \left|\int_{-N}^{N} \frac{1}{(\alpha + 2N + i\beta - \lambda)^{p+1}} J(\alpha + 2N + i\beta)\, \mathrm{d}\beta\right| \leqq$$

$$= \int_{-N}^{N} \frac{1}{((\lambda - \alpha + 2N)^2 + \beta^2)^{(p+1)/2}} [1 + ((\alpha + 2N)^2 + \beta^2)^{1/2}]^k \, \mathrm{d}\beta =$$

$$= \int_{-N}^{N} \frac{1}{(\lambda - \alpha + 2N)^{p+1}} [1 + ((\alpha + 2N)^2 + N^2)^{1/2}]^k =$$

$$= \frac{2N}{(\lambda - \alpha + 2N)^{p+1}} [1 + ((\alpha + 2N)^2 + N^2)^{1/2}]^k ,$$

$$\left|\int_{0}^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta + iN)\, \mathrm{d}\eta\right| \leqq$$

$$\leqq \int_{0}^{2N} \frac{1}{((\lambda + \eta - \alpha)^2 + N^2)^{(p+1)/2}} [1 + (\alpha^2 + N^2)^{1/2}]^k \, \mathrm{d}\eta \leqq$$

$$\leqq \int_{0}^{2N} \frac{1}{N^{p+1}} [1 + (\alpha^2 + N^2)^{1/2}]^k \leqq \frac{2}{N^p} [1 + (\alpha^2 + N^2)^{1/2}]^k ,$$

$$\int_{0}^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta - iN)\, \mathrm{d}\eta \leqq \frac{2}{N^p} [1 + (\alpha^2 + N^2)^{1/2}]^k$$

for every $p \in \{0, 1, \ldots\}$ and $N > \frac{1}{2}\lambda$.

Letting $N \to \infty$, we see from (2) that

$$(3)\qquad \int_{-N}^{N} \frac{1}{(\alpha + 2N + i\beta - \lambda)^{p+1}} J(\alpha + 2N + i\beta)\, \mathrm{d}\beta \to_{N\to\infty} 0 ,$$

$$\int_{0}^{2N} \frac{1}{(\alpha + \eta + iN - \lambda)^{p+1}} J(\alpha + \eta + iN)\, \mathrm{d}\eta \to_{N\to\infty} 0 ,$$

$$\int_{0}^{2N} \frac{1}{(\alpha + \eta - iN - \lambda)^{p+1}} J(\alpha + \eta - iN)\, \mathrm{d}\eta \to_{N\to\infty} 0$$

for every $p \in \{k + 1, k + 2, \ldots\}$.

Now we conclude from (1) and (3) by means of (ε) that

$$(4)\qquad J^{(p)}(\lambda) = -\frac{p!}{2\pi} \int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)^{p+1}}\, \mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p \in \{k + 1, k + 2, \ldots\}$.

On the other hand, let us define, on the basis of (ε)

$$J_0(\lambda) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)}\,d\beta \quad \text{for every} \quad \lambda > \alpha\,.$$

It is easy to verify that

(6) the function J_0 is infinitely differentiable on (α, ∞),

(7) $$J_0^{(p)}(\lambda) = -\frac{p!}{2\pi}\int_{-\infty}^{\infty} \frac{J(\alpha + i\beta)}{(\alpha + i\beta - \lambda)^{p+1}}\,d\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \ldots\}$,

(8) $$J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty)\,.$$

By (4)–(7),

(9) $$J^{(k+1)}(\lambda) = J_0^{(k+1)}(\lambda)$$

for every $\lambda > \alpha$. Consequently, by (9)

(10) $$J - J_0 \quad \text{is a polynomial}\,.$$

Taking (γ) and (8) into account, we see that

(11) $$J(\lambda) - J_0(\lambda) \to 0 \quad (\lambda > \alpha,\ \lambda \to \infty)\,.$$

Hence (10) and (11) imply $J = J_0$ and the conclusion of Lemma 5 follows immediately from (7).

6. Theorem. *Let $f \in R^+ \to C$ and $\alpha > 0$. If*

(α) *the function f is measurable,*

(β) *there exist two nonnegative constants M, ω so that $\omega < \alpha$ and $|f(t)| \leqq Me^{\omega t}$ for almost all $t \in R^+$,*

(γ) $$\int_{-\infty}^{\infty} \left| \int_0^{\infty} e^{-(\alpha + i\beta)\tau} f(\tau)\,d\tau \right| d\beta < \infty\,,$$

then

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha + i\beta)t} \left(\int_0^{\infty} e^{-(\alpha + i\beta)\tau} f(\tau)\,d\tau \right) d\beta\,.$$

for almost all $t \in R^+$.

Proof. Let us first fix the constants M, ω so that the assumption (β) holds.

Further let us define a function $F \in (\omega, \infty) \to C$ by

(1) $$F(\lambda) = \int_0^{\infty} e^{-\lambda\tau} f(\tau)\,d\tau \quad \text{for} \quad \lambda > \omega\,.$$

By Theorem 4 we have

$$(2) \qquad \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to f(t) \quad (p \to \infty,\ p+1 > \alpha t)$$

for almost all $t \in R^+$.

On the other hand, let J be the function defined by

$$(3) \qquad J(z) = \int_0^\infty e^{-z\tau} f(\tau)\,\mathrm{d}\tau$$

for every $\operatorname{Re} z \geqq \alpha$.

It is easy to deduce from our assumptions that

(4) the function J has the properties 5 $(\alpha)-(\varepsilon)$.

Hence by (4), we obtain from Lemma 5 that

$$(5) \qquad J^{(p)}(\lambda) = (-1)^p \frac{p!}{2\pi}\int_{-\infty}^\infty \frac{J(\alpha + i\beta)}{(\lambda - \alpha - i\beta)^{p+1}}\,\mathrm{d}\beta$$

for every $\lambda > \alpha$ and $p \in \{0, 1, \dots\}$.

Now it follows from (1), (3) and (5) that

(6)

$$\frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\left(\frac{p+1}{t}\right)^{p+1}}{\left(\frac{p+1}{t} - \alpha - i\beta\right)^{p+1}} J(\alpha + i\beta)\,\mathrm{d}\beta =$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{\left(1 - \frac{(\alpha + i\beta)\,t}{p+1}\right)^{p+1}} J(\alpha + i\beta)\,\mathrm{d}\beta$$

for every $t \in R^+$ and $p + 1 > \alpha t$.

By Lemma 2 we see that

$$(7) \qquad \left|\frac{1}{\left(1 - \frac{(\alpha + i\beta)\,t}{p+1}\right)^{p+1}}\right| = \frac{1}{\left[\left(1 - \frac{\alpha t}{p+1}\right)^2 + \left(\frac{\beta t}{p+1}\right)^2\right]^{(p+1)/2}} \leqq$$

$$\leqq \frac{1}{\left(1 - \frac{\alpha t}{p+1}\right)^{p+1}} \leqq e^{\alpha t} e^{\alpha t/(p+1-\alpha t)}$$

for every $t \in R^+$ and $p + 1 > \alpha t$.

By Lemma 3,

$$(8)\qquad \frac{1}{\left(1 - \frac{(\alpha + i\beta)\,t}{p+1}\right)^{p+1}} \to e^{(\alpha+i\beta)t} \quad (p \to \infty,\ p + 1 > \alpha t)\,.$$

Now we get from (6)–(8)

$$(9)\qquad \frac{(-1)^p}{p!}\left(\frac{p+1}{t}\right)^{p+1} F^{(p)}\left(\frac{p+1}{t}\right) \to \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+i\beta)t}\, J(\alpha + i\beta)\,\mathrm{d}\beta$$

$$(p \to \infty,\ p + 1 > \alpha t)$$

for every $t \in R^+$.

We see from (2) and (9) that the assertion of our theorem is fulfilled and this completes the proof.

Note. In the continuation of this paper, we shall study different complex inversion formulas for the Laplace transform as relatively simple consequences of Theorem 6 from a new unified point of view.

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ISOMORPHISM OF PROJECTIVE PLANES AND ISOTOPISM OF PLANAR TERNARY RINGS

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In the last years isotopism of planar ternary rings was studied especially by G. E. MARTIN. The theory of planar ternary rings was developed together with the theory of projective planes so that there is the close connection between isotopism of planar ternary rings and isomorphism of projective planes. This article is supposed to be a contribution to that problem.

I. PROJECTIVE PLANES, PLANAR TERNARY RINGS

Under the term *projective plane* we understand an ordered triple $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ where $\mathscr{P}$, $\mathscr{L}$ are disjoint sets and $\mathscr{I}$ is a binary relation from $\mathscr{P}$ to $\mathscr{L}$ (i.e. $\mathscr{I} \subseteq \subseteq \mathscr{P} \times \mathscr{L}$) and the following axioms are satisfied: $(\mathbf{P}_1)$ to every two distinct elements $\boldsymbol{p}, \boldsymbol{q} \in \mathscr{P}$ there exists just one element $\boldsymbol{L} \in \mathscr{L}$ such that $\boldsymbol{p}, \boldsymbol{q} \mathscr{I} \boldsymbol{L}$, $(\mathbf{P}_2)$ to every two distinct elements $\boldsymbol{L}, \boldsymbol{M} \in \mathscr{L}$ there exists just one element $\boldsymbol{p} \in \mathscr{P}$ such that $\boldsymbol{p} \mathscr{I} \boldsymbol{L}, \boldsymbol{M}$, and $(\mathbf{P}_3)$ there exist four elements from $\mathscr{P}$ such that no three of them are in the relation $\mathscr{I}$ with the same element of $\mathscr{L}$. The elements of $\mathscr{P}$ are called points, the elements of $\mathscr{L}$ lines, and the relation $\mathscr{I}$ is called incidence. The phrases "lies on" or "passes through" are also used for the relation $\mathscr{I}$. From the axioms of the projective plane it follows that the number of points on every line is equal to the number of lines passing through each point.

Let $\boldsymbol{\mathscr{P}}_1 = (\mathscr{P}_1, \mathscr{L}_1, \mathscr{I}_1)$, $\boldsymbol{\mathscr{P}}_2 = (\mathscr{P}_2, \mathscr{L}_2, \mathscr{I}_2)$ be projective planes. Then each ordered couple of bijections (π, λ), $\pi : \mathscr{P}_1 \to \mathscr{P}_2$, $\lambda : \mathscr{L}_1 \to \mathscr{L}_2$ such that $\boldsymbol{p} \mathscr{I}_1 \boldsymbol{L}$ holds if and only if $\boldsymbol{p}^\pi \mathscr{I}_2 \boldsymbol{L}^\lambda$ is called an isomorphism of projective plane $\boldsymbol{\mathscr{P}}_1$ onto the projective plane $\boldsymbol{\mathscr{P}}_2$. With respect to the symmetry of the relation "to be isomorphic onto" it is possible to speak about isomorphic projective planes.

A *planar ternary ring* PTR is an ordered couple $(\mathcal{S}, \mathrm{T})$ where $\mathcal{S}$ is a set, $|\mathcal{S}| \geqq 2$, and T is a ternary operation defined on $\mathcal{S}$, i.e. a mapping $\mathrm{T} : \mathcal{S}^3 \to \mathcal{S}$ satisfying the following axioms:

(i) $\forall a, b, c \in \mathcal{S}\ \exists!\ x \in \mathcal{S}\quad \mathrm{T}(a, b, x) = c$,

(ii) $\forall a, b, c, d \in \mathcal{S},\ a \neq c\ \exists!\ x \in \mathcal{S}\quad \mathrm{T}(x, a, b) = \mathrm{T}(x, c, d)$,

(iii) $\forall a, b, c, d \in \mathcal{S},\ a \neq c\ \exists!\ (x, y) \in \mathcal{S}^2\quad \mathrm{T}(a, x, y) = b,\ \mathrm{T}(c, x, y) = d$.

If it holds in a PTR that

(iv) $\exists O^l \in \mathcal{S}\ \forall b \in \mathcal{S}\ \exists b' \in \mathcal{S}\ \forall m \in \mathcal{S}\quad \mathrm{T}(O^l, m, b) = b'$,

(v) $\exists O^r \in \mathcal{S}\ \forall b \in \mathcal{S}\ \exists b'' \in \mathcal{S}\ \forall x \in \mathcal{S}\quad \mathrm{T}(x, O^r, b) = b''$,

then the PTR is called an intermediate ternary ring (ITR). The elements O^l and O^r are called respectively the left and the right quasizero of PTR. The left (right) quasizero is unique in the PTR $\mathbf{R} = (\mathcal{S}, \mathrm{T})$. If there exists an element $O \in \mathcal{S}$ such that $\mathrm{T}(O, m, b) = \mathrm{T}(x, O, b) = b$ for all $x, m, b \in \mathcal{S}$, then O is called the zero of the PTR.

Lemma 1. *If O^l and O^r are respectively the left and the right quasizero of* PTR $\mathbf{R} = (\mathcal{S}, \mathrm{T})$, *then* $\mathrm{T}(O^l, m, b) = \mathrm{T}(x, O^r, b)$ *for any* $b, x, m \in \mathcal{S}$.

Proof. For each $b \in \mathcal{S}$, $\mathrm{T}(O^l, O^r, b) = b'$ and $\mathrm{T}(O^l, O^r, b) = b''$ according to the axioms (iv) or (v). Since T is a ternary operation, we have $b' = b''$.

Lemma 2. *If* $\mathbf{R} = (\mathcal{S}, \mathrm{T})$ *is an* ITR, *then the left (right) quasizero induces in $\mathcal{S}$ a permutation φ for which it holds: for each $b \in \mathcal{S}$ $\mathrm{T}(O^l, m, b) = b^\varphi$ for all $m \in \mathcal{S}$ $(\mathrm{T}(x, O^r, b) = b^\varphi$ for all $x \in \mathcal{S})$.*

Proof. The fact that φ is injective in $\mathcal{S}$ follows from the definition of the PTR. If $c \in \mathcal{S}$, then there is $b \in \mathcal{S}$ such that $\mathrm{T}(O^l, O^r, b) = c$ according to the axiom (i). Hence $b^\varphi = c$ and φ is surjective, too.

We consider all the PTR's of the same cardinality to be always constructed over the same set $\mathcal{S}$.

If $(\mathcal{S}, \mathrm{T}_1)$ is a PTR and $\alpha, \beta, \gamma, \delta$ arbitrary permutations in $\mathcal{S}$, then the system $(\mathcal{S}, \mathrm{T}_2)$ defined by the relation

$$(*)\qquad [\mathrm{T}_2(x, m, b)]^\alpha = \mathrm{T}_1(x^\beta, m^\gamma, b^\delta)$$

holding for all $x, m, b \in \mathcal{S}$ is called isotopic to the PTR $(\mathcal{S}, \mathrm{T}_1)$. By verifying the axioms (i)–(iii), $(\mathcal{S}, \mathrm{T}_2)$ may be easily shown to be a PTR, too. With respect to the symmetry of the relation "to be isotopic to" it is possible to speak about isotopic PTR's. An ordered quadruple of permutations $(\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta, \gamma, \delta$ are permutations in $\mathcal{S}$ satisfying the relation $(*)$, is called an isotopism of the PTR $(\mathcal{S}, \mathrm{T}_1)$ onto the PTR $(\mathcal{S}, \mathrm{T}_2)$.

2. ISOMORPHISM OF PROJECTIVE PLANES AND ISOTOPISM OF PLANAR TERNARY RINGS

Theorem 1. *Let* $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ *be a* PTR, *then the ordered triple* $(\mathscr{P}, \mathscr{L}, \mathscr{I})$ *defined in the following way is a projective plane*:

$$\mathscr{P} = \{(x, y) \mid (x, y) \in \mathscr{S}^2\} \cup \{(z) \mid z \in \mathscr{S}\} \cup \{(\infty)\}, \quad \infty \notin \mathscr{S},$$
$$\mathscr{L} = \{[m, b] \mid (m, b) \in \mathscr{S}^2\} \cup \{[n] \mid n \in \mathscr{S}\} \cup \{[\infty]\}, \quad \infty \notin \mathscr{S},$$

$\mathscr{I} \subseteq \mathscr{P} \times \mathscr{L}$ *and the following holds*:

(a) $(x, y)\, \mathscr{I}[m, b] \Leftrightarrow \mathrm{T}(x, m, b) = y$,

(b) $(x, y)\, \mathscr{I}[x]$ *for all* $x, y \in \mathscr{S}$,

(c) $(m)\, \mathscr{I}[m, b]$ *for all* $m, b \in \mathscr{S}$,

(d) $(m)\, \mathscr{I}[\infty]$ *for all* $m \in \mathscr{S}$,

(e) $(\infty)\, \mathscr{I}[x]$ *for all* $x \in \mathscr{S}$,

(f) $(\infty)\, \mathscr{I}[\infty]$.

The proof may be done by verifying the validity of the axioms $(\mathbf{P}_1)-(\mathbf{P}_3)$ in $(\mathscr{P}, \mathscr{L}, \mathscr{I})$.

The projective plane from the above theorem will be called the projective plane over the PTR $\mathbf{R}$ and denoted by $\mathscr{P}_{\mathbf{R}}$.

Now let us have an ITR $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ and a projective plane $\mathscr{P}_{\mathbf{R}}$ over it. According to Theorem 1 all the points incident with the line $[\infty]$ are of the (m) shape for all $m \in \mathscr{S}$, and all the lines passing through the point (∞) are of the $[x]$ shape for all $x \in \mathscr{S}$. Now let us choose a significant point (O^r) and a significant line $[O^l]$ where $O^l, O^r \in \mathscr{S}$, O^l is the left quasizero and O^r is the right quasizero of the PTR $\mathbf{R}$. According to Theorem 1, each point incident with the line $[O^l]$ is of the (O^l, a) shape. However, to each $a \in \mathscr{S}$ there exist just one $b \in \mathscr{S}$ such that $b^\varphi = a$ where φ is the permutation in $\mathscr{S}$ induced by the quasizeros. Each point (O^l, a) may be recorded in the (O^l, b^φ) shape. Each line passing through the point (O^r) is of the $[O^r, a]$ shape. Since $\mathrm{T}(O^l, O^r, a) = a^\varphi$, it holds $(O^l, a^\varphi)\, \mathscr{I}[O^r, a]$. The line passing through the points (O^l, c^φ) and (m) is denoted by $[m, c]$ because $\mathrm{T}(O^l, m, c) = c^\varphi$. The point lying on the lines $[O^r, d]$ and $[x]$ is denoted by (x, d^φ) because $\mathrm{T}(x, O^r, d) = d^\varphi$.

The following two theorems showing the reciprocal connection between isomorphisms of projective planes and isotopisms of PTR's are given without any proof.

Theorem 2. *If the* PTR's $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic, then the projective planes* $\mathscr{P}_{\mathbf{R}_1}, \mathscr{P}_{\mathbf{R}_2}$ *are isomorphic.*

Proof. See [4], page 288, Theorem 9.3.3.

However, the inversion of this theorem is impossible. An example of isomorphic projective planes over nonisotopic PTR's is mentioned in [4], Chapter 11.4. The following theorem answers the question which isomorphisms of projective planes over the PTR's with zero make the corresponding PTR's isotopic.

Theorem 3. *If there exists an isomorphism (π, λ) of a projective plane $\mathcal{P}_{\mathbf{R}_1}$ onto a projective plane $\mathcal{P}_{\mathbf{R}_2}$ such that $(\infty)^\pi = (\infty)$, $(O_1)^\pi = (O_2)$, $(O_1, O_1)^\pi = (O_2, O_2)$ where O_1 is the zero of the* PTR $\mathbf{R}_1$ *and O_2 is the zero of the* PTR $\mathbf{R}_2$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Proof. See [2].

A suitable criterion can be formulated also for ITR's.

Theorem 4. *If there exists an isomorphism (π, λ) of a projective plane $\mathcal{P}_{\mathbf{R}_1} = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ onto a projective plane $\mathcal{P}_{\mathbf{R}_2} = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ such that $(\infty_1)^\pi = (\infty_2)$, $(O_1^r)^\pi = (O_2^r)$, $[O_1^l]^\lambda = [O_2^l]$ where O_1^l and O_1^r is the left and the right quasizero of the* PTR $\mathbf{R}_1 = (\mathcal{S}, \mathrm{T}_1)$, *while O_2^l and O_2^r is the left and the right quasizero of the* PTR $\mathbf{R}_2 = (\mathcal{S}, \mathrm{T}_2)$, *then $\mathbf{R}_1$ and $\mathbf{R}_2$ are isotopic.*

Proof. The assumptions of this theorem imply that:

(1) there exists a bijection π' that is the restriction of π on $\{(x) \mid x \in \mathcal{S}\}$ such that $(x)^{\pi'} = (y)$. It means that π' induces on $\mathcal{S}$ a permutation γ for which $x^\gamma = y$.

(2) there exists a bijection π'' that is the restriction of π on $\{(O_1^l, x) \mid x \in \mathcal{S}\}$ such that $(O_1^l, x)^{\pi''} = (O_2^l, y)$. It means that π'' induces on $\mathcal{S}$ a permutation α for which $x^\alpha = y$.

(3) there exists a bijection λ' that is the restriction of λ on $\{[x] \mid x \in \mathcal{S}\}$ such that $[x]^{\lambda'} = y$. It means that λ' induces on $\mathcal{S}$ a permutation β for which $x^\beta = y$.

Thus $(m)^\pi = (m^\gamma)$, $(O_1^l, b^{\varphi_1})^\pi = (O_2^l, b^{\varphi_1\alpha})$, $[x]^\lambda = [x^\beta]$ for all $x, m, b \in \mathcal{S}$.*)

It is necessary to determine the images of the remaining points and lines with respect to the permutations α, β, γ. Since $(O_1^r)^\pi = (O_2^r)$ and $(\infty_1)^\pi = (\infty_2)$, it is $[\infty_1]^\lambda = [\infty_2]$. Since $(m)^\pi = (m^\gamma)$ and $(O_1^l, b^{\varphi_1})^\pi = (O_2^l, b^{\varphi_1\alpha})$, it is $[m, b]^\lambda = [m^\gamma, c]$. Furthermore, it holds: $\mathrm{T}_2(O_2^l, m^\gamma, c) = b^{\varphi_1\alpha}$ and $\mathrm{T}_2(O_2^l, m^\gamma, c) = c^{\varphi_2}$ which implies $c = b^{\varphi_1\alpha\varphi_2^{-1}}$ and thus $[m, b]^\lambda = [m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}]$. Since $[x]^\lambda = [x^\beta]$ and $[O_1^r, y]^\lambda = [O_2^r, y^{\varphi_1\alpha\varphi_2^{-1}}]$, it is $(x, y^{\varphi_1})^\pi = (x^\beta, z)$. It holds that $\mathrm{T}_2(x^\beta, O_2^r, y^{\varphi_1\alpha\varphi_2^{-1}}) = z$ and $\mathrm{T}_2(x^\beta, O_2^r, z^{\varphi_2^{-1}}) = z$. This implies that $z = y^{\varphi_1\alpha}$ and thus $(x, y^{\varphi_1})^\pi = (x^\beta, y^{\varphi_1\alpha})$.

Since (π, λ) is an isomorphism of $\mathcal{P}_{\mathbf{R}_1}$ onto $\mathcal{P}_{\mathbf{R}_2}$, it is true that $(x, y^{\varphi_1})\, \mathcal{I}_1 [m, b] \Leftrightarrow (x, y^{\varphi_1})^\pi\, \mathcal{I}_2 [m, b]^\lambda \Rightarrow (x^\beta, y^{\varphi_1\alpha})\, \mathcal{I}_2 [m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}]$. This means that for all $x, y^{\varphi_1}, m, b \in \mathcal{S}$ for which the above mentioned relation holds, $\mathrm{T}_1(x, m, b) = y^{\varphi_1}$ and $\mathrm{T}_2(x^\beta, m^\gamma, b^{\varphi_1\alpha\varphi_2^{-1}}) = y^{\varphi_1\alpha}$. Let $\varphi_1\alpha\varphi_2^{-1} = \delta$, then we can write $[\mathrm{T}_1(x, m, b)]^\alpha = \mathrm{T}_2(x^\beta, m^\gamma, b^\delta)$ for all $x, m, b \in \mathcal{S}$.

Let us state some corollaries of this theorem. Let be given two ITR's: $\mathbf{R}_1 = (\mathcal{S}, \mathrm{T}_1)$ with the left quasizero O_1^l and the right quasizero O_1^r, and $\mathbf{R}_2 = (\mathcal{S}, \mathrm{T}_2)$ with the left quasizero O_2^l and the right quasizero O_2^r. Let $\mathcal{P}_{\mathbf{R}_1} = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$, $\mathcal{P}_{\mathbf{R}_2} = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ be the projective planes over the given PTR's.

*) φ_1, φ_2 are the permutations in $\mathcal{S}$ induced by the quasizeros from $\mathbf{R}_1, \mathbf{R}_2$.

Corollary 1. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $(\infty_1)^\pi = (\infty_2)$, $(O_1^l, b)^\pi = (O_2^l, c)$, $(O_1^r)^\pi = (O_2^r)$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Corollary 2. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $[\infty_1]^\lambda = [\infty_2]$, $[O_1^l]^\lambda = [O_2^l]$, $(O_1^r)^\pi = (O_2^r)$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Corollary 3. *If there exists an isomorphism* (π, λ) *of the projective plane* $\mathscr{P}_{\mathbf{R}_1}$ *onto* $\mathscr{P}_{\mathbf{R}_2}$ *such that* $[\infty_1]^\lambda = [\infty_2]$, $[O_1^l]^\lambda = [O_1^l]$, $[O_1^r, b]^\lambda = [O_2^r, c]$, *then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isotopic.*

Now let us apply the results established in Theorem 4 to two suitable PTR's and the corresponding projective planes over them. Let us introduce several concepts needed in the example below. In a PTR $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ in which elements $O, 1 \in \mathscr{S}$, $O \neq 1$ exist where O is the zero in $\mathbf{R}$ and $\mathrm{T}(x, 1, O) = \mathrm{T}(1, x, O) = x$ for all $x = \mathscr{S}$, two binary operations $+, \circ$ may be defined by the relations $\mathrm{T}(x, m, O) = x \circ m$, $\mathrm{T}(1, m, b) = m + b$ for all $x, m, b \in \mathscr{S}$. If $\mathrm{T}(x, m, b) = x \circ m + b$ for all $x, m, b \in \in \mathscr{S}$ where the operation $\circ$ has the priority, then $\mathbf{R} = (\mathscr{S}, \mathrm{T})$ is called a linear PTR and we can write $\mathbf{R} = (\mathscr{S}, +, \circ)$.

Example. Let us have linear PTR's $\mathbf{R}_1 = (\mathscr{S}, +_1, \circ_1)$ and $\mathbf{R}_2 = (\mathscr{S}, +_2, \circ_2)$ where $\mathscr{S} = \{0, 1, 2, a, b, c, d, e, f\}$, in which the binary operations are given by means of the following tables:

Table 1

$+_1, +_2$	0	1	2	a	b	c	d	e	f
0	0	1	2	a	b	c	d	e	f
1	1	2	0	b	c	a	e	f	d
2	2	0	1	c	a	b	f	d	e
a	a	b	c	d	e	f	0	1	2
b	b	c	a	e	f	d	1	2	0
c	c	a	b	f	d	e	2	0	1
d	d	e	f	0	1	2	a	b	c
e	e	f	d	1	2	0	b	c	a
f	f	d	e	2	0	1	c	a	b

Table 2

$\circ_1$	0	1	2	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	b	c	d	e	f
2	0	2	1	d	f	e	a	c	b
a	0	a	d	b	1	f	c	2	e
b	0	b	f	e	c	1	2	d	a
c	0	c	e	1	d	a	f	b	2
d	0	d	a	f	2	b	e	1	c
e	0	e	c	2	a	d	b	f	1
f	0	f	b	c	e	2	1	a	d

Table 3

$\circ_2$	0	1	2	a	b	c	d	e	f
0	0	0	0	0	0	0	0	0	0
1	0	1	2	a	b	c	d	e	f
2	0	2	1	d	f	e	a	c	b
a	0	a	d	e	c	1	f	b	2
b	0	b	f	1	d	a	c	2	e
c	0	c	e	b	1	f	2	d	a
d	0	d	a	c	e	2	b	f	1
e	0	e	c	f	2	b	1	a	d
f	0	f	b	2	a	d	e	1	c

Let us remark that the element O has the property of both the left nd the right quasizero of the PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$. According to Theorem 1 the projective planes $\mathscr{P}_{\mathbf{R}_1} = (\mathscr{P}_1, \mathscr{L}_1, \mathscr{I}_1)$ and $\mathscr{P}_{\mathbf{R}_2} = (\mathscr{P}_2, \mathscr{L}_2, \mathscr{I}_2)$ can be constructed in the above mentioned way. We show that if there exists an isomorphism of $\mathscr{P}_{\mathbf{R}_1}$ onto $\mathscr{P}_{\mathbf{R}_2}$ satisfying the assumptions of Theorem 4, then $\mathbf{R}_1$ and $\mathbf{R}_2$ are isotopic.

Let us define bijections $\pi : \mathscr{P}_1 \to \mathscr{P}_2$ and $\lambda : \mathscr{L}_1 \to \mathscr{L}_2$ such that $(x, y)^\pi = (x \circ_1 2, y)$, $[m, b]^\lambda = [m \circ_1 2, b]$, $(m)^\pi = (m \circ_1 2)$ and $[x]^\lambda = [x \circ_1 2]$. It is possible to verify that $x \circ_1 m = (x \circ_1 2) \circ_2 (m \circ_1 2)$ for all $x, m \in \mathscr{S}$. This statement together with the condition (a) of Theorem 1 imply the following results: if $(x, y) \mathscr{I}_1[m, b]$, then $(x, y)^\pi \mathscr{I}_2[m, b]^\lambda$. For the other couples of incident points and lines from $\mathscr{P}_{\mathbf{R}_1}$ the incidence in $\mathscr{P}_{\mathbf{R}_2}$ is obviously preserved. It means that (π, λ) is an isomorphism of the projective planes $\mathscr{P}_{\mathbf{R}_1}, \mathscr{P}_{\mathbf{R}_2}$. The following holds: $(\infty_1)^\pi = (\infty_2)$, $(O)^\pi = (O)$ and $[O]^\lambda = [O]$. The isomorphism (π, λ) satisfies the assumptions of Theorem 4.

If we construct an isotopism $(\alpha, \beta, \gamma, \delta)$ of PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$ in the same way as in Theorem 4, then $x^\gamma = x \circ_1 2$, $x^\beta = x \circ_1 2$, $x^\alpha = x$ for all $x \in \mathscr{S}$. Since O is the zero in the PTR's $\mathbf{R}_1$ and $\mathbf{R}_2$, we have $\varphi_1 = \varphi_2 = \mathrm{id}_\mathscr{S}$. Then $\alpha = \delta = \mathrm{id}_\mathscr{S}$ and $\beta = \gamma$. Since (π, λ) is an isomorphism and $(x, y) \mathscr{I}_1[m, b] \Leftrightarrow (x, y)^\pi \mathscr{I}_2[m, b]^\lambda \Rightarrow (x^\beta, y) \mathscr{I}_2[m^\beta, b]$, it means that for each $x, y, m, b \in \mathscr{S}$ for which the above mentioned relation holds, $x \circ_1 m +_1 b = y$ and $x^\beta \circ_2 m^\beta +_2 b = y$. We can write $x \circ_1 m +_1 b = x^\beta \circ_2 m^\beta +_2 b$ which holds for each $x, m, b \in \mathscr{S}$, as can be verified from Tables 1—3. Hence the planar ternary rings $\mathbf{R}_1$ and $\mathbf{R}_2$ are isotopic.

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CATERPILLARS

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A caterpillar is a tree C with the property that after deleting all terminal edges and all terminal vertices of C a snake (a tree consisting of one simple path) or the null-graph (a graph without vertices and without edges) is obtained. This concept was introduced by F. HARARY and A. J. SCHWENK in [1].

Evidently caterpillars together with the one-vertex graph form a class of trees which is closed under taking subtrees and under connected homomorphisms. Every star and every snake is a caterpillar.

If C is a caterpillar, then we denote by $B(C)$ the graph obtained from C by deleting all terminal vertices and all terminal edges. If $B(C)$ is the null-graph, then C is a tree with two vertices; this case is trivial. In the other cases $B(C)$ is a snake; we shall call it the body of C.

The vertices of $B(C)$ will be denoted by $v_0, v_1, \ldots, v_d$, where d is the length of $B(C)$ and the vertices v_i, v_{i+1} for $i = 0, 1, \ldots, d - 1$ are adjacent. If t_i is the number of terminal edges of C incident with v_i for $i = 0, 1, \ldots, d$, then C is uniquely determined by the vector $[t_0, t_1, \ldots, t_d]$. Note that t_0 and t_d must be different from zero; otherwise v_0 or v_d would be a terminal vertex of C and this would contradict the fact that v_0 and v_d belong to $B(C)$. Nevertheless, t_i for $1 \leqq i \leqq d - 1$ may be equal to zero. Evidently each $(d + 1)$-dimensional vector whose co-ordinates are non-negative integers and the first and the last of them are different from zero determines uniquely a caterpillar in which t_i have the described meaning. In general, two vectors may correspond to every caterpillar with at least three vertices; this depends on the choice of v_0 (for v_0 we choose one of the two terminal vertices of the body of C). The vector of a caterpillar does not depend on the choice of v_0, if and only if there exists an automorphism of C whose restriction onto the body of C is not an identity mapping. If we want to assign a unique vector to every caterpillar, we may take that one which precedes the other in the lexicographical ordering of the set of all $(d + 1)$-dimensional vectors. However, in the sequel, when we speak about the vector of a caterpillar, we mean anyone of the two vectors which are assigned to that caterpillar. A caterpillar with at least three vertices is a snake, if and only if its vector $[t_0, t_1, \ldots, t_d]$ has the property that $t_0 = t_d = 1$, $t_i = 0$ for $i = 1, \ldots, d - 1$. A caterpillar is a star, if and only if its vector is one-dimensional.

There exist various ways how to determine a tree. We shall mention some of them and show characterizations of caterpillars in terms of them.

L. Nebeský [3] has defined tree algebras. A tree algebra (M, P) is an algebra with an element set M and with a ternary operation P which satisfies the following axioms:

I. $P(u, u, v) = u$;

II. $P(u, v, w) = P(v, u, w) = P(u, w, v)$;

III. $P(P(u, v, w), v, x) = P(u, v, P(w, v, x))$;

IV. $P(u, v, x) \neq P(v, w, x) \neq P(u, w, x) \Rightarrow P(u, v, x) = P(u, w, x)$.

Every finite tree T determines uniquely a tree algebra (M, P), whose element set is the vertex set of T and in which $P(u, v, w)$ is the common vertex of the path connecting u and v, the path connecting u and w and the the path connecting v and w in T. Conversely, every finite tree algebra determines uniquely a tree. Thus there is a one-to-one correspondence between finite trees and finite tree algebras. This was proved by L. Nebeský.

Theorem 1. *Let T be a finite tree with at least three vertices, let (M, P) be the tree algebra corresponding to T. The tree T is a caterpillar, if and only if for any nine elements $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$, where $x_1 \neq x_2 \neq x_3 \neq x_1$, $y_1 \neq y_2 \neq y_3 \neq y_1$, $z_1 \neq z_2 \neq z_3 \neq z_1$, the vertex $P(P(x_1, x_2, x_3), P(y_1, y_2, y_3), P(z_1, z_2, z_3))$ coincides with some of the vertices $P(x_1, x_2, x_3)$, $P(y_1, y_2, y_3)$, $P(z_1, z_2, z_3)$.*

Proof. Let T be a caterpillar. As $x_1 \neq x_2 \neq x_3 \neq x_1$, $P(x_1, x_2, x_3)$ cannot be a terminal vertex of T, because a terminal vertex of T can be contained only in such a path whose terminal vertex it is. Thus $P(x_1, x_2, x_3)$ and analogously also $P(y_1, y_2, y_3)$ and $P(z_1, z_2, z_3)$ belong to the body of T. The body of T is a snake, therefore for any three of its vertices there exists one of them which lies between the other two. This implies that at least one of the vertices $P(x_1, x_2, x_3)$, $P(y_1, y_2, y_3)$, $P(z_1, z_2, z_3)$ lies on the path connecting the other two and thus it is equal to $P(P(x_1, x_2, x_3), P(y_1, y_2, y_3), P(z_1, z_2, z_3))$. If T is not a caterpillar, then it contains a subtree isomorphic to the tree in Fig. 1; this was mentioned in [1]. If $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ are such as is denoted in Fig. 1, then $P(x_1, x_2, x_3) = x_2$, $P(y_1, y_2, y_3) = y_2$, $P(z_1, z_2, z_3) = z_2$, but $P(x_2, y_2, z_2) = x_3$, which is different from x_2, y_2, z_2.

Another way of determining trees was described by E. A. Smolenskiî [4]. If $u_1, \ldots, u_n$ are terminal vertices of a tree T and d_{ij} is the distance between u_i nad u_j in T for $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, then the matrix $\|d_{ij}\|$ is called the distance matrix of T. The tree T is uniquely (up to an isomorphism) determined by its distance matrix. In the following theorem the letter u with subscripts has this meaning and the letter v with subscripts has the meaning as in the definition of the vector of a caterpillar.

Theorem 2. *Let T be a tree with n terminal vertices, let $\mathbf{D} = \|d_{ij}\|$ be its distance matrix. The tree T is a caterpillar, if and only if any three pairwise distinct numbers i, j, k from the numbers $1, \ldots, n$ satisfy*

$$\min(d_{ij} + d_{jk} - d_{ik},\ d_{ij} + d_{ik} - d_{jk},\ d_{ik} + d_{jk} - d_{ij}) = 2\,. \tag{1}$$

Proof. Let T be a caterpillar with the vector $[t_0, t_1, \ldots, t_d]$. Let $v_{l(i)}, v_{l(j)}, v_{l(k)}$ be the vertices of the body of T which are adjacent to u_i, u_j, u_k respectively. Without loss of generality let $l(i) \leqq l(j) \leqq l(k)$. Then

$$d_{ij} = 2 + l(j) - l(i)\,, \quad d_{jk} = 2 + l(k) - l(j)\,, \quad d_{ik} = 2 + l(k) - l(i)$$

and therefore

$$d_{ij} + d_{jk} - d_{ik} = 2\,,$$
$$d_{ij} + d_{ik} - d_{jk} = 2 + 2\,l(j) - 2\,l(i) \geqq 2\,,$$
$$d_{ik} + d_{jk} - d_{ij} = 2 + 2\,l(k) - 2\,l(j) \geqq 2\,.$$

Thus the equality (1) holds. Now suppose that (1) holds and prove that T is a caterpillar. Let again u_i, u_j, u_k be three pairwise distinct terminal vertices of T. Let $v = P(u_i, u_j, u_k)$; this symbol is taken from the tree algebra defined above. Evidently

$$d_{ij} = d(u_i, v) + d(u_j, v)\,, \quad d_{jk} = d(u_j, v) + d(u_k, v)\,, \quad d_{ik} = d(u_i, v) + d(u_k, v)\,,$$

where $d(x, y)$ denotes the distance of vertices x and y in T. Then

$$d_{ij} + d_{jk} - d_{ik} = 2d(u_j, v)\,, \quad d_{ij} + d_{ik} - d_{jk} = 2d(u_i, v)\,,$$
$$d_{ik} + d_{jk} - d_{ij} = 2d(u_k, v)\,.$$

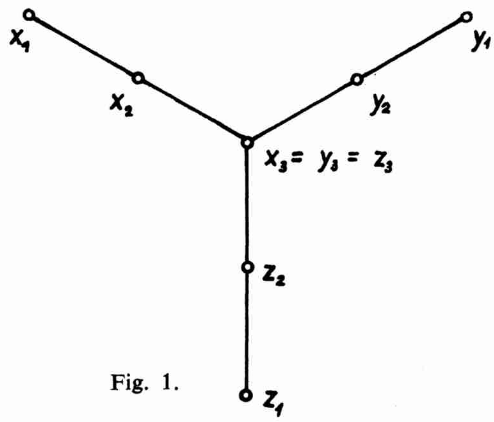

Fig. 1.

If (1) is fulfilled, then at least one of the vertices u_i, u_j, u_k has the distance 1 from $v = P(u_i, u_j, u_k)$. But this excludes the existence of a tree isomorphic to that in Fig. 1 and T must be a caterpillar.

Now we shall study some problems of embedding.

L. Nebeský has defined a completely separable tree as a tree which can be embedded into every block graph which has exactly two blocks and the same number of vertices as this tree. (A block graph is a graph, each of whose blocks is a clique. The problems of embedding trees into block graphs were studied in [5]. Here we shall define a stronger concept of a completely separable rooted tree.

A rooted graph is a graph in which one vertex is chosen and called the root of this graph. If this graph is a tree, it is called a rooted tree. A rooted tree is called completely separable, if it can be embedded into every rooted block graph which has exactly two blocks and the same number of vertices as this tree and its root is not a cut-vertex in it, in such a way the root of the tree and the root of the block graph coincide.

Theorem 3. *A rooted tree is completely separable, if and only if it is a caterpillar whose root is a terminal vertex adjacent to a terminal vertex of its body.*

Proof. Let C be a rooted caterpillar whose root is a terminal vertex adjacent to a terminal vertex of its body. Let the vector of C be $[t_0, t_1, \ldots, t_d]$, let the root of C be a vertex r adjacent to v_0. Let the number of vertices of C be n; then $d + 1 + \sum_{i=0}^{d} t_i = n$. For $j = 0, 1, \ldots, d$ let $n_j = j + 1 + \sum_{i=0}^{j} t_i$. Let G be a block graph having exactly two blocks, one with k vertices, the other with $n - k + 1$ vertices, where k is a positive integer, $2 \leqq k \leqq n - 1$. Let B_1 be the block of G with k vertices and B_2 the block of G with $n - k + 1$ vertices. Let a be a cut-vertex of G. Let the root r_0 of G be in B_1. If $k - 1 \leqq n_0$, then we identify the vertex v_0 of C with the vertex a of G, choose $k - 1$ vertices from the n_0 vertices of C adjacent with v_0, one of them being r, and identify them with the vertices of B_1 so that r is identified with r_0. The remaining vertices will be identified with vertices of B_2. The embedding is complete. If $k - 1 > n_0$, then there exists j such that $n_j < k - 1 \leqq n_{j+1}$. Then we identify v_{j+1} with a. From the t_{j+1} terminal vertices adjacent to v_{j+1} we choose $k - 1 - n_j$ ones. These vertices, the vertices $v_0, \ldots, v_j$ and all terminal vertices adjacent to some of the vertices $v_0, \ldots, v_j$ will be identified with the vertices of B_1, r being identified with r_0, and the remaining vertices will be identified with the vertices of B_2.

Now we shall prove that no other rooted tree is completely separable. A root of a completely separable rooted tree must be its terminal vertex. If we have a block graph with n vertices and two blocks, one of which has only two vertices and the root of this block graph is the vertex of the two-vertex block which is not a cut-vertex, then this root has the degree one. If we embed a tree with n vertices into this block graph, only a terminal vertex of this tree can be identified with this root. Let us have a rooted tree T which is not a caterpillar and suppose that it is completely separable. Then T contains a vertex w such that there exist at least three branches A_1, A_2, A_3 outgoing from w, each of which contains at least three vertices including w. The vertex w is not a root of T, because it is not terminal. Thus the root r of T belongs to a branch A_0 outgoing from w. The branch A_0 may coincide with some of the branches A_1, A_2, A_3; without loss of generality suppose $A_0 \neq A_1$, $A_0 \neq A_2$. Let A_0 have k vertices. Take a rooted block graph G with n vertices and two blocks, one of which has $k + 1$ vertices, contains the root r_0 of G and is denoted by B_1. No vertex of A_0 can be identified with the cut-vertex a of G; otherwise some vertices of B_1 would be identified with no vertex of T. Thus the whole A_0 is embedded into B_1 and a must be identified with a vertex a_0 of T adjacent to w and not belonging to A_0. Without loss of generality suppose that a_0 does not belong to A_2. Then A_2 must be embedded into the same block of G as B_0, but this is not possible, because this block has only $k + 1$ vertices and they are identified with the vertices of A_0 and with the vertex a_0. This is a contradiction with the assumption that T is completely separable. Now let C be a caterpillar with n vertices with the vector $[t_0, t_1, \ldots, t_d]$ and let its root be a terminal vertex adjacent to some v_j, where $j \neq 0$, $j \neq d$. Suppose that C is completely separable. Take a rooted block graph G with n vertices and two blocks, one of which has three vertices and contains the root of G. Then either v_{j-1}, or v_{j+1} must be identified with the cut-vertex of G. Now by a similar argument as in the preceding case we obtain a contradiction.

Now we shall consider embedding caterpillars into the graphs of n-dimensional cubes (or shortly n-cubes). A graph of the n-cube, where n is a positive integer, is the graph whose vertices are all n-dimensional vectors whose co-ordinates are zeros and ones and in which two vertices are joined by an edge if and only if they differ from each other in exactly one co-ordinate. Embedding trees into n-cubes was studied by I. Havel and P. Liebl [2]. Every finite tree is embeddable into an n-cube for some n. If T is a finite tree, then the minimal n such that T is embeddable into the n-cube will be called the dimension of T and denoted by dim T.

Theorem 4. *Let T be a tree with $k \geqq 2$ vertices. Then*

$$]\log_2 k[\; \leqq \dim T \leqq k - 1 \, .$$

These bounds cannot be improved.

Remark. The symbol $]x[$ denotes the least integer which is greater than or equal to x; some authors call it "the post-office function".

Proof. If l is a positive integer and $l < \,]\log_2 k[$, then $l < \log_2 k$. The number of vertices of the l-cube is $2^l < k$ and thus a graph with k vertices cannot be embedded into it. Therefore $]\log_2 k[\; \leqq \dim T$. The upper bound will be proved by induction. If $k = 2$, then T is isomorphic to the one-dimensional cube and $\dim T = 1$; thus the assertion is true. Now let $k > 2$. Let u be a terminal vertex of T, let e be the edge incident with u, let v be the other end vertex of e. By deleting u and e from T we obtain a tree T' with $k - 1$ vertices. Let $m = \dim T'$. According to the induction assumption, $m \leqq k-2$. Consider a graph Q_{k-1} of the $(k-1)$-dimensional cube. Its vertices are $(k-1)$-dimensional vectors whose co-ordinates are zeros and ones. Let Q_{k-2} be the subgraph of Q_{k-1} induced by the set of all vertices of Q_{k-1} whose last co-ordinate is 0; this graph Q_{k-2} is a graph of the $(k-2)$-dimensional cube. We embed T' into Q_{k-2}. Let $[a_1, \ldots, a_{k-2}, 0]$ be the vertex of Q_{k-2} with which v is identified in this embedding (the co-ordinates $a_1, \ldots, a_{k-2}$ are zeros or ones). Then we identify u with $[a_1, \ldots, a_{k-2}, 1]$ and T is embedded into Q_{k-1}. Therefore $\dim T \leqq \dim T' + 1 \leqq k - 1$. The bounds cannot be improved, because a snake with k vertices can be embedded into the cube of the dimension $]\log_2 k[$ (as a part of its Hamiltonian path) and a star with k vertices cannot be embedded into the cube of the dimension smaller than $k-1$ (in such a cube there exists no vertex of the degree at least $k-1$).

Theorem 5. *Let k, m be positive integers such that $k \geqq 2$ and*

$$]\log_2 k[\ \leqq m \leqq k-1\,.$$

Then there exists a caterpillar C with k vertices such that $\dim C = m$.

Proof. For any positive integer h such that $2 \leqq h \leqq k-2$ let $C(h)$ be a caterpillar with the vector $[t_0, t_1, \ldots, t_d]$, where $d = k - h - 1$, $t_0 = h - 1$, $t_d = 1$ and $t_i = 0$ for $i = 1, \ldots, d-1$. The caterpillar $C(2)$ is a snake with k vertices and $\dim C(2) =]\log_2 k[$. The caterpillar $C(k-2)$ can be emebedded into the $(k-2)$-dimensional cube so that v_0 is identified with $[0, \ldots, 0]$, v_1 with $[1, 0, \ldots, 0]$, the terminal vertices adjacent to v_0 are identified with $[0, 1, 0, \ldots, 0]$, $[0, 0, 1, 0, \ldots, 0]$, ..., $[0, \ldots, 0, 1]$, the terminal vertex adjacent to v_1 is identified with $[1, 1, 0, \ldots, 0]$. But $C(k-2)$ cannot be embedded into the $(k-3)$-dimensional cube, because it contains the vertex v_0 of the degree $k-2$. Therefore $\dim C(k-2) = k-2$. Now let us take the caterpillars $C(h)$ and $C(h+1)$ for some h, $2 \leqq h \leqq k-3$. The caterpillar $C(h+1)$ is obtained from $C(h)$ by deleting one terminal vertex and adding another. In the proof of Theorem 4 we have proved that by adding one terminal vertex the dimension of a tree increases at most by one; by deleting a vertex obviously it cannot increase. Thus $\dim C(h+1) \leqq 1 + \dim C(h)$. This implies that $\dim C(h)$ for $2 \leqq h \leqq k-2$ attains all integral values in the interval $\langle]\log_2 k[, k-2\rangle$. There exists a caterpillar C with k vertices and $\dim C = k-1$, namely a star. Thus the assertion is proved.

In the end we propose two problems.

Problem 1. A universal caterpillar for caterpillars with n vertices is a caterpillar into which each caterpillar with n vertices can be embedded. Determine the least number of vertices of a universal caterpillar for caterpillars with n vertices.

Problem 2. Characterize graphs whose spanning tree is a caterpillar. (This is a generalization of graphs with Hamiltonian paths.)

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KONFIGURACE BODŮ ROVINNÉ KUBIKY III

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V článcích [1] a [2] jsme převedli hledání konfigurací bodů rovinné kubiky na určování podgrup grupy G – tj. grupy bodů rovinné kubiky. Odvodili jsme, že existují podgrupy řádu $9.2^n, 2^{n+1}, 2^{n+2}$, 3.2^n pro kubiku rodu 1 a 2.2^n, 3.2^n pro kubiku s bodem uzlovým. V tomto článku dokážeme daleko obecnější větu.

Věta 1. *Grupa G bodů rovinné kubiky, která má aspoň tři inflexní body, má podgrupy všech konečných řádů.*

Poznámka 1. V dalším budeme písmenem G označovat grupu z věty 1 a operaci této grupy nazývejme násobení – abychom v dalším mohli použít mocnin a odmocnin.

Věta 2. *Každý prvek grupy G je jistá n-tá odmocnina z jednotkového prvku. Existují n-té odmocniny z jednotkového prvku pro každé přirozené n.*

Důkaz. Nechť inflexní bod J je jednotkový prvek a X je libovolný bod uvažované kubiky $(X \neq J)$. Konstruujme mocniny bodu X. Tečnový bod bodu X je $1/X^2$ (body označujeme pomocí symbolů pro násobení). Třetí průsečík kubiky s přímkou $J(1/X^2)$ je X^2. Třetí průsečík přímky XX^2 s kubikou je $1/X^3$. Na přímce $J(1/X^3)$ leží bod X^3 atd. Je zřejmé, že uvažované přímky vytvářejí jednak svazek přímek se středem v J a jednak svazek přímek se středem v X. Nyní může nastat:

1) Přímka XX^k je tečnou v bodě X^k, potom je $X^k = 1/X^{k+1}$ a tedy $X^{2k+1} = J$.

2) Přímka $J(1/X^k)$ je tečnou v bodě $1/X^k$, potom je $X^k = 1/X^k$ a tedy $X^{2k} = J$.

Jeden z těchto případů musí nastat (jak X, tak i J jsou tečnové body), protože postup při konstruování bodu X můžeme algebraicky zapsat, navíc můžeme dát podmínku, aby $X^k = 1/X^{k+1}$ resp. $X^k = 1/X^k$ pro libovolné k. Vyjdeme-li např. od bodu $X = (x_1, x_2, 1)$ (nevlastní body $(y_1, y_2, 0)$ jsou jenom tři a pro tyto body můžeme naší úvahu provést speciálně) povede předcházející podmínka k soustavě dvou rovnic o dvou neznámých (např. $1/X^{k+1} = (x'_1, x'_2, 1)$, tj. $x_1 = x'_1$, $x'_2 = x_2$) a protože uvažujeme kubiku v projektivní rovině rozšířené o komplexní elementy, má tato soustava vždy řešení.

Poznámka 2. Zřejmě body, které mají za svůj tečnový bod, bod J, jsou druhé odmocniny z J. Inflexní body jsou třetí odmocniny z J. Body, které mají za svůj tečnový bod některý inflexní bod jsou šesté odmocniny z J. Podobně body, které mají za svůj tečnový bod druhou odmocninu z J jsou čtvrté odmocniny z J, atd.

Poznámka 3. Z předcházejícího je vidět, jestliže G je grupa bodů kubiky s bodem uzlovým, existují právě dvě $\sqrt{J}$, tři $\sqrt[3]{J}$, čtyři $\sqrt[4]{J}$ a šest $\sqrt[6]{J}$. Je-li G grupa bodů kubiky bez singulárního bodu, existují čtyři $\sqrt{J}$, devět $\sqrt[3]{J}$, šestnáct $\sqrt[4]{J}$ a třicetšest $\sqrt[6]{J}$. Pravděpodobně platí, že pro kubiku s bodem uzlovým existuje právě n n-tých odmocnin z J a pro kubiku bez singulárního bodu existuje právě n^2 n-tých odmocnin z J. Toto bychom dokázali podrobnějším zkoumáním algebraického vyjádření v důkaze věty 2. Pro naše další úvahy stačí dokázat, že těchto $\sqrt{J}$ je aspoň n a těchto n n-tých odmocnin určuje podgrupu grupy G. Uvědomíme si, že o uvažovaných prvcích grupy G platí všechny výsledky, které platí o n-tých odmocninách z jedné v tělese komplexních čísel tj. o komplexních jednotkách. Pro každé přirozené n existuje tzv. primitivní n-tá odmocnina z jedné a v našem případě existuje tedy primitivní n-tá odmocnina z J a cyklická podgrupa určená touto primitivní odmocninou je podgrupa splňující větu 1. Věta 1 je tedy dokázaná.

Užitím věty 2 z článku [1] a věty 1 dostáváme:

Věta 3. *Existuje konfigurace bodů rovinné kubiky, která má aspoň tři inflexní body, typu:* $(3n_n, n_3^2)$, *přičemž n je přirozené číslo větší než 2.*

V článcích [1] a [2] jsme mimo základních typů konfigurací (jako v předcházející větě 3) dostávali další konfigurace tím, že jsme podrobně zkoumali podgrupy grupy G (vynecháním bodů resp. přímek jsme dostávali další typy). V této práci si všimneme podrobněji podgrup pro případ kdy n je prvočíslo větší než tři.

Věta 4. *Nechť p je prvočíslo větší než tři. V podgrupě $G_p \subset G$ platí:*

1) *Každý bod grupy G_p je tečnovým bodem bodu grupy G_p.*
2) *Body grupy G_p různé od J (J je inflexní a jednotkový) lze rozdělit do disjunktních dvojic, přičemž spojnice bodů každé dvojice prochází bodem J.*
3) *Každým bodem $T \in G_p$ $(T \neq J)$ prochází právě $(p-3)/2$ přímek na nichž leží právě tři vesměs různé body grupy G_p.*

Důkaz. Podle předcházejícího je každý bod ležící v G_p tzv. primitivní $\sqrt[n]{J}$ a každé dvě primitivní $\sqrt[n]{J}$ mají různé druhé mocniny (tečnový bod každého bodu leží také v G_p) a z toho vyplývá tvrzení 1). Dvojice bodů z tvrzení dvě jsou navzájem inverzní body. Každý bod T má v G_p jednak svůj jediný tečnový bod T' a jednak je tečnovým bodem jediného bodu $\bar{T}$ (podle již dokázaného tvrzení 1). Body $\bar{T}$, T', T jsou navzájem různé, neboť bod T není inflexní. Spojnice bodu T s každým dalším bodem grupy G_p prochází ještě jediným bodem grupy G_p. Těchto spojnic je tedy $p - 3$ a vždy dvě splynou. Tím je dokázáno tvrzení 3.

Uvažujme nyní body grupy G_p různé od bodu J a všechny spojnice těchto bodů. Dostáváme:

Věta 5. *Existují konfigurace bodů rovinné kubiky, která má aspoň tři inflexní body, typu:* $(p - 1_{(p-5)/2}, \frac{1}{2}(p - 5)(p - 1)_3)$, *kde p je prvočíslo větší než sedm.*

Literatura

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Zusammenfassung

KONFIGURATIONEN VON PUNKTEN AN EINER KUBIK, III

JAROMÍR KRYS, Hradec Králové

Das wichtigste Ergebnis des Artikels ist der Beweis des Satzes: *Die Gruppe G von Punkten einer ebenen Kubik, welche mindestens drei inflexe Punkte hat, besitzt Untergruppen der Ordnung n, wo n eine beliebige natürliche Zahl ist.*

Dadurch ist bewiesen, das es Konfigurationen der gegebenen ebenen Kubik des Typs $(3g_g\ g_3^2)$ für alle natürliche Zahl $g > 2$ gibt.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON AN INTEGRAL OPERATOR IN THE SPACE OF FUNCTIONS WITH BOUNDED VARIATION, II

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In this note the considerations from [3] concerning the Fredholm-Stieltjes integral equations in the space $BV_n[0, 1]$ of all n-vector functions of bounded variation on the interval $[0, 1]$ are continued.

Let us denote by R_n the n-dimensional real space of all column n-vectors. By a star the transpose of a vector or a matrix will be denoted. For $\mathbf{x} = (x_1, \ldots, x_n)^* \in R_n$ we define the norm $\|\mathbf{x}\| = \max_{i=1,\ldots,n} |x_i|$. The set of all $n \times n$-matrices let be denoted by $L(R_n)$. For an $n \times n$-matrix $\mathbf{A} = (a_{ij})$, $i, j = 1, \ldots, n$ we set $\|\mathbf{A}\| = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |a_{ij}|$. The relation for $\|\mathbf{A}\|$ defines the usual operator norm which corresponds to the norm in R_n given above.

We denote by $BV_n[0, 1] = BV_n$ the set of all column n-vector functions $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^*$, $t \in [0, 1]$ for which

$$\|\mathbf{x}\|_{BV_n} = \|\mathbf{x}(0)\| + \operatorname{var}_0^1 \mathbf{x} < \infty$$

where $\operatorname{var}_0^1 \mathbf{x}$ means the usual variation of the function $\mathbf{x}$ on the interval $[0, 1]$. By $\|\cdot\|_{BV_n}$ a norm in BV_n is given and the linear space BV_n equipped with this norm is a Banach space. If $\varphi \in BV_n$ then the one-sided limits $\lim_{\tau \to t+} \varphi(\tau) = \varphi(t+)$, $t \in [0, 1)$ and $\lim_{\tau \to t-} \varphi(\tau) = \varphi(t-)$, $t \in (0, 1]$ exist. Further, let NBV_n be the subspace of all elements $\varphi \in BV_n$ for which $\varphi(t+) = \varphi(t)$ if $t \in (0, 1)$ and $\varphi(0) = \mathbf{0}$. NBV_n is a closed subspace in BV_n and, consequently, NBV_n is also a Banach space if it is equipped with the norm of BV_n, i.e. $\|\varphi\|_{NBV_n} = \operatorname{var}_0^1 \varphi$.

Let us set

$$(1) \qquad \langle \mathbf{x}, \varphi \rangle = \int_0^1 \mathbf{x}^*(t)\, d\varphi(t) = \sum_{i=1}^{n} \int_0^1 x_i(t)\, d\varphi_i(t)$$

for $\mathbf{x} \in BV_n$, $\varphi \in NBV_n$ where the integration is taken in the Perron-Stieltjes sense. The integrals occurring in this definition exist (see [4]).

The relation $\langle \cdot, \cdot \rangle$ evidently defines a bilinear form on $BV_n \times NBV_n$.

1. Lemma. *If $\varphi \in NBV_n$ and $\langle \mathbf{x}, \varphi \rangle = 0$ for every $\mathbf{x} \in BV_n$ then $\varphi = \mathbf{0}$. If $\mathbf{x} \in BV_n$ and $\langle \mathbf{x}, \varphi \rangle = 0$ for every $\varphi \in NBV_n$ then $\mathbf{x} = \mathbf{0}$.*

Proof. Assume that $\varphi \neq \mathbf{0}$. Then there exists an index $i = 1, \ldots, n$ such that either a) there is an $\alpha \in (0, 1)$ such that $\varphi_i(\alpha-) \neq \varphi_i(\alpha)$ or b) $\varphi_i(t-) = \varphi_i(t)$ for all $t \in (0, 1)$ and

1) $\varphi_i(0+) \neq 0 = \varphi_i(0)$

or

2) $\varphi_i(0+) = 0$, $\varphi_i(1) \neq \varphi_i(1-)$

or

3) φ_i is continuous on $[0, 1]$ and there exist $0 \leqq \beta < \gamma \leqq 1$ such that $\varphi_i(\beta) = \varphi_i(\gamma)$.

For the cases a), b1), b2) let us define $x_j(t) = 0, j \neq i, t \in [0, 1]$, $x_i(t) = 0$, $t \in [0, 1]$, $t \neq \alpha$, $t \neq 0$, $t \neq 1$ and $x_i(\alpha) = 1$, $x_i(0) = 1$, $x_i(1) = 1$ respectively. Then we have

$$\langle \mathbf{x}, \varphi \rangle = \int_0^1 x_i(t)\, \mathrm{d}\varphi_i(t) = x_i(\alpha)\,[\varphi_i(\alpha+) - \varphi_i(\alpha-)] = \varphi_i(\alpha) - \varphi_i(\alpha-) \neq 0$$

by Proposition 2,1 from [3] in the case a) and similarly $\langle \mathbf{x}, \varphi \rangle \neq 0$ in the cases b1) and b2). In the case b3) let us set $x_i(t) = 1$ for $t \in [\beta, \gamma]$, $x_i(t) = 0$ for $t \in [0, 1] \setminus [\beta, \gamma]$. By the same Proposition 2,1 from [3] it can be easily shown that in this case we have also $\langle \mathbf{x}, \varphi \rangle \neq 0$. Hence the first assertion of our lemma is proved.

For proving the second part let us assume that $\mathbf{x} \in BV_n$, $\mathbf{x} \neq \mathbf{0}$. Then for some $i = 1, \ldots, n$ either there exists an $\alpha \in (0, 1]$ such that $x_i(\alpha) \neq 0$ or $x_i(t) = 0$ for every $t \in (0, 1]$ and $x_i(0) \neq 0$. In the first case we set $\varphi_i(t) = 0$ for $t \in [0, \alpha)$, $\varphi_i(t) = 1$ for $t \in [\alpha, 1]$ and $\varphi_j(t) = 0$ for all $t \in [0, 1]$ and $j = 1, \ldots, n$, $j \neq i$. Evidently $\varphi \in NBV_n$ and by Proposition 2,1 from [3] we get $\langle \mathbf{x}, \varphi \rangle = \int_0^1 x_i(t)\, \mathrm{d}\varphi_i(t) = x_i(\alpha) \neq 0$. In the second case we set $\varphi_i(t) = 1$, $t \in (0, 1]$, $\varphi_i(0) = 0$ and Proposition 2,1 [3] implies also in this case $\langle \mathbf{x}, \varphi \rangle = x_i(0) \neq 0$.

2. Proposition. *The pair of the spaces BV_n, NBV_n forms a dual system (BV_n, NBV_n) with respect to the bilinear form $\langle \mathbf{x}, \varphi \rangle$ given by the relation (1).*

This proposition is an immediate consequence of Lemma 1 and the definition of a dual system, see [1], § 15.

Let us denote $J = [0, 1] \times [0, 1]$ and assume that $\mathbf{K}(t, s) : J \to L(R_n)$ is an $n \times n$-matrix valued function defined on the square J such that

$$v_J(\mathbf{K}) < \infty \tag{2}$$

where $v_J(\mathbf{K})$ denotes the two-dimensional (Vitali) variation of $\mathbf{K}$ on J (see [3]). Further, we assume that

$$\operatorname{var}_0^1 \mathbf{K}(0, \cdot) < \infty\,. \tag{3}$$

These assumptions assure that for every fixed $t \in [0, 1]$ the variation $\operatorname{var}_0^1 \mathbf{K}(t, \cdot)$ is finite and, consequently, for any $\mathbf{x} \in BV_n$ the integral

$$\int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{K}\mathbf{x} \tag{4}$$

exists for every $t \in [0, 1]$. In this way the relation (4) defines a linear operator on the space BV_n which maps BV_n into itself (see [3], Proposition 2,3).

The function $\mathbf{K}(t, s) : J \to L(R_n)$ which determines the operator $\mathbf{K}$ by the relation (4) is called the kernel of the operator $\mathbf{K}$. In some situations the operator $\mathbf{K}$ remains unchanged if the kernel $\mathbf{K}(t, s) : J \to L(R_n)$ is altered.

3. Proposition. *Let us assume that* $\mathbf{K}(t, s)\ J: \to L(R_n)$ *satisfies* (2) *and* (3) *and define a new kernel* $\mathbf{K}^*(t, s)$ *by the relations* [1])

$$\mathbf{K}^*(t, s) = \mathbf{K}(t, s+) - \mathbf{K}(t, 0) = \lim_{\sigma \to s+} \mathbf{K}(t, \sigma) - \mathbf{K}(t, 0) \quad \textit{if} \quad s \in (0, 1), \tag{5}$$

$$\mathbf{K}^*(t, 0) = 0, \quad \mathbf{K}^*(t, 1) = \mathbf{K}(t, 1) - \mathbf{K}(t, 0).$$

Then

(i) $v_J(\mathbf{K}^*) < \infty$, $\operatorname{var}_0^1 \mathbf{K}^*(0, \cdot) < \infty$, $\operatorname{var}_0^1 \mathbf{K}^*(\cdot, 0) < \infty$,

(ii) $\mathbf{K}\mathbf{x} = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \int_0^1 \mathrm{d}_s[\mathbf{K}^*(t, s)]\, \mathbf{x}(s)$ *for every* $\mathbf{x} \in BV_n$,

(iii) *the integral* $\int_0^1 \mathbf{K}^*(t, s)\, \mathrm{d}\psi(t)$ *exists for every* $\psi \in BV_n$, $s \in [0, 1]$ *and*

$$\int_0^1 \mathbf{K}^*(t, 0)\, \mathrm{d}\psi(t) = \mathbf{0}, \tag{6}$$

$$\lim_{\delta \to 0+} \int_0^1 \mathbf{K}^*(t, s + \delta)\, \mathrm{d}\psi(t) = \int_0^1 \mathbf{K}^*(t, s)\, \mathrm{d}\psi(t) \quad \textit{for any} \quad s \in (0, 1). \tag{7}$$

Proof. Let us assume that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ is an arbitrary subdivision of the interval $[0, 1]$ and let us create the corresponding net-type subdivision

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j], \quad i.j = 1, \ldots, k$$

of the interval J. Let us set $\mathbf{K}(t, s) = \mathbf{K}(t, 1)$ for every $t \in [0, 1]$, $s > 1$. For any given $\delta > 0$ we have

$$\sum_{i=1}^{k} \|\mathbf{K}(\alpha_i, \alpha_1 + \delta) - \mathbf{K}(\alpha_i, \alpha_0) - \mathbf{K}(\alpha_{i-1}, \alpha_1 + \delta) + \mathbf{K}(\alpha_{i-1}, \alpha_0)\| +$$
$$+ \sum_{j=2}^{k} \sum_{i=1}^{k} \|\mathbf{K}(\alpha_i, \alpha_j + \delta) - \mathbf{K}(\alpha_i, \alpha_{j-1} + \delta) - \mathbf{K}(\alpha_{i-1}, \alpha_j + \delta) +$$
$$+ \mathbf{K}(\alpha_{i-1}, \alpha_{j-1} + \delta)\| \leqq v_J(\mathbf{K}).$$

---

[1]) Let us mention that the limit $\mathbf{K}(t, s+)$ exists for every $t \in [0, 1]$, $s \in [0, 1)$ if $\mathbf{K}(t, s)$ satisfies (2) and (3) since for every $t \in [0, 1]$ $\mathbf{K}(t, s)$ is of bounded variation in the second variable.

Passing to the limit $\delta \to 0+$ we get by the definition (5) of $\mathbf{K}^*$ the inequality

$$\sum_{j=1}^{k} \sum_{i=1}^{k} \|\mathbf{K}^*(\alpha_i, \alpha_j) - \mathbf{K}^*(\alpha_i, \alpha_{j-1}) - \mathbf{K}^*(\alpha_{i-1}, \alpha_j) + \\ + \mathbf{K}^*(\alpha_{i-1}, \alpha_{j-1})\| \leqq v_J(\mathbf{K}).$$

This holds for every net-type subdivision J_{ij} of J and, consequently, by the definition of the Vitali variation we obtain

$$v_J(\mathbf{K}^*) \leqq v_J(\mathbf{K}) < \infty .$$

Further, we have

$$\operatorname{var}_0^1 \mathbf{K}^*(0, \cdot) = \operatorname{var}_0^1 (\mathbf{K}(0, t+) - \mathbf{K}(0, 0)) =$$
$$= \operatorname{var}_0^1 (\mathbf{K}(0, t+) - \mathbf{K}(0, t) + \mathbf{K}(0, t) - \mathbf{K}(0, 0)) \leqq$$
$$\leqq \operatorname{var}_0^1 (\mathbf{K}(0, t+) - \mathbf{K}(0, t)) + \operatorname{var}_0^1 \mathbf{K}(0, t) \leqq 2 \operatorname{var}_0^1 \mathbf{K}(0, \cdot) < \infty .$$

Clearly also $\operatorname{var}_0^1 \mathbf{K}^*(\cdot, 0) = 0$. In this way (i) is proved.

Since $\operatorname{var}_0^1 \mathbf{K}(t, \cdot) < \infty$ for every $t \in [0, 1]$ (see (2,14 a) in [3]) we obtain from the well known properties of functions with bounded variation that $\mathbf{K}(t, s+) - \mathbf{K}(t, s) = = \mathbf{0}$ holds for every $s \in (0, 1)$ except an at most countable set of points in the interval $(0, 1)$. Hence for the difference $\mathbf{W}(t, s) = \mathbf{K}^*(t, s) - \mathbf{K}(t, s)$ we have $\mathbf{W}(t, s+) - - \mathbf{W}(t, s-) = \mathbf{0}$ for any $s \in (0, 1)$ and it can be shown also that $\mathbf{W}(t, 0+) = \mathbf{W}(t, 0)$, $\mathbf{W}(t, 1) = \mathbf{W}(t, 1-)$. By Corollary 2,2 in [3] we obtain

$$\int_0^1 d_s[\mathbf{W}(t, s)]\, \mathbf{x}(s) = \int_0^1 d_s[\mathbf{K}^*(t, s)]\, \mathbf{x}(s) - \int_0^1 d_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) = \mathbf{0}$$

for all $t \in [0, 1]$ and for any $\mathbf{x} \in BV_n$. Hence (ii) is proved.

Since by (i) we have $v_J(\mathbf{K}^*) < \infty$ and $\operatorname{var}_0^1 \mathbf{K}^*(\cdot, 0) < \infty$, it is also $\operatorname{var}_0^1 \mathbf{K}^*(\cdot, s) < < \infty$ for every $s \in [0, 1]$ and the integral $\int_0^1 \mathbf{K}^*(t, s)\, d\psi(t)$ exists for every $\psi \in BV_n$ (see e.g. [4]). The relation (6) is clear from $\mathbf{K}^*(t, 0) = 0$, $t \in [0, 1]$. For every $t \in [0, 1]$, $s \in (0, 1)$ we have $\|\mathbf{K}^*(t, s + \delta) - \mathbf{K}^*(t, s)\| \leqq \|\mathbf{K}^*(0, s + \delta) - \mathbf{K}^*(0, s)\| + + \operatorname{var}_0^1 (\mathbf{K}^*(\cdot, s + \delta) - \mathbf{K}^*(\cdot, s))$. Hence $\lim_{\delta \to 0+} \sup_{t \in [0,1]} \|\mathbf{K}^*(t, s + \delta) - \mathbf{K}^*(t, s)\| = 0$ (see Remark 2,3 in [3]) and consequently

$$\lim_{\delta \to 0+} \left\| \int_0^1 (\mathbf{K}^*(t, s + \delta) - \mathbf{K}^*(t, s))\, d\psi(t) \right\| \leqq$$
$$\leqq \lim_{\delta \to 0+} \sup_{t \in [0,1]} \|\mathbf{K}^*(t, s + \delta) - \mathbf{K}^*(t, s)\| \operatorname{var}_0^1 \psi = 0 .$$

This proves (iii) and also the proposition.

4. Corollary. *Let us assume that* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3). *Let us define*

$$\mathbf{K}'\varphi = \int_0^1 (\mathbf{K}^\#)^* (t, s) \,\mathrm{d}\varphi(t)\,, \quad \varphi \in BV_n$$

where $(\mathbf{K}^\#)^*$ *is the transposed matrix to* $\mathbf{K}^\#$ *defined by* (5). *Then* $\mathbf{K}'$ *is a linear operator which maps* BV_n *into* NBV_n.

Proof. For an arbitrary subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ of the interval $[0, 1]$ we have

$$\sum_{i=1}^k \left\| \int_0^1 ((\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1}))\, \mathrm{d}\varphi(t) \right\| \leqq$$

$$\leqq \sum_{i=1}^k \sup_{t \in [0,1]} \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1})\| \operatorname{var}_0^1 \varphi \leqq$$

$$\leqq \operatorname{var}_0^1 \varphi \,.\, (v_J((\mathbf{K}^\#)^*) + \operatorname{var}_0^1 (\mathbf{K}^\#)^* (0, \cdot))$$

since (see (2,12) in [3]) we have

$$\sum_{i=1}^k \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1})\| \leqq$$

$$\leqq \sum_{i=1}^k \|(\mathbf{K}^\#)^* (t, \alpha_i) - (\mathbf{K}^\#)^* (t, \alpha_{i-1}) - (\mathbf{K}^\#)^* (0, \alpha_i) + (\mathbf{K}^\#)^* (0, \alpha_{i-1})\| +$$

$$+ \sum_{i=1}^k \|(\mathbf{K}^\#)^* (0, \alpha_i) - (\mathbf{K}^\#)^* (0, \alpha_{i-1})\| \leqq$$

$$\leqq \sum_{i=1}^k v_{[0,1] \times [\alpha_{i-1}, \alpha_i]}((\mathbf{K})^\#)^* + \operatorname{var}_0^1 ((\mathbf{K})^\#)^* \leqq v_J((\mathbf{K}^\#)^*) + \operatorname{var}_0^1 (\mathbf{K}^\#)^* (0, \cdot)\,.$$

This implies $\operatorname{var}_0^1 \int_0^1 (\mathbf{K}^\#)^* (t, s)\, \mathrm{d}\varphi(t) < \infty$ because $(\mathbf{K}^\#)^*$ evidently satisfies (i) from Proposition 3. From (iii) of the same proposition and from the definition of NBV_n we obtain that for every $\varphi \in BV_n$ the integral $\int_0^1 (\mathbf{K}^\#)^* (t, s)\, \mathrm{d}\varphi(t)$ as a function of the variable s belongs to NBV_n.

From the results of [3], the following result can be easily deduced:

5. Theorem. *If* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3) *then the relation*

$$\mathbf{K}\mathbf{x} = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s)\,, \quad t \in [0, 1]\,, \quad \mathbf{x} \in BV_n \tag{8}$$

defines a completely continuous operator on BV_n.

The relation

$$\mathbf{K}'\varphi = \int_0^1 (\mathbf{K}^\#)^* (t, s)\, \mathrm{d}\varphi(t)\,, \quad s \in [0, 1]\,, \quad \varphi \in NBV_n \tag{9}$$

where $\mathbf{K}^\#$ *is given by* (5) *defines a completely continuous operator on* NBV_n.

Moreover, if $\langle \cdot, \cdot \rangle$ *is the bilinear form on* $BV_n \times NBV_n$ *given by* (1) *then*

$$\langle \mathbf{K}\mathbf{x}, \varphi \rangle = \langle \mathbf{x}, \mathbf{K}'\varphi \rangle \tag{10}$$

for every $\mathbf{x} \in BV_n$ *and* $\varphi \in NBV_n$.

Proof. The complete continuity of $\mathbf{K}$ given by (8) is proved in Theorem 3,1 from [3]. Theorem 3,2 from [3] states that the operator

$$\mathbf{K}'\psi = \int_0^1 (\mathbf{K}^*)^* (t, s) \, \mathrm{d}\psi(t), \quad \psi \in BV_n$$

is completely continuous on BV_n. Since NBV_n is a closed subspace of BV_n the restriction of this operator onto NBV_n (i.e. the operator $\mathbf{K}'$ given by (9)) is also completely continuous and maps NBV_n into itself (cf. Corollary 4). Hence the second statement is also valid.

By (ii) from Proposition 3 we have $\mathbf{K}\mathbf{x} = \mathbf{K}^\#\mathbf{x}$, where $\mathbf{K}^\#\mathbf{x} = \int_0^1 \mathrm{d}_s[\mathbf{K}^\#(t, s)] \, \mathbf{x}(s)$, $\mathbf{x} \in BV_n$ and $\mathbf{K}^\#$ is given by (5). Hence $\langle \mathbf{K}\mathbf{x}, \varphi \rangle = \langle \mathbf{K}^\#\mathbf{x}, \varphi \rangle$ for every $\mathbf{x} \in BV_n$, $\varphi \in NBV_n$. Using Lemma 2,2 from [3] we interchange the order of integrations and by an easy computation we obtain the equality

$$\langle \mathbf{K}^\#\mathbf{x}, \varphi \rangle = \langle \mathbf{x}, \mathbf{K}'\varphi \rangle$$

where $\mathbf{K}'$ is given by (9) and $\mathbf{x} \in BV_n$, $\varphi \in NBV_n$ are arbitrary, i.e. (10) holds for all $\mathbf{x} \in BV_n$, $\varphi \in NBV_n$.

In the subsequent considerations we use the usual notation: for a given linear operator $\mathbf{A}$ acting on a Banach space X we set

$$N(\mathbf{A}) = \{\mathbf{x} \in X; \; \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

(the null space of $\mathbf{A}$) and

$$R(\mathbf{A}) = \{\mathbf{y} \in X; \; \mathbf{y} = \mathbf{A}\mathbf{x}, \; \mathbf{x} \in X\}$$

(the range of $\mathbf{A}$). We define the index $\operatorname{ind} \mathbf{A}$ of the operator $\mathbf{A}$ by the relation

$$\operatorname{ind} \mathbf{A} = \dim N(\mathbf{A}) - \operatorname{codim} R(\mathbf{A})$$

if the difference on the right hand side of this equality is defined.

Using this notation we state the following

6. Theorem. *If* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2) *and* (3) *then*

$$\operatorname{ind}(\mathbf{I} - \mathbf{K}) = \operatorname{ind}(\mathbf{I} - \mathbf{K}') = 0 \tag{11}$$

where $\mathbf{I}$ *stands for the identity operator in the corresponding Banach space and the operators* $\mathbf{K}$, $\mathbf{K}'$ *are given by* (8), (9) *respectively.*

Moreover, we have

$$\dim N(\mathbf{I} - \mathbf{K}) = \dim N(\mathbf{I} - \mathbf{K}') \tag{12}$$

and the Fredholm-Stieltjes integral equation

$$\mathbf{x}(t) = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s) + \mathbf{f}(t), \quad t \in [0, 1], \quad \mathbf{f} \in BV_n \tag{13}$$

has a solution in BV_n *if and only if*

$$\langle \mathbf{f}, \boldsymbol{\varphi} \rangle = 0$$

for all solutions $\boldsymbol{\varphi} \in NBV_n$ *of the equation*

$$\boldsymbol{\varphi}(s) = \int_0^1 (\mathbf{K}^{\#})^* (t, s)\, \mathrm{d}\boldsymbol{\varphi}(t), \quad s \in [0, 1]. \tag{14}$$

Similarly, the equation

$$\boldsymbol{\varphi}(s) = \int_0^1 (\mathbf{K}^{\#})^* (t, s)\, \mathrm{d}\boldsymbol{\varphi}(t) + \boldsymbol{\psi}(s), \quad s \in [0, 1], \quad \boldsymbol{\psi} \in NBV_n \tag{15}$$

has a solution in NBV_n *if and only if*

$$\langle \mathbf{x}, \boldsymbol{\psi} \rangle = 0$$

for every solution $\mathbf{x} \in BV_n$ *of the homogeneous Fredholm-Stieltjes integral equation*

$$\mathbf{x}(t) = \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)]\, \mathbf{x}(s), \quad t \in [0, 1]. \tag{16}$$

Proof. The equality (11) follows immediately from the complete continuity of the operators $\mathbf{K}$, $\mathbf{K}'$ stated in Theorem 5 (see e.g. [1], Theorem 40,1).

Since (BV_n, NBV_n) is a dual system with respect to the bilinear form (1) and (10) is satisfied we have

$$\langle \mathbf{x} - \mathbf{K}\mathbf{x}, \boldsymbol{\varphi} \rangle = \langle \mathbf{x}, \boldsymbol{\varphi} \rangle - \langle \mathbf{K}\mathbf{x}, \boldsymbol{\varphi} \rangle = \langle \mathbf{x}, \boldsymbol{\varphi} \rangle - \langle \mathbf{x}, \mathbf{K}'\boldsymbol{\varphi} \rangle = \langle \mathbf{x}, \boldsymbol{\varphi} - \mathbf{K}'\boldsymbol{\varphi} \rangle.$$

All the assumptions of Satz 40.2 from [1] are satisfied and, consequently, the result follows immediately from this Satz.

Remark. Theorem 6 is essentially a comprehensive version of the results from [3]. In [3], the quotient space BV_n/S_n was used instead of NBV_n. The version of the Fredholm theory for the equation (13) and the corresponding conjugate equation (15) given in Theorem 6 seems to be more natural than the version given in [3].

For the linear operator $\mathbf{K} : BV_n \to BV_n$ defined by (8) we have $\operatorname{ind} (\mathbf{I} - \mathbf{K}) = 0$ and consequently, if $\dim N(\mathbf{I} - \mathbf{K}) = 0$, i.e. if $N(\mathbf{I} - \mathbf{K}) = \mathbf{0}$ then $BV_n/R(\mathbf{I} - \mathbf{K}) = \mathbf{0}$

and also $R(\mathbf{I} - \mathbf{K}) = BV_n$. In this situation the Bounded Inverse Theorem applies, i.e. the inverse operator $(\mathbf{I} - \mathbf{K})^{-1}$ exists and is bounded (see [2]). This yields the following

7. Lemma. *Let us assume that* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2), (3) *and that* $N(\mathbf{I} - \mathbf{K}) = \mathbf{0}$, *i.e. the homogeneous integral equation* (16) *has only the trivial solution* $\mathbf{x} = \mathbf{0}$ *in* BV_n. *Then there exists a constant* $C \geqq 0$ *such that for every* $\mathbf{f} \in BV_n$ *the inequality*

$$\|\mathbf{x}\|_{BV_n} \leqq C\|\mathbf{f}\|_{BV_n}$$

holds for the unique solution $\mathbf{x} \in BV_n$ *of the nonhomogeneous equation* (13). (*Let us mention that* $C = \|(\mathbf{I} - \mathbf{K})^{-1}\|$.)

Remark. As was mentioned above, when the assumptions of Lemma 7 are satisfied the inverse operator $(\mathbf{I} - \mathbf{K})^{-1}$ exists. In the sequel we prove that this inverse operator has the form $\mathbf{I} + \boldsymbol{\Gamma}$ where $\boldsymbol{\Gamma} : BV_n \to BV_n$ is a linear integral operator of the same type as the operator $\mathbf{K}$ given by (8).

8. Theorem. *Let us assume that* $\mathbf{K} : J \to L(R_n)$ *satisfies* (2), (3). *If the homogeneous equation* (16) *has only the trivial solution* $\mathbf{x} = \mathbf{0} \in BV_n$ *then there exists a uniquely determined* $n \times n$*-matrix valued function* $\boldsymbol{\Gamma} : J \to L(R_n)$ *such that*

$$\boldsymbol{\Gamma}(t, s) = \mathbf{K}(t, s) - \mathbf{K}(t, 0) + \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)]\, \boldsymbol{\Gamma}(r, s) \tag{17}$$

for all $t, s \in [0, 1]$,

$$\mathrm{var}_0^1\, \boldsymbol{\Gamma}(0, \cdot) < \infty\,, \tag{18}$$

$$\boldsymbol{\Gamma}(t, 0) = \mathbf{0} \quad \textit{for every} \quad t \in [0, 1]\,, \tag{19}$$

$$v_J(\boldsymbol{\Gamma}) < \infty \tag{20}$$

and for any $\mathbf{f} \in BV_n$ *the unique solution* $\mathbf{x} \in BV_n$ *of* (13) *is given by the resolvent formula*

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)]\, \mathbf{f}(s)\,. \tag{21}$$

Proof. Let us denote by $\mathbf{y}_l$ the l-th column of the $n \times n$-matrix $\mathbf{y} \in L(R_n)$. Then the relation (17) can be written in the form

$$\boldsymbol{\Gamma}_l(t, s) = \mathbf{K}_l(t, s) = \mathbf{K}_l(t, 0) + \int_0^1 \mathrm{d}_r[\mathbf{K}(i, r)]\, \boldsymbol{\Gamma}_l(r, s)\,, \quad l = 1, 2, \ldots, n\,. \tag{21}$$

We have evidently

$$\mathrm{var}_0^1\, (\mathbf{K}(\cdot, s) - \mathbf{K}(\cdot, 0)) \leqq v_J(\mathbf{K}) < \infty$$

for every $s \in [0, 1]$. Hence for any fixed $s \in [0, 1]$ and $l = 1, \ldots, n$ we have $\operatorname{var}_0^1 (\boldsymbol{K}_l(\cdot, s) - \boldsymbol{K}_l(\cdot, 0)) < \infty$. This implies by the assumptions and by Theorem 6 that for any $l = 1, \ldots, n$, $s \in [0, 1]$ the relation (21) determines uniquely the n-vector $\boldsymbol{\Gamma}_l(t, s)$ and, consequently, also the $n \times n$-matrix valued function $\boldsymbol{\Gamma}(t, s)$ is uniquely determined by (17) for every fixed $s \in [0, 1]$. Moreover, by Lemma 7 we have

$$\|\boldsymbol{\Gamma}_l(\cdot, 0)\|_{BV_n} \leqq C\|\boldsymbol{K}_l(\cdot, 0) - \boldsymbol{K}_l(\cdot, 0)\| = 0\,.$$

Hence $\boldsymbol{\Gamma}(t, 0) = \boldsymbol{0}$ for every $t \in [0, 1]$. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ be an arbitrary subdivision of the interval $[0, 1]$. For $\boldsymbol{\Gamma}(t, s) : J \to L(R_n)$ satisfying (17) we have

$$\boldsymbol{\Gamma}(t, \alpha_j) - \boldsymbol{\Gamma}(t, \alpha_{j-1}) =$$
$$= \boldsymbol{K}(t, \alpha_j) - \boldsymbol{K}(t, \alpha_{j-1}) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(t, r)]\,(\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1}))$$

for $t \in [0, 1]$, $j = 1, 2, \ldots, k$. Using Lemma 7 and the obvious fact that $\operatorname{var}_0^1 (\boldsymbol{K}(\cdot, \alpha_j) - \boldsymbol{K}(\cdot, \alpha_{j-1})) < \infty$ we get

$$\text{(22)} \qquad \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{\Gamma}(\cdot, \alpha_j) - \boldsymbol{\Gamma}(\cdot, \alpha_{j-1})) \leqq$$
$$\leqq C[\|\boldsymbol{K}(0, \alpha_j) - \boldsymbol{K}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{K}(\cdot, \alpha_j) - \boldsymbol{K}(\cdot, \alpha_{j-1}))]$$

where $C \geqq 0$ is a constat. Hence

$$\sum_{j=1}^k \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| \leqq C(\operatorname{var}_0^1 \boldsymbol{K}(0, \cdot) + v_J(\boldsymbol{K}))\,.$$

Since this inequality holds for any subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ we obtain (18). The inequality (20) can be shown as follows. For the subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ we define the net-type subdivision

$$J_{ij} = [\alpha_{i-1}, \alpha_i] \times [\alpha_{j-1}, \alpha_j]$$

$i, j = 1, \ldots, k$ of the interval J. For $\boldsymbol{\Gamma} : J \to L(R_n)$ defined by (17) we have ($i, j = 1, \ldots, k$)

$$m_{\boldsymbol{\Gamma}}(J_{ij}) = m_{\boldsymbol{K}}(J_{ij}) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}(\alpha_i, r) - \boldsymbol{K}(\alpha_{i-1}, r)]\,(\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1}))$$

where $m_{\boldsymbol{\Gamma}}(J_{ij}) = \boldsymbol{\Gamma}(\alpha_i, \alpha_j) - \boldsymbol{\Gamma}(\alpha_i, \alpha_{j-1}) - \boldsymbol{\Gamma}(\alpha_{i-1}, \alpha_j) + \boldsymbol{\Gamma}(\alpha_{i-1}, \alpha_{j-1})$ and similarly for $m_{\boldsymbol{K}}(J_{ij})$. Usual estimates for the Perron-Stieltjes integral lead to the inequality (see [3], [4])

$$\|m_{\boldsymbol{\Gamma}}(J_{ij})\| \leqq \|m_{\boldsymbol{K}}(J_{ij})\| +$$
$$+ \sup_{r \in [0,1]} \|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \operatorname{var}_0^1 (\boldsymbol{K}(\alpha_i, \cdot) - \boldsymbol{K}(\alpha_{i-1}, \cdot))$$

for every $i, j = 1, 2, \ldots, k$ and also to the inequality

$$\sum_{i,j=1}^{k} \|m_\Gamma(J_{ij})\| \leqq v_J(\mathbf{K}) +$$
$$+ \sum_{i=1}^{k} \operatorname{var}_0^1 (\mathbf{K}(\alpha_i, \cdot) - \mathbf{K}(\alpha_{i-1}, \cdot)) \, . \sum_{j=1}^{k} \sup_{r\in[0,1]} \|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \, .$$

Since

$$\|\boldsymbol{\Gamma}(r, \alpha_j) - \boldsymbol{\Gamma}(r, \alpha_{j-1})\| \leqq$$
$$\leqq \|\boldsymbol{\Gamma}(0, \alpha_j) - \boldsymbol{\Gamma}(0, \alpha_{j-1})\| + \operatorname{var}_0^1 (\boldsymbol{\Gamma}(\cdot, \alpha_{j-1})) - \boldsymbol{\Gamma}(\cdot, \alpha_{j-1}))$$

for every $r \in [0, 1]$, we have by (22)

$$\sum_{i,j=1}^{k} \|m_\Gamma(J_{ij})\| \leqq v_J(\mathbf{K}) + v_J(\mathbf{K})\, C[\sum_{j=1}^{k} \|\mathbf{K}(0, \alpha_j) - \mathbf{K}(0, \alpha_{j-1})\| +$$
$$+ \operatorname{var}_0^1 (\mathbf{K}(\cdot, \alpha_j) - \mathbf{K}(\cdot, \alpha_{j-1}))] \leqq v_J(\mathbf{K}) [1 + C(\operatorname{var}_0^1 \mathbf{K}(0, \cdot) + v_J(\mathbf{K}))] < \infty \, .$$

Since the subdivision $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ of $[0, 1]$ is arbitrary we obtain by the definition of the Vitali variation v_J the inequality (20). [2])

It remains to show that by the formula (21) the unique solution of the equation (13) is given. The integral $\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)] \, \boldsymbol{f}(s)$ exists for every $\boldsymbol{f} \in BV_n$ and $t \in [0, 1]$ since (18) and (20) are satisfied (see Proposition 2,3 in [3]). Let us put $\boldsymbol{x}(t)$ from (21) into the expression $\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \boldsymbol{x}(s)$. We obtain

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \boldsymbol{x}(s) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s)] \, \boldsymbol{f}(s) -$$
$$- \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \left(\boldsymbol{f}(r) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(r, s)] \, \boldsymbol{f}(s) \right) =$$
$$= \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s)] \, \boldsymbol{f}(s) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \left(\int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}(r, s)] \, \boldsymbol{f}(s) \right).$$

Interchanging the order of integrations in the last integral by Lemma 2,2 in [3] and using (17) we obtain

$$\boldsymbol{x}(t) - \int_0^1 \mathrm{d}_s[\mathbf{K}(t, s)] \, \boldsymbol{x}(s) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s \Big\{ \boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s) -$$
$$- \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \, \boldsymbol{\Gamma}(r, s) \Big\} \, \boldsymbol{f}(s) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s \Big\{ \boldsymbol{\Gamma}(t, s) - \mathbf{K}(t, s) +$$
$$+ \mathbf{K}(t, 0) - \int_0^1 \mathrm{d}_r[\mathbf{K}(t, r)] \, \boldsymbol{\Gamma}(r, s) \Big\} \, \boldsymbol{f}(s) = \boldsymbol{f}(t) \, ,$$

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[2]) The fact that only net-type subdivisions of J are taken into account is not essential since evidently every subdivision of J can be refined to a net-type one.

i.e. $\mathbf{x}(t)$ given by (21) is really the unique solution of the equation (13) and the theorem is completely proved.

Let us now consider the case when $\mathbf{K} : J \to L(R_n)$ satisfies (2) and (3) but the assumption $N(\mathbf{I} - \mathbf{K}) = \{\mathbf{0}\}$ is not satisfied. By Theorem 6 we know that $\dim N(\mathbf{I} - \mathbf{K}) = \operatorname{codim} R(\mathbf{I} - \mathbf{K}) = \dim N(\mathbf{I} - \mathbf{K}') = \operatorname{codim} R(\mathbf{I} - \mathbf{K}') = r$ where $r > 0$ is an integer. In this case $R(\mathbf{I} - \mathbf{K}) \neq BV_n$ and the inverse operator $(\mathbf{I} - \mathbf{K})^{-1}$ cannot be defined on the whole space BV_n. The equation (13) has solutions only for $\mathbf{f} \in R(\mathbf{I} - \mathbf{K})$. Our aim is to show that in this situation there exists also an operator $\mathbf{\Gamma}^0$ acting on BV_n such that if $\mathbf{f} \in R(\mathbf{I} - \mathbf{K})$ then $\mathbf{f} + \mathbf{\Gamma}^0\mathbf{f}$ is a solution of the equation (13) and, moreover, that the operator $\mathbf{\Gamma}^0$ is an integral operator of the same type as $\mathbf{K}$. We prove this fact following a general scheme known from functional analysis.

In the sequel we assume that $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^r \in BV_n$ is a given basis of the r – dimensional null space $N(\mathbf{I} - \mathbf{K})$ (linearly independent solutions of the homogeneous integral equation (16)) and $\boldsymbol{\varphi}^1, \ldots, \boldsymbol{\varphi}^r \in NBV_n$ is a given basis of $N(\mathbf{I} - \mathbf{K}')$ (linearly independent solutions of the equation (14)). It is known (see e.g. [1], Satz 15.1) that there exist linearly independent elements $\boldsymbol{\eta}^i$ in NBV_n and $\mathbf{y}^i$ in BV_n, $i = 1, \ldots, r$ such that

$$\langle \mathbf{x}^j, \boldsymbol{\eta}^i \rangle = \delta_{ij}\,, \quad i, j = 1, \ldots, r\,,$$

$$\langle \mathbf{y}^j, \boldsymbol{\varphi}^i \rangle = \delta_{ij}\,, \quad i, j = 1, \ldots, r$$

($\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$).

Let us define the projections

$$\mathbf{P}\mathbf{x} = \sum_{i=1}^{r} \langle \mathbf{x}, \boldsymbol{\eta}^i \rangle\, \mathbf{x}^i\,, \quad \mathbf{x} \in BV_n\,,$$

$$\mathbf{Q}\mathbf{x} = \sum_{i=1}^{r} \langle \mathbf{x}, \boldsymbol{\varphi}^i \rangle\, \mathbf{y}^i\,, \quad \mathbf{x} \in BV_n\,.$$

It is easy to show that $\mathbf{P}$, $\mathbf{Q}$ are bounded projection operators. Further, evidently $R(\mathbf{P}) = N(\mathbf{I} - \mathbf{K})$ and by Theorem 6 also

$$N(\mathbf{Q}) = \{\mathbf{x} \in X;\ \langle \mathbf{x}, \boldsymbol{\varphi} \rangle = 0 \text{ for every } \boldsymbol{\varphi} \in N(\mathbf{I} - \mathbf{K}')\} = R(\mathbf{I} - \mathbf{K})\,.$$

The projections $\mathbf{P}$, $\mathbf{Q}$ generate decompositions of the Banach space BV_n into direct sums

$$BV_n = R(\mathbf{P}) \oplus N(\mathbf{P}) = N(\mathbf{I} - \mathbf{K}) \oplus N(\mathbf{P})\,, \tag{23}$$

$$BV_n = R(\mathbf{Q}) \oplus N(\mathbf{Q}) = R(\mathbf{Q}) \oplus R(\mathbf{I} - \mathbf{K})\,. \tag{24}$$

Let us now define the linear operator

$$\text{(25)} \qquad \boldsymbol{L}\boldsymbol{x} = \sum_{i=1}^{r} \langle \boldsymbol{x}, \boldsymbol{\eta}^i \rangle \, \boldsymbol{y}^i = \sum_{i=1}^{r} \boldsymbol{y}^i(t) \int_0^1 \boldsymbol{x}^*(s) \, \mathrm{d}\boldsymbol{\eta}^i(s) =$$

$$= \int_0^1 \mathrm{d}_s\Big[\sum_{i=1}^{r} \boldsymbol{y}^i(t) \, \boldsymbol{\eta}^{i*}(s)\Big] \, \boldsymbol{x}(s) \, .$$

$\boldsymbol{L}$ is evidently a bounded finite-dimensional (and consequently completely continuous) operator on BV_n and

$$N(\boldsymbol{L}) = \{\boldsymbol{x} \in BV_n;\ \langle \boldsymbol{x}, \boldsymbol{\eta}^i \rangle = 0 \text{ for every } i = 1, \ldots, r\} = N(\boldsymbol{P}) \, ,$$

$$R(\boldsymbol{L}) \subset R(\boldsymbol{Q}) \, .$$

Let us set

$$\text{(26)} \qquad \boldsymbol{K}^\circ = \boldsymbol{K} + \boldsymbol{L}$$

where $\boldsymbol{K}$ is the operator corresponding to the kernel $\boldsymbol{K} : J \to L(R_n)$ via the relation (4). $\boldsymbol{K}^\circ$ is evidently a completely continuous operator on BV_n and ind $(\boldsymbol{I} - \boldsymbol{K}^\circ) = 0$. Let us assume that $\boldsymbol{x} \in N(\boldsymbol{I} - \boldsymbol{K}^\circ)$. Then

$$(\boldsymbol{I} - \boldsymbol{K}^\circ) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x} - \boldsymbol{L}\boldsymbol{x} = \boldsymbol{0}$$

and by (24) necessarily $(\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{L}\boldsymbol{x} = \boldsymbol{0}$ because $R(\boldsymbol{L}) \subset R(\boldsymbol{Q})$. Hence $\boldsymbol{x} \in N(\boldsymbol{I} - \boldsymbol{K}) \cap N(\boldsymbol{L}) = N(\boldsymbol{I} - \boldsymbol{K}) \cap N(\boldsymbol{P})$ and, consequently, by (23) we obtain $\boldsymbol{x} = \boldsymbol{0}$. This yields $N(\boldsymbol{I} - \boldsymbol{K}^\circ) = \{\boldsymbol{0}\}$ and $\dim N(\boldsymbol{I} - \boldsymbol{K}^\circ) = 0$. Using the complete continuity of the operator $\boldsymbol{K}^\circ$ we obtain $R(\boldsymbol{I} - \boldsymbol{K}^\circ) = BV_n$ and by the Bounded Inverse Theorem also the existence of a bounded inverse operator $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1}$. Since $\boldsymbol{x}^i \in N(\boldsymbol{I} - \boldsymbol{K})$ we have $(\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{P}\boldsymbol{x} = \sum_{i=1}^{r} \langle \boldsymbol{x}, \boldsymbol{\eta}^i \rangle \, (\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x}^i = \boldsymbol{0}$ for all $\boldsymbol{x} \in BV_n$ and

$$\text{(27)} \qquad (\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} \, .$$

Since $\boldsymbol{P}$ is a projection we have $R(\boldsymbol{I} - \boldsymbol{P}) = N(\boldsymbol{P}) = N(\boldsymbol{L})$. Hence $\boldsymbol{L}(\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} = \boldsymbol{0}$ for every $\boldsymbol{x} \in BV_n$ and also

$$(\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} - \boldsymbol{L}(\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}^\circ) (\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x}$$

for every $\boldsymbol{x} \in BV_n$. Multiplying from the left by $(\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1}$ and using (27) we obtain further

$$\text{(28)} \qquad (\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} (\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} (\boldsymbol{I} - \boldsymbol{K}^\circ) (\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} =$$

$$= (\boldsymbol{I} - \boldsymbol{K}) (\boldsymbol{I} - \boldsymbol{P}) \, \boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{K}) \, \boldsymbol{x}$$

for every $\boldsymbol{x} \in BV_n$. Hence

$$(\boldsymbol{I} - \boldsymbol{K})(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f} = \boldsymbol{f}$$

for every $\boldsymbol{f} \in R(\boldsymbol{I} - \boldsymbol{K})$, i.e. $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f}$ is a solution of the equation $(\boldsymbol{I} - \boldsymbol{K})\, \boldsymbol{x} = \boldsymbol{f}$. It is easy to see that if we set

$$\boldsymbol{K}^\circ(t, s) = \boldsymbol{K}(t, s) + \sum_{i=1}^{r} \boldsymbol{y}^i(t)\, \boldsymbol{\eta}^{i*}(s)$$

then for the operator $\boldsymbol{K}^\circ$ given by (26) we have

$$\boldsymbol{K}^\circ \boldsymbol{x} = \int_0^1 \mathrm{d}_s[\boldsymbol{K}^\circ(t, s)]\, \boldsymbol{x}(s)$$

and $v_J(\boldsymbol{K}^\circ) < v_J(\boldsymbol{K}) + \sum_{i=1}^{r} \operatorname{var}_0^1 \boldsymbol{y}^i \,.\, \operatorname{var}_0^1 \boldsymbol{\eta}^i < \infty$, $\operatorname{var}_0^1 \boldsymbol{K}^\circ(0, \cdot) \leqq \operatorname{var}_0^1 \boldsymbol{K}(0, \cdot) + \sum_{i=1}^{r} \|\boldsymbol{y}^i(0)\| \operatorname{var}_0^1 \boldsymbol{\eta}^i < \infty$. Hence the kernel $\boldsymbol{K}^\circ(t, s) : J \to L(R_n)$ satisfies all assumptions of Theorem 8 and, consequently, by this theorem there exists a $\boldsymbol{\Gamma}^\circ(t, s) : J \to L(R_n)$ which satisfies the equation

$$\boldsymbol{\Gamma}^\circ(t, s) = \boldsymbol{K}^\circ(t, s) - \boldsymbol{K}^\circ(t, 0) + \int_0^1 \mathrm{d}_r[\boldsymbol{K}^\circ(t, r)]\, \boldsymbol{\Gamma}^\circ(r, s)\,, \quad t, s \in [0, 1] \tag{29}$$

and $\boldsymbol{\Gamma}^\circ(t, 0) = \boldsymbol{0}$ for every $t \in [0, 1]$, $\operatorname{var}_0^1 \boldsymbol{\Gamma}^\circ(0, \cdot) < \infty$, $v_J(\boldsymbol{\Gamma}^\circ) < \infty$. Moreover, for every $\boldsymbol{f} \in BV_n$ the unique solution $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} \boldsymbol{f}$ of the equation

$$\boldsymbol{x} - \boldsymbol{K}^\circ \boldsymbol{x} = \boldsymbol{f}$$

is given by the relation

$$\boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)]\, \boldsymbol{f}(s)\,,$$

i.e. $(\boldsymbol{I} - \boldsymbol{K}^\circ)^{-1} = \boldsymbol{I} + \boldsymbol{\Gamma}^\circ$ where $\boldsymbol{\Gamma}^\circ \boldsymbol{x} = \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)]\, \boldsymbol{x}(s)$ for $\boldsymbol{x} \in BV_n$.

Let us now summarize the above results.

9. Theorem. *Let $\boldsymbol{K} : J \to L(R_n)$ satisfy (2) and (3). Then there exists an $n \times n$-matrix valued function $\boldsymbol{\Gamma}^\circ(t, s) : J \to L(R_n)$ such that $\operatorname{var}_0^1 \boldsymbol{\Gamma}^\circ(0, \cdot) < \infty$, $v_J(\boldsymbol{\Gamma}^\circ) < \infty$, $\boldsymbol{\Gamma}^\circ(t, 0) = \boldsymbol{0}$ for all $t \in [0, 1]$, $\boldsymbol{\Gamma}^\circ(t, s)$ satisfies (29) for all $t, s \in [0, 1]$ and the relation*

$$\boldsymbol{x}(t) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)]\, \boldsymbol{f}(s)\,, \quad t \in [0, 1] \tag{30}$$

defines a solution of the Fredholm-Stieltjes integral equation (13) provided $\boldsymbol{f} \in BV_n$ belongs to $R(\boldsymbol{I} - \boldsymbol{K})$ (i.e. when the equation (13) has a solution for the given $\boldsymbol{f} \in BV_n$).

If $\boldsymbol{f} \in R(\boldsymbol{I} - \boldsymbol{K})$ *then the general form of solutions of the equation* (13) *is given by*

$$\boldsymbol{x}(t) = \boldsymbol{f}(t) + \int_0^1 \mathrm{d}_s[\boldsymbol{\Gamma}^\circ(t, s)]\, \mathrm{f}(s) + \sum_{i=1}^{r} \alpha_i\, \boldsymbol{x}^i(t)$$

where $\boldsymbol{x}^i \in BV_n$, $i = 1, \ldots, r$ *are all the linearly independent solutions of the homogeneous Fredholm-Stieltjes integral equation* 16) *and* $\alpha_1, \ldots, \alpha_r$ *are arbitrary real constants.*

Remark. The last part of the theorem follows from the well-known properties of linear equations. The theorem includes also the statement of the previous Theorem 8 and gives in the general situation the desired "solving kernel result". Naturally, for the case $\dim N(\boldsymbol{I} - \boldsymbol{K}) > 0$ the construction of the solving kernel $\boldsymbol{\Gamma}^\circ$ depends upon the knowledge of the structure of the null-spaces of the operators $\boldsymbol{I} - \boldsymbol{K}$ and $\boldsymbol{I} - \boldsymbol{K}^\circ$.

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ON PARTITION GRAPHS AND GENERALIZATIONS OF LINE GRAPHS

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By a graph we mean a graph in the sense of BEHZAD and CHARTRAND [1] or HARARY [2]. If G is a graph, then we denote by $V(G)$, $E(G)$, $\delta(G)$ and $L(G)$ its vertex set, edge set, minimum degree and line graph, respectively. If G is a graph and $u \in V(G)$, then we denote

$$E(G, u) = \{x \in E(G);\ x \text{ is incident with } u\}\ .$$

Let G and H be graphs, let $E(G) \neq \emptyset$, and let M be a graph-theoretical property (of graphs). We shall say that H is an M-extension of G if there exists a 1-1-mapping g from $E(G)$ onto $V(H)$ such that the following conditions hold:

$(1_{G,g,H})$ if $x, y \in E(G)$ and $g(x)\, g(y) \in E(H)$, then x and y are adjacent edges of G;

$(2_{G,g,H}/M)$ if $u \in V(G)$ and $E(G, u) \neq \emptyset$, then the subgraph of H induced by $\{g(z); z \in E(G, u)\}$ has the property M.

We denote by A_1, A_2, and A_3 the properties
"either to be trivial or to contain no vertex of degree 0",
"to be connected",
and
"to be complete",
respectively.

It is clear that for every graph G with $E(G) \neq \emptyset$, $L(G)$ is the only A_3-extension of G. This means that the concepts of an A_1-extension and an A_2-extension are generalizations of the concept of the line graph of a graph.

Let F and G be graphs. We say that G is a partition graph of F if there exists a mapping f from $V(F)$ onto $V(G)$ such that the following condition holds:

$(3_{F,f,G})$ if u and v are distinct vertices of G, then u and v are adjacent if and only if there exist $r \in f^{-1}(u)$ and $s \in f^{-1}(v)$ such that $rs \in E(F)$.

We say that G is a contraction of F if there exists a mapping f from $V(F)$ onto $V(G)$ such that $(3_{F,f,G})$ and the following condition holds:

$(4_{F,f,G})$ if $w \in V(G)$, then the subgraph of F induced by $f^{-1}(w)$ is connected.

The concept of a partition graph of a graph was studied by E. SAMPATHKUMAR and V. N. BHAVE [3]. (The concept of a contraction of a graph can be found in [1], p. 92.)

The following theorem is the main result of the present note:

Theorem. *Let G, H, and J be graphs, and let $\delta(G) \geqq 2$. Then*

(I) *if H is an A_1-extension of G, and J is an A_2-extension of H, then G is a partition graph of J;*

(II) *if H is an A_2-extension of G, and J si the A_3-extension of H, then G is a contraction of J.*

Proof. Let H be an A_1-extension of G (resp. an A_2-extension of G), and let J be an A_2-extension of H (resp. the A_3-extension of H). There exists a 1-1-mapping $g : E(G) \to V(H)$ such that $(1_{G,g,H})$ and $(2_{G,g,H}/A_1)$ (resp. $(2_{G,g,H}/A_2)$) hold. Similarly, there exists a 1-1-mapping $h : E(H) \to V(J)$ such that $(1_{H,h,J})$ and $(2_{H,h,J}/A_2)$ (resp. $(2_{H,h,J}/A_3)$) hold.

First, we assume that $(2_{G,g,H}/A_1)$ and $(2_{H,h,J}/A_2)$ hold.

Let r be an arbitrary vertex of G. We denote by $H(r)$ the subgraph of H induced by $\{g(x_r);\ x_r \in E(G, r)\}$. Since $\delta(G) \geqq 2$, we have that $H(r)$ is nontrivial. From $(2_{G,g,H}/A_1)$ it follows that $\delta(H(r)) \geqq 1$. We denote by $J(r)$ the subgraph of J induced by $\{h(y_r);\ y_r \in E(H(r))\}$.

We introduce a mapping f from $V(J)$ into $V(G)$. Let v be an arbitrary vertex of J. Then there are adjacent vertices t and u of H such that $h^{-1}(v) = tu$. From $(1_{G,g,H})$ it follows that $g^{-1}(t)$ and $g^{-1}(u)$ are adjacent edges of G. We denote by $f(v)$ the vertex of G incident both with $g^{-1}(t)$ and $g^{-1}(u)$. Since t and u are vertices of $H(f(v))$, we have that v is a vertex of $J(f(v))$.

Let s be an arbitrary vertex of G. It is easy to see that $f(w) = s$ for each $w \in V(J(s))$. This means that f is a mapping from $V(J)$ onto $V(G)$, and that $f^{-1}(s_0) = V(J(s_0))$ for every $s_0 \in V(G)$.

Let v_1 and v_2 be adjacent vertices of J and let $f(v_1) \neq f(v_2)$. There exist $y_1, y_2 \in E(H)$ such that $h(y_1) = v_1$ and $h(y_2) = v_2$. From $(1_{H,j,J})$ it follows that y_1 and y_2 are adjacent. This means that there exist distinct vertices u_0, u_1 and u_2 of H such that $y_1 = u_0u_1$ and $y_2 = u_0u_2$. It is clear that $f(v_1)$ is incident both with $g^{-1}(u_0)$ and $g^{-1}(u_1)$, and that $f(v_2)$ is incident both with $g^{-1}(u_0)$ and $g^{-1}(u_2)$. Since $f(v_1) \neq f(v_2)$, we have that $g^{-1}(u_0) = f(v_1)f(v_2)$. Hence $f(v_1)$ and $f(v_2)$ are adjacent.

Let s_1 and s_2 be adjacent vertices of G. We shall prove that there exist $w_1 \in f^{-1}(s_1)$ and $w_2 \in f^{-1}(s_2)$ such that w_1 and w_2 are adjacent vertices of J. Denote $x_0 = s_1s_2$. Obviously, $V(H(s_1)) \cap V(H(s_2)) = \{g(x_0)\}$ and $E(H(s_1)) \cap E(H(s_2)) = \emptyset$. From $(2_{G,h,H}/A_1)$ it follows that there exist $u' \in V(H(s_1))$ and $u'' \in V(H(s_2))$ such that

$g(x_0)\, u'$ and $g(x_0)\, u''$ are edges of H. Hence $E(H, g(x_0)) \cap E(H(s_1)) \neq \emptyset \neq$ $\neq E(H, g(x_0)) \cap E(H(s_2))$. Clearly, $E(H, g(x_0)) \subseteq E(H(s_1)) \cup E(H(s_2))$. From $(2_{H,h,J}/A_2)$ it follows that there exist $y^* \in E(H, g(x_0)) \cap E(H(s_1))$ and $y^{**} \in E(H, g(x_0)) \cap$ $\cap E(H(s_2))$ such that $h(y^*)$ and $h(y^{**})$ are adjacent in J. Denote $w_1 = h(y^*)$ and $w_2 = h(y^{**})$. It is obvious that $w_1 \in f^{-1}(s_1)$ and $w_2 \in f^{-1}(s_2)$.

We have proved that $(3_{J,f,G})$ holds. Hence G is a partition graph of J.

Now we assume that $(2_{G,g,H}/A_2)$ and $(2_{H,h,J}/A_3)$ hold. This implies that also $(2_{G,g,H}/A_1)$ and $(2_{H,h,J}/A_2)$ hold. Let r' be an arbitrary vertex of G. From $(2_{G,g,H}/A_2)$ it follows that $H(r')$ is nontrivial connected. From $(2_{H,h,J}/A_3)$ it follows that $J(r')$ is also connected. Since $V(J(r')) = f^{-1}(r')$, we have that $(4_{J,f,G})$ holds. Hence G is a contraction of J, which completes the proof.

Corollary. *Let G be a graph such that $\delta(G) \geqq 2$. Then G is a contraction of $L(L(G))$.*

Note that if G is a graph without a triangle which can be obtained from a cycle of a length at least six by adding one new edge, then G is not a partition graph of $L(G)$.

References

[1] *M. Behzad, G. Chartrand:* Introduction to the Theory of Graphs. Allyn and Bacon, Boston 1971.

[2] *F. Harary:* Graph Theory. Addison-Wesley, Reading (Mass.) 1969.

[3] *E. Sampathkumar, V. N. Bhave:* Partition graphs of a graph. Progress of Mathematics *9* (1975), 33—42.

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STRUČNÉ CHARAKTERISTIKY ČLÁNKŮ OTIŠTĚNÝCH V TOMTO ČÍSLE V CIZÍM JAZYKU

ILJA ČERNÝ, Praha: *Extension of a homeomorphism of a topological circumference.* (O rozšíření homeomorfismu topologické kružnice.)

V článku se dokazuje, že každé homeomorfní zobrazení h topologické kružnice $T \subset \boldsymbol{S}$ do $\boldsymbol{S}$ (kde $\boldsymbol{S}$ je uzavřená Gaussova rovina) lze rozšířit na homeomorfní zobrazení celého $\boldsymbol{S}$ na $\boldsymbol{S}$. Z hlubších vět topologie roviny se užívá pouze Jordanovy věty a věty o θ-křivkách.

MILAN ŠTĚDRÝ, Praha: *Periodic solutions of a weakly nonlinear wave equation.* (Periodická řešení slabě nelineární vlnové rovnice.)

V článku jsou odvozeny postačující podmínky, které zaručují, že slabě nelineární vlnová rovnice obsahující parametr ε má řešení, které je periodické s periodou $2\pi + \varepsilon\lambda$ a to pro všechna ε z jistého okolí 0. Je ukázáno, že tyto postačující podmínky jsou splněny pro jistou třídu vlnových rovnic.

VÁCLAV HAVEL, Brno: *Kleine Desargues-Bedingung in Geweben.* (Malá Desarguesova podmínka v tkáních.)

Předmětem článku je studium různých specializací Desarguesovy podmínky (s nevlastní osou i centrem) v tkáních libovolného stupně na příslušnou souřadnicovou algebru.

MIROSLAV SOVA, Praha: *On inversion of Laplace transform* (I). (O inversní Laplaceově transformaci.)

V článku autor ukazuje, jak lze odvodit komplexní vzorec pro inverzní transformaci z obecné Postovy-Widderovy věty o inversi.

PAVEL BURDA, Ostrava: *Isomorphism of projective planes and isotopism of planar ternary rings.* (Isomorfismus projektivních rovin a isotopie planárních ternárních okruhů.)

V článku jsou diskutovány problémy týkající se vzájemné souvislosti isomorfismu projektivních rovin a isotopie planárních ternárních okruhů. Je formulována věta, určující podmínky, které je nutno klást na isomorfismus projektivních rovin, aby příslušné planární ternární okruhy byly isotopické.

BOHDAN ZELINKA, Liberec: *Caterpillars.* (Housenky.)

Housenka je strom C, z kterého vynecháním všech koncových hran a všech koncových vrcholů vznikne strom skládající se z jediné jednoduché cesty, nebo prázdný graf. Autor podává charakterizaci housenek pomocí stromových algeber L. Nebeského a distančních matic E. A. Smolenského a studuje vnoření housenek do blokových grafů a do grafů n-rozměrných krychlí.

Štefan Schwabik, Praha: *On an integral operator in the space of functions with bounded variation,* II. (O integrálním operátoru v prostoru funkcí s ohraničenou variací, II.)

V práci autor ukazuje, že řešení Fredholmovy-Stieltjesovy rovnice lze vyjádřit pomocí integrálního operátoru stejného typu jako je operátor vystupující v rovnici samotné.

Ladislav Nebeský, Praha: *On partition graphs and generalizations of line graphs.* (O rozkladových grafech a zobecněních hranových grafů.)

V této poznámce je dokázána věta o rozkladových grafech. Věta má tento důsledek: Je-li G graf neobsahující žádný uzel stupně 0 nebo 1, potom G je kontrakcí hranového grafu hranového grafu grafu G.

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

RECENSE

Oscar Zariski - Pierre Samuel: COMMUTATIVE ALGEBRA, Volume 1. Graduate Texts in Mathematics, Vol. 28. Springer-Verlag, New York—Heidelberg—Berlin 1975. Stran VIII + + 329, cena DM 34,50.

Kniha je téměř nezměněné druhé vydání 1. dílu dvousvazkové Zariského a Samuelovy monografie o komutativní algebře, která vyšla v roce 1958 (1. díl) a 1960 (2. díl) v nakladatelství Van Nostrand, Princeton, N. J. Oba díly byly přeloženy do ruštiny a vyšly v Izdat. inostr. lit. v Moskvě roku 1963.

Rozvoj metod komutativní algebry je těsně spjat s rozvojem abstraktní algebraické geometrie a tato skutečnost je patrna i v této učebnici, jejíž autoři jsou ostatně dobře známí algebraičtí geometři. V 1. díle učebnice je tato spojitost zatím méně zřetelná, neboť zde se teprve rozvíjejí základy komutativní algebry.

První díl obsahuje pět kapitol. První kapitola seznamuje čtenáře s úvodními pojmy jako jsou grupy, okruhy, tělesa, okruhy polynomů a vektorové prostory. Podrobněji jsou tu vyšetřeny podílové okruhy vzhledem k dané multiplikativně uzavřené množině prvků okruhu, potřebné v algebraické geometrii.

Druhá kapitola detailněji rozvíjí teorii těles. Standardní obsah je obohacen podrobnějším vyšetřením transcendentních těles (separabilita, lineární disjunktnost).

Klasický je i obsah třetí kapitoly o ideálech v komutativních okruzích a modulech nad těmito okruhy. Kapitola seznamuje čtenáře se základními vlastnostmi ideálů a modulů, s operacemi na těchto objektech, s významem podmínek konečnosti řetězců a s komposičními řadami v modulech. Podrobněji jsou tu vyšetřeny direktní rozklady modulů a okruhů. Pro komutativní algebry nad tělesem je zaveden tensorový součin, jehož je zde pak použito k vyšetření volné vnořitelnosti dvou oborů integrity nad daným tělesem do tělesa.

Vlastní komutativní algebra začíná ve čtvrté kapitole věnované noetherovským okruhům. Úvodem je dokázána základní Hilbertova věta o bázi a vyšetřena struktura okruhů s klesající podmínkou řetězců. Hlavní obsah kapitoly tvoří primární rozklady ideálů v noetherovských okruzích a jejich aplikace, vztahy mezi ideály okruhu a jeho podílového okruhu, vlastnosti prvoideálů a jejich řetězců v noetherovských okruzích. Krátký dodatek ke čtvrté kapitole zobecňuje primární rozklady pro noetherovské moduly.

Pátá kapitola je věnována Dedekindovým okruhům a klasické teorii ideálů. Nejprve se tu vyšetřuje jeden ze základních pojmů komutativní algebry, totiž pojem celistvé závislosti okruhů. Čtenář tu nalezne např. normalisační lemma, Cohenovu-Seidenbergovu větu o vztazích mezi prvoideály okruhu a jeho celistvého nadokruhu. Následuje pak vlastní studium Dedekindových okruhů. Ty jsou definovány jako obory integrity, v nichž každý ideál je součin prvoideálů a je odvozena jejich charakterisace jakožto noetherovských celistvě uzavřených okruhů, v nichž každý vlastní prvoideál je maximální. Kapitolu uzavírá studium nadokruhů nad Dedekindovými okruhy a aplikace dosažených výsledků, např. v teorii kvadratických těles.

Zariského a Samuleova kniha je v podstatě první ucelenou a přístupně psanou učebnicí komutativní algebry. Za dobu své existence se stala známým, hojně používaným i citovaným dílem. I když se od doby jejího vzniku objevila řada znamenitých knih o komutativní algebře, její potřebnost jako základní učebnice o komutativní algebře trvá stále.

Václav Vilhelm, Praha

Dale Husemoller: FIBRE BUNDLES, Second Edition, Graduate Texts in Mathematics, 20. Springer-Verlag, New York—Heidelberg—Berlin 1975, XV + 327 stran, cena DM 41,20.

První vydání recenzované knihy vyšlo r. 1966 a bylo přeloženo do ruštiny r. 1970. Druhé vydání je rozšířeno pouze o stručný popis několika nezávislých důkazů Adamsovy hypotézy o vztahu stabilních tříd fibrové homotopie ke K-teorii a o 25tistránkový dodatek, v němž se ukazuje využití dvojné suspenze při studiu Hopfova invariantu modulo prvočíslo. Protože ruský překlad je všeobecně dostupný, uvedeme obsah všech tří částí knihy jen stručně. První část je samostatným výkladem obecné teorie topologických fibrovaných prostorů a lze ji chápat jako moderní ekvivalent první poloviny klasického Steenrodova díla „Topology of Fibre Bundles" z r. 1951. U čtenáře se předpokládá znalost základů homotopické topologie. Pro vektorové bandly se dokazují homotopické klasifikační věty a pro hlavní fibrované prostory se podává Milnorova konstrukce universálního bandlu. Přitom se probírají i základní vlastnosti fibrovaných prostorů se strukturní grupou a jejich „souřadnicové" konstrukce pomocí přechodových funkcí, což patří k základnímu aparátu diferenciální geometrie. Výklad je moderní a systematický, takže tuto část lze doporučit k samostatnému studiu např. aspirantům.

Druhá část je úvodem do K-teorie. Základní pojmy jsou ještě vyloženy systematicky, dále se však výklad podstatně zrychluje. Řadu důkazů autor vypouští nebo jen naznačuje, což mu však umožňuje uvést některé hluboké výsledky, kterých bylo pomocí K-teorie dosaženo. Dokazuje se (jen v komplexním případě) Bottova věta o periodičnosti. Uvádí se klasická konstrukce, jak pomocí Cliffordovy algebry lze na n-rozměrné sféře jednoduše sestrojit jistý počet globálně nezávislých vektorových polí, a vysvětlují se hlavní rysy proslulého Adamsova důkazu tvrzení, že více takovýchto polí již neexistuje. Každý hlavní fibrovaný prostor s bází X a strukturní grupou G určuje přirozený morfizmus z reprezentačního okruhu $R(G)$ grupy G do okruhu $K(X)$. Pro klasické grupy se takto dostává řada silných vztahů mezi $R(G)$ a $K(X)$, např. reprezentační okruhy spinorových grup určují zcela K-okruhy sfér. Uvádí se rovněž Atiyahův důkaz neexistence elementů s Hopfovým invariantem rovným jedné. Při prvním vydání knihy byla její druhá část vysoce ceněna proto, že je v ní shromážděno mnoho důležitých výsledků, které byly roztroušeny po časopisech a cyklostylovaných přednáškách. Situace se během doby změnila a recenzent se nemůže zbavit dojmu, že by bylo nutno tyto partie podstatně přepracovat, aby se kniha opravdu mohla stát učebnicí pro graduované studenty, jako to odpovídá názvu řady, v níž nyní vychází. Třetí část knihy je věnována charakteristickým třídám. I když výklad klasických Stiefelových-Whitneyových a Chernových tříd je celkem souvislý, není dostatečně podrobný k tomu, aby začátečník z něho získal ucelený obraz o jejich významu. Závěrečná kapitola obsahuje velmi obecný přístup k pojmu charakteristické třídy, která se chápe jako morfismus z vhodného faktorového prostoru vektorových bandlů do nějakého kohomologického funktoru. V několika významných případech autor dospívá k úplnému popisu všech charakteristických tříd.

Druhé vydání recenzované knihy svědčí o stále rostoucím zájmu o teorii fibrovaných prostorů, která nachází četné aplikace v diferenciální a algebraické geometrii, globální analýze aj. Hlavní cena knihy spočívá i nadále v širokém okruhu otázek, které jsou v ní vyloženy. Citelným nedostatkem pak je to, že v druhé polovině knihy autor nejen vypouští mnohé důkazy, ale je nadmíru stručný i při zavádění nových pojmů, takže četné jeho definice jsou, přesně vzato, neúplné. (Velkou práci zde vykonal redaktor ruského překladu, který text na řadě míst doplnil a upravil. Bohužel, při druhém anglickém vydání nebylo k těmto podstatným zlepšením přihlédnuto.) Po prostudování celé knihy se však čtenáři otevře přístup do nitra dnešní algebraické topologie, kam, jak se zdá, žádná „královská" cesta zatím nevede.

Ivan Kolář, Brno

Serge Lang: $SL_2(R)$. Addison-Wesley Publishing Company 1975, str. X + 428, cena 19,50 $.

Předmětem knihy jsou representace grupy unimodulárních transformací reálné roviny, tj. grupy matic 2 × 2 nad tělesem reálných čísel o determinantu 1. Je to nejjednodušší, netriviální grupa lineárních transformací. Její velká důležitost plyne z jejího dalšího významu — je to grupa konformních zobrazení poloroviny Im $z > 0$ komplexní roviny C na sebe.

Vzhledem k tomu, že tato grupa může být grupou symetrie lineárních systémů (a to i nekonečné dimense), je zajímavé a důležité znát její lineární representace a to, jak jsou konstruovány z irreducibilních. Pro sestrojení representací je použita metoda indukovaných representací (pocházející v diskretním případě od Frobenia a rozšířena na spojitý případ Mackeyem). Při ní se vyjde z podgrupy, pro níž je representace známa, a representace celé grupy je integrálem přes homogenní prostor levých tříd. V případě $SL_2(R)$ je možné vzít za tuto podgrupu grupu rotací roviny, která má navíc výhodu, že je komutativní a tedy její irreducibilní representace jsou jednorozměrné. Zároveň s representací π grupy se vyšetřuje i representace $d\pi$ její Lieovy algebry.

Druhá polovina knihy se zabývá případem, kdy podgrupa o známé representaci je diskretní $SL_2(Z)$. Vyšetřování těsně souvisí s teorií funkce komplexní proměnné (eliptické funkce, Laplaceův operátor).

Kniha je zakončena 5 dodatky, kde je pro čtenáře shrnuta nutná látka z teorie operátorů a Sobolevových prostorů.

Kniha nejen systematicky vykládá dosažené výsledky v teorii representace grupy $SL_2(R)$ systematickým způsobem a z jednotného hlediska, ale je zároveň ilustrací použití obecné teorie indukovaných representací.

Bibliografie (i když si autor nečiní nárok na úplnost) je rozsáhlá (Achiezer je soustavně přepisován Akniezer).

Úprava a tisk jsou velmi pěkné. *Václav Alda,* Praha

Paul R. Halmos: NAIVE SET THEORY. Springer-Verlag, Berlin—Heidelberg—New York 1974, VII + 104 str., cena DM 13,50.

Uveďme nejprve několik autorových myšlenek z úvodu: Všichni matematici souhlasí, že každý matematik musí znát něco z teorie množin, nesouhlas začíná při pokusu rozhodnout kolik je toto „něco". Kniha je autorovou odpovědí na rozsah tohoto „něco" při čemž je rozsah stanovován s ohledem na začínajícího matematika, který chce studovat např. grupy a nebo integrály. Autor se snaží vystačit s minimem logického formalismu a filosofického rozboru. Látka je podávána z axiomatického hlediska v tom smyslu, že jsou formulovány axiomy a používají se v důkazech. Mnohem více však autor zdůrazňuje druhou „naivní" stránku svého přístupu, čímž rozumí jednak používání běžného, neformálního jazyka, ale hlavně to, že teorii množin považuje za sourhn faktů, jejichž krátkým a vhodným shrnutím jsou axiomy.

Kniha je psána formou 25 poměrně krátkých kapitol, z nichž každá je věnována úzce vymezenému oddílu — pojmu, axiomu nebo větě. Autor věnuje velkou pozornost vysvětlování, proč se zavádí ten který pojem nebo axiom, právě toto je velikou předností práce. V některých případech (např. v důkazu, že z axiomu výběru plyne, že každou množinu lze dobře uspořádat), je v knize uvedeno klasické pojetí, i když v axiomatické teorii množin již byly vytvořeny jednodušší metody. V poslední kapitole se o hypotéze kontinua píše, že je známa pouze konsistence s axiomy teorie množin. Toto však již od Cohenových výsledků (1964) neplatí, neboť je již známa i konsistence negace hypotézy kontinua.

Snaze o pečlivé vysvětlení je podřízen i rozsah knihy, který je pro autora minimem toho, co by měl znát každý matematik; zdá se, že pro práci v mnoha směrech moderní matematiky tento rozsah nebude postačující. Shrňme alespoň heslovitě obsah knihy: uspořádané a neuspořádané dvojice, základní množinové operace, přirozená čísla a Peanova axiomatika, axiom výběru, Zornovo lemma, uspořádání a dobré uspořádání, ordinální a kardinální čísla a základy jejich aritmetiky, Cantorova-Bernsteinova věta.

Autor píše velmi přehledně a vtipně a používá hezkých přirovnání. Knihu je možno vřele doporučit každému, kdo se chce obeznámit se základy teorie množin a neklade si za cíl dosáhnout v tomto oboru hlubokých znalostí.

Antonín Sochor, Praha

D. Ivaşcu: INTRODUCERE ÎN TEORIA GRUPURILOR KLEIN, Ed. Acad. RSR, Bucureşti 1973, 169 str., cena Lei 7,25.

Kniha se skládá ze tří kapitol. V první se probírají geometrické aspekty teorie Kleinových grup včetně konstrukce fundamentálních oblastí. Ve druhé kapitole je uvedena teorie funkcí a forem, invariantních vzhledem ke Kleinově grupě; v poslední teorie Teichmüllerových prostorů.

Kniha je velmi pěkně napsaná (i když rumunský text bude jistě řadě čtenářů působit potíže). Vznikla v semináři vedeném prof. Cabiria Andreian Cazacu; její výsledky se opírají o práce L. V. Ahlforse a L. Berse.

Alois Švec, Olomouc

R. Roşca: VARIETĂŢI IZOTROPE ŞI PSEUDOIZOTROPE INCLUSE ÎNTR-O VARIETATE RELATIVISTA, Ed. Acad. RSR, Bucureşti 1972, 148 str., cena Lei 5,25.

Práce je z největší části založena na autorových původních výsledcích. Relativistickou varietou V_L nazývá autor čtyřrozměrnou diferencovatelnou varietu s metrikou $ds^2 = g_{ij}\, dx_i\, dx_j$ signatury $(---+)$; Minkowskiho prostor M^4 je pak plocha V_L. Knížka se zabývá systematickou lokální diferenciální geometrií ploch a nadploch prostorů M^4 resp. V_L; je probrána i teorie nadploch vícerozměrného Minkowskiho prostoru M^{n+1}. Studium je provedeno klasickým Cartanovým aparátem.

Alois Švec, Olomouc

Nathan Jacobson: LECTURES IN ABSTRACT ALGEBRA, III. Theory of Fields and Galois Theory. Graduate Texts in Mathematics, 32. Springer-Verlag, New York—Heidelberg—Berlin 1964; druhé opravené vydání. Str. XI + 323, cena 36,20 DM.

Prvním účelem knihy je presentování teorie těles, důležité k porozumění moderní algebraické teorie čísel, teorie okruhů a algebraické geometrie. Těmito aspekty se zabývají hlavně kapitoly I, IV a V, které probírají konečně dimensionální rozšíření těles a Galoisovu teorii, obecnou teorii struktury těles a teorii valuací. Také výsledky třetí kapitoly o abelovských rozšířeních nacházejí aplikace v teorii čísel. Velká pozornost je věnována vztahům mezi současnou teorií těles a klasickými problémy, které vedly k jejímu vytvoření. Příkladem toho je druhá kapitola, která podává Galoisovu teorii řešitelnosti algebraických rovnic, a kapitola šestá, kde je probráno Artinovo použití teorie reálných uzavřených těles na řešení Hilbertova problému o positivně definitních racionálních funkcích. Kniha obsahuje velkou řadu cvičení a je napsána velmi dobře, jak to již odpovídá Jacobsonovu standartu. Názvy kapitol: Introduction, Finite dimensional extension fields, Galois theory of equations, Abelian extensions, Structure theory of fields, Valuation theory, Artin-Schreier theory.

Alois Švec, Olomouc

A. A. Borovkov: STOCHASTIC PROCESSES IN QUEUEING THEORY (Stochastické procesy v teorii hromadné obsluhy). Vyšlo jako 4. svazek edice Applications of Mathematics v nakladatelství Springer, New York—Heidelberg—Berlín 1976; 290 stran, cena 72.80 DM.

Ruský originál této knihy — Вероятностные процессы в теории массового обслуживания — vyšel v moskevském nakladatelství Nauka již v r. 1972 a je tedy dostatečně znám i u nás. Autor, známý sovětský odborník v teorii hromadné obsluhy, se tu pokusil podat ucelenou teorii pojednávající o různých typech systémů hromadné obsluhy z jednotícího strukturálního hlediska. Tuto snahu jistě uvítal každý, kdo zná roztříštěnost přístupů, metod i způsobů interpretace myšlenek a výsledků teorie hromadné obsluhy v dosud běžném pojetí. Cenou, kterou za jednotu teorie musel autor zplatit, byla ovšem ztráta názornosti a průzračnosti některých klasických postupů specifických pro určité typy systémů hromadné obsluhy. Kdežto z hlediska *matematické teorie* jde o nesporný pokrok, prakticky orientovaní zájemci patrně zaváhají před úkolem ovládnout tak abstraktní obecnou teorii bez přímé vazby k aplikacím, resp. k reálným modelům a interpretacím, a zůstanou asi i nadále příznivci klasického přístupu ke studiu systémů hromadné obsluhy. Kromě toho zůstaly některé zajímavé a významné aspekty — např. vliv frontového režimu — zatím mimo dosah Borovkovovy teorie.

Vydání Borovkovovy monografie v angličtině je bezpochyby možno jen uvítat, neboť přispěje k značnému rozšíření okruhu čtenářů, kteří se tak mohou bezprostředně seznámit s originálními myšlenkami autorovými. Přitom však není anglické vydání pouhým doslovným překladem ruského originálu. Autor knihy spolupracoval s překladatelem a poskytl mu některé nové, resp. upravené výsledky a dodal řadu doplňků, mj. tři celé nové paragrafy. Uvádí-li ovšem překladatel v předmluvě, že autor dodal asi sto stran nového materiálu, nelze to chápat tak, že v anglickém vydání je sto nových stran navíc: skutečný rozsah *přidaného* textu je zhruba čtyřikrát menší, zbytek dlužno přičíst na vrub oprav a úprav jednotlivých pasáží. Nepochybně však autorovými zásahy hodnota překladu podstatně vzrostla.

Vcelku lze konstatovat, že Borovkovova kniha patří dnes k stěžejním dílům literatury o teorii hromadné obsluhy.

František Zítek, Praha

F. Spitzer: PRINCIPLES OF RANDOM WALK (Principy náhodných procházek). Vyšlo jako 34. svazek edice Graduate Texts in Mathematics ve Springerově nakladatelství, New York—Heidelberg—Berlín 1976; 420 stran, cena 48,40 DM.

Jde o druhé vydání známé monografie, jejíž první vydání vyšlo již v r. 1964 v princetonském nakladatelství D. Van Nostrand. Autor využil příležitosti nového vydání k opravě chyb, zjednodušení některých důkazů a k doplnění seznamu literatury. Základní struktura a obsah knihy zůstaly ovšem zachovány.

I když v mezidobí došlo k poměrně prudkému a rozsáhlému rozvoji teorie náhodných procházek, a to jak v souvislosti s teorií potenciálu tak i směrem k různým zobecněním klasické náhodné procházky — tj. procházky *jediné* částice, která se pohybuje s danými *pevnými* pravděpodobnostmi v daných směrech — zůstává Spitzerova monografie dosud nepřekonaným souhrnným dílem, v němž lze najít poučení o hlavních myšlenkách a výsledcích této nesporně zajímavé partie teorie pravděpodobnosti.

Jednou z předností náhodných procházek je jejich relativní jednoduchost a názornost, která dovoluje pomocí vhodných interpretací snáze proniknout do hlubší problematiky. Ač jde o dosti speciální oblast, je velmi vhodná pro účely úvodního studia, neboť lze na ní demonstrovat některé základní problémy a výsledky teorie stochastických procesů, zejména markovovských, které jsou v případě obecných procesů daleko méně přístupné.

Spitzerova kniha si tedy plným právem zaslouží zařazení mezi standardní učební texty na postgraduální úrovni a lze ji bez váhání doporučit jako vhodnou četbu např. pro aspiranty a vůbec všechny seriosní zájemce o moderní teorii stochastických procesů.

František Zítek, Praha

J. T. Oden, J. N. Reddy: VARIATIONAL METHODS IN THEORETICAL MECHANICS (Variační metody v teoretické mechanice). Universitext, Springer-Verlag, Berlin-Heidelberg-New York 1976. 302 stran, 2 obr., cena DM 29,80.

Otevřeme-li tuto knihu a přečteme-li si obsah, získáme dojem, že půjde o výjimečně skvělou monografii. Výběr látky je nesporně velice slibný: historický úvod, základní definice a věty z klasického i moderního variačního počtu, přehled základů mechaniky kontinua včetně thermodynamického hlediska, množství variačních principů z různých odvětví mechaniky a fyziky, teorie eliptických okrajových úloh, monotonních operátorů, variačních nerovnic, aproximační vlastnosti a konvergence metody konečných prvků. Pustíme-li se však do čtení, nadšení se ztrácí až nakonec z původního obdivu pro knihu a její autory zbude jen směsice dvojakých pocitů. Důvodů je hned několik: nepřesné formulace četných vět a definic, neúplnost a jindy zase zbytečnost některých částí důkazů, mezery v přehledu literatury článkové i knižní a konečně poměrně mnoho tiskových chyb.

Jako ilustrativní příklady uvádím: na str. 25 (Example 2.13) — z toho, že nejsou splněny postačující podmínky věty 2.3 ještě neplyne, že tvrzení této věty neplatí; v lemmatech 5.1 až 5.4 místo předpokladu „dostatečně hladké funkce" stačí spojitá funkce a místo předpokladu rovnosti nule pro obecný argument t stačí pro koncový bod časového intervalu; na str. 107 (Corollary 4.1.1) — zdá se, že značně obecná formulace činí autorům potíže a proto zbytečně omezují variace tím, že požadují splnění homogenních okrajových podmínek; na str. 109 výklad odvození „doplňkového principu" je chybný: $Tu = v$ se nedosazuje, nýbrž vyplyne teprve jako Eulerova podmínka — správně má být „existuje $u \in H$ tak, že $Tu = v$"; na str. 164 předpoklad spojité diferencovatelnosti koeficientů E_{ijkl} je zbytečně omezující — stačí omezenost a měřitelnost; v důkazu lemmatu 6.3.2 (který je převzat z práce I. Babušky a A. K. Azize), má být pomocná úloha $\Delta^2 U = 0$ v Ω, $U = 0$ a $\partial U/\partial n = g$ na $\partial\Omega$ místo chybného $\Delta U = 0$ v Ω a $\partial U/\partial n = g$ na $\partial\Omega$; na str. 239 u Brouwerovy věty chybí předpoklad, že zobrazení je „do sebe"; na str. 279 v důkazu věty 7.1 schází přechod od konečného elementu na celou oblast, zato (7.56) je nadbytečné. Mezi citovanou literaturou chybí např. jedna z nejlepších monografií o konečných prvcích od Fixe a Stranga, dále články o variačních principech od četných autorů ze socialistických zemí; u konvolučních principů schází zmínka o A. Schaperym, který zavedl tento druh variační formulace ještě před M. Gurtinem. Tiskových chyb je tolik, že se zdá jakoby kniha vyšla bez korektury.

Není sice nebezpečí, že by tato kniha dosáhla u nás většího rozšíření, ale přesto varuji před studiem této verze. Snad by po opravě závažnějších a tiskových chyb bylo možno knihu doporučit v novém, přepracovaném vydání. Zatím slouží spíše jako odstrašující příklad, jak se knihy psát nemají.

Ivan Hlaváček, Praha

Ferenc Szász: RADIKALE DER RINGE. VEB Deutscher Verlag der Wissenschaften, Berlin — Akadémiai Kiadó, Budapest, 1975. Stran 300, cena 50,— M.

Obsahem této knihy je teorie radikálů asociativních okruhů. Od čtenáře nevyžaduje žádných speciálních znalostí kromě základních algebraických pojmů z úvodních univerzitních přednášek, např. pojem okruhu, jednostranných a oboustranných ideálů, faktorového okruhu, homomorfismu okruhů apod. Jistá sběhlost v abstraktním uvažování a zkušenost z práce s matematickými pojmy se ovšem předpokládá. Algebraik, zájemce o problematiku z teorie okruhů, zde nalezne hojnost nových výsledků, které se nakupily od vyjítí knihy „Rings and Radicals". (N. Divinsky). Má možnost pokusit se o vyřešení některých otevřených problémů, které jsou vždy položeny ke konci každé kapitoly a kterých kniha obsahuje celkem 111. Podle autorových slov by řešení zadaných úloh mohlo být stimulem k dalšímu rozvinutí teorie radikálů.

Jedním z hlavních úkolů teorie okruhů je popsat strukturu všech okruhů. Např. známá Wedderburnova-Artinova věta charakterizuje polojednoduché okruhy. Jistou mírou „jednoduchosti" (přesně polojednoduchosti) okruhu je nulovost jeho radikálu. Zhruba řečeno, čím je větší radikál okruhu, tím je jeho struktura méně jednoduchá, např. je-li okruh roven svému radikálu. Při konstrukci různých typů radikálů v okruzích si všimneme, že je možné celý postup chápat obecněji.

Budiž dána nějaká třída R okruhů. Ideál I okruhu A nazveme R-ideálem právě tehdy, je-li $I \in R$. Existuje-li takový R-ideál R(A) okruhu A, že zahrnuje všechny R-ideály okruhu A, potom tento ideál R(A) nazýváme R-radikálem okruhu A. R-polojednoduchým okruhem A budeme rozumět každý okruh A, jehož R-radikál R(A) existuje a je roven nulovému ideálu okruhu A. Radikálová třída je pak každá třída R okruhů splňující tyto požadavky: každý homomorfní obraz R-okruhu je opět R-okruh, každý okruh obsahuje R-radikál a faktorový okruh každého okruhu podle svého R-radikálu je R-polojednoduchý.

Tento obecný pohled na radikály v okruzích je autorem dodržen. O podrobnějším obsahu knihy si uděláme jistou představu podle názvů jednotlivých kapitol: Allgemeine Theorie der Radikale. Theorie der supernilpotenten und speziellen Radikale. Nilradikale. Das Jacobsonsche Radikal. Das Brown-McCoysche Radikal. Weitere konkrete Radikale und Zeroid-Pseudoradikale.

Bedřich Pondělíček, Poděbrady

Stein, Sherman K.: MATHEMATICS. The man-made universe. 3. vydání, W. H. Freeman and Comp., San Francisco 1976. Str. xv + 573, cena US $ 12,50.

Dílo, které se svým obsahem i pojetím řadí na hranici mezi to, co nepřesně nazýváme rekreační matematikou, a mezi učební texty, vychází již ve třetím vydání. Autor je upravil na základě zkušeností svých i řady čtenářů a učitelů, kteří knihu používali. Nejrozsáhlejší změnou je doplnění kapitoly o pravděpodobnosti, která se však nijak neliší od většiny standardních výkladů (házení kostek, ruleta apod.).

Recenzi druhého vydání najde čtenář v Časopise pro pěstování matematiky *96* (1971), str. 220—221.

Jiří Jarník, Praha

Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ZPRÁVY

OBHAJOBY A DISERTAČNÍ PRÁCE KANDIDÁTŮ VĚD

Před komisemi pro obhajoby kandidátských disertačních prací obhájili dne 1. listopadu 1976 JAROSLAV BARTÁK práci na téma: „Stabilita abstraktních diferenciálních rovnic" a JIŘÍ CERHA práci na téma: „O některých problémech v teorii Volterrovy-Stieltjesovy rovnice", dne 12. listopadu 1976 VĚROSLAV JURÁK práci na téma „Turnaje a λ-roviny", dne 17. listopadu 1976 RNDr. BRANISLAV ROVAN práci na téma „Nutné podmienky inklúzie hlavných abstraktných tried jazykov s ohraničenými generátormi" a 16. prosince 1976 RNDr. SVATOPLUK ŠVARC práci na téma: Realizace topologických prostorů konvergenčními strukturami".

Redakce