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HARMONIC FUNCTIONS ON CONVEX SETS AND SINGLE LAYER POTENTIALS

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Introduction. Consider a domain G in the m-dimensional euclidean space R^m $(m > 2)$. A class of harmonic functions on G is formed by restricting Newtonian potentials associated with signed measures on ∂G. The problem that we are going to investigate here, can be formulated as follows: Given a harmonic function h on G, does there exist a signed measure μ with support in ∂G such that h coincides on G with the Newtonian potential of μ? If it were not the case, the question arises how to characterize the class $\mathscr{P}_m(G)$ consisting of all harmonic functions on $G \subset R^m$ which are representable by means of potentials mentioned above. This class of functions occurs in a natural way in connection with the Neumann and Robin problems treated by the method of integral equations.

An analogous question may be, of course, formulated also for plane domains. In this case Newtonian potentials are replaced by logarithmic potentials. In [3], Chap. IV, G. C. EVANS characterizes the system $\mathscr{P}_2(G)$ for $G = U_r$ where U_r denotes a circle with a radius r. His proof depends on the complex functions theory and Herglotz's theorem. The plane case is also investigated in [16]. DE LA VALLÉ POUSSIN gives in § 2, Chap. 9, sufficient conditions (see Théorème 260) for $h \in \mathscr{P}_2(U_r)$.

The results of Evans were extended by G. A. GARRETT in [4]. In addition to the above mentioned representation he investigated also a representation by means of double layer potentials. (Compare [15] where some other kinds of representations of harmonic functions in R^3 can be found as well.) In [4] the system $\mathscr{P}_3(G)$ is studied for G with a very smooth boundary (it is assumed that the normal satisfies a Lipschitz condition). In this connection a system of special sets $G_\alpha \subset G$ which exhaust G in an exactly determined sense is introduced and functions $h \in \mathscr{P}_3(G)$ are characterized in terms of a growth condition imposed on the total variation of the flow of grad h on G_α. To prove these facts, Garrett makes essential use of smoothness of the boundary in order to get information about the kernel of the corresponding integral equations.

Methods introduced in [4] are not applicable even for sets with boundaries of the class C^1. In particular, they give no results for such simple geometric bodies as cubes, cylindres, cones etc.

For an investigation of the above mentioned problem, it is convenient to apply methods developed for solving the boundary value problems in potential theory by J. KRÁL. We make essential use of the results connected with the Neumann problem obtained in [8], [13] and [14].

We shall prove a characterization of $\mathscr{P}_m(G)$ in the case that G is an open bounded *convex* set. The main result is presented by Theorem 10. The example given in Sec. 16 shows that there are "many" harmonic functions not contained in $\mathscr{P}_m(G)$.

The problem studied in this paper can be also understood as an inverse problem in potential theory. The case when $R^m \setminus G$ is a k-manifold is investigated in [2]. It should be noted here that a characterization of an essentially different type is given by I. N. KARCIVADZE (compare [7]).

1. Notation. Throughout this paper $m > 2$ will be a fixed integer. The closure of a set $M \subset R^m$ is denoted by $\overline{M}$, its boundary by ∂M and its interior by $\operatorname{int} M$. We shall write $\Omega_r(x)$ for $\{z \in R^m;\ |z - x| < r\}$. For each positive integer k and $M \subset R^m$, $H_k M$ will denote the outer Hausdorff k-dimensional measure on R^m defined by

$$H_k M = 2^{-k}\alpha_k \lim_{\varepsilon\to 0+} \inf \sum (\operatorname{diam} M_n)^k$$

where α_k is the volume of the unit k-ball and the infimum is taken over all sequences $\{M_n\}$ of sets M_n with $\bigcup_n M_n = M$ such that $\operatorname{diam} M_n \leqq \varepsilon$ for all n. H_m thus coincides with the Lebesgue measure in R^m (see [9]). If K is a compact subset of R^m, we shall write $C = C(K)$ for the Banach space of all continuous functions on K. The dual space of C is denoted by $C' = C'(K)$; the elements of C' are called Radon measures on K. For a Radon measure $\nu \in C'$ and $f \in C$ we shall sometimes write $\int_K f \, d\nu$ instead of $\nu(f)$. If $A \subset K$ is measurable and χ_A is its characteristic function, then we write $\nu(A)$ instead of $\nu(\chi_A)$.

For $\nu \in C'$ we shall consider its potential

$$U\nu : x \mapsto \int_K p(x - y) \, d\nu(y)$$

corresponding to the Newtonian kernel $p(z) = |z|^{2-m}/(m - 2)$. For a positive superharmonic function v and a set $A \subset R^m$, $\hat{R}_v^A$ will denote the balayage of v relative to A in R^m (for the definition see [5]).

2. Lemma. *Let G be a convex subset of R^m with a nonempty interior. Then for any $x_0 \in R^m$ there exists an m-dimensional density*

$$d_G(x_0) = \lim_{r\to 0+} \frac{H_m(\Omega_r(x_0) \cap G)}{H_m(\Omega_r(x_0))}$$

and $d_G(x_0) > 0$ for all $x_0 \in \overline{G}$.

Proof. Obviously, $d_G(x_0) = 0$ for each $x_0 \in R^m \setminus \bar{G}$, $d_G(x_0) = 1$ for $x_0 \in \text{int } G$. Consider now $x_0 \in \partial G$. Without loss of generality we can assume $x_0 = 0$.

Let $0 < r_1 < r_2$ and let φ be defined by

$$\varphi : x \longmapsto (r_1/r_2)\, x\,, \quad x \in R^m\,.$$

Then $\varphi(G) \subset \bar{G}$ and $\varphi(\Omega_{r_2}(0)) = \Omega_{r_1}(0)$. Using the fact that $H_m(\bar{G} \setminus G) = 0$, we obtain

$$\frac{H_m(\Omega_{r_1}(0) \cap G)}{H_m(\Omega_{r_1}(0))} = \frac{H_m(\Omega_{r_1}(0) \cap \bar{G})}{H_m(\Omega_{r_1}(0))} \geqq \frac{H_m(\Omega_{r_1}(0) \cap \varphi(G))}{H_m(\Omega_{r_1}(0))} =$$

$$= \frac{r_1^m}{r_2^m} \cdot \frac{H_m(\Omega_{r_2}(0) \cap G)}{H_m(\Omega_1(0)) \cdot r_1^m} = \frac{H_m(\Omega_{r_2}(0) \cap G)}{H_m(\Omega_{r_2}(0))} > 0\,.$$

It follows that there exists a positive

$$\lim_{r \to 0+} \frac{H_m(\Omega_r(x_0) \cap G)}{H_m(\Omega_r(x_0))}\,.$$

3. Lemma. *Let G be a bounded convex subset of R^m with a nonempty interior and let s be a non-negative continuous superharmonic function on R^m. Then $\hat{R}_s^G$ is a continuous potential of a positive Radon measure.*

Proof. The only fact which should be verified here is that $\hat{R}_s^G$ is continuous. According to Lemma 2, G is not thin at any of its boundary points. This can be shown in a similar way as in [5] (see Corollary 10.5). Consequently, we have $\hat{R}_s^G = s$ on $\bar{G}$. (See Theorem 10.7 in [5].)

Since $\hat{R}_s^G$ is harmonic on $R^m \setminus \bar{G}$, it follows from the Riesz decomposition theorem and Theorem 6.9 in [5] that $\hat{R}_s^G$ is a potential of a positive Radon measure $\mu \in C'(\bar{G})$. Now we can apply Evans-Vasilesco's theorem to obtain continuity of $\hat{R}_s^G$ on R^m.

4. Remark. It should be noted that in our special case $\hat{R}_s^G$ coincides with the réduite R_s^G.

In view of the introductory remarks we shall suppose troughout this note that G is a fixed open bounded convex subset of R^m. We shall investigate the system $\mathscr{P}_m(G)$ defined in the introduction.

Without loss of generality we suppose $0 \in G$ and for $\varrho > 0$ we define

$$G_\varrho = \{\varrho x;\, x \in G\}\,.$$

In what follows we shall write C and C' instead of $C(\partial G)$ and $C'(\partial G)$, respectively.

5. Proposition. *Let h be harmonic on a neighborhood of $\bar{G}$. Then there is a constant $c > 0$ and a positive Radon measure $\mu \in C'$ such that $h + c = U\mu$ on G and $U\mu$ is a continuous potential.*

Proof. Since h is harmonic on a neighborhood of a compact set $\bar{G}$, there is $\varrho > 1$ such that h is harmonic on a neighborhood of $\bar{G}_\varrho$. Choose $\alpha_1 > \sup h(\partial G)$, $\alpha_2 <$

$< \inf h(\partial G_\varrho)$ and let u be the capacitary potential for $\bar{G}$. Since $0 \leqq u \leqq 1$ on R^m and $\lim_{\|y\| \to \infty} u(y) = 0$, we have by the maximum principle

$$u(x) - 1 \leqq \beta < 0 \quad \text{on} \quad \partial G_\varrho .$$

It follows that there is $\gamma > 0$, $\gamma > \alpha_1$ such that

$$\gamma(u(x) - 1) \leqq \alpha_2 - \alpha_1$$

whenever $x \in \partial G_\varrho$.

Put $q = \gamma u$. Obviously, q is a potential. It follows from Lemma 3 that $q = \gamma$ on $\bar{G}$ and q is a continuous potential on R^m. Let $c = \gamma - \alpha_1$. Then

$$q(x) - c = \alpha_1 > h(x) \quad \text{on} \quad \partial G , \tag{1}$$

$$q(x) - c \leqq \alpha_2 < h(x) \quad \text{on} \quad \partial G_\varrho . \tag{2}$$

Define a function p on R^m as follows:

$$p = \begin{cases} h + c & \text{on} \quad \bar{G} \\ \inf(h + c, q) & \text{on} \quad G_\varrho \setminus \bar{G} \\ q & \text{on} \quad R^m \setminus G_\varrho . \end{cases}$$

We shall prove that p is the potential of a positive Radon measure. According to the Riesz decomposition theorem, it is sufficient to show that p is a non-negative superharmonic function dominated by a potential. It follows from the continuity of q and h and from (1) that for each $x \in \partial G$ there is a neighborhood $V(x)$ such that

$$p(y) = h(y) + c \quad \text{for all} \quad y \in V(x) .$$

Analogously for each $x \in \partial G_\varrho$ there is a neighborhood $W(x)$ such that

$$p(y) = q(y) \quad \text{on} \quad W(x) .$$

Continuity of p on $R^m \setminus (\partial G_\varrho \cup \partial G)$ is clear from the definition of p. Therefore p is a continuous superharmonic function on R^m. Applying the minimum principle to p, which is non-negative on ∂G_ϱ, we obtain that $p \geqq 0$ on G_ϱ and hence $p \geqq 0$ on R^m (note that $p = q$ on $R^m \setminus G_\varrho$).

Since p is dominated by the potential q, it follows that p is a continuous potential. By the Riesz decomposition theorem there is a measure μ such that $U\mu = \hat{R}_p^G$.

Since $U\mu$ is harmonic on $R^m \setminus \partial G$, it follows from Theorem 6.9 of [5] that $\mu \in C'$. By Lemma 3, $U\mu = \hat{R}_p^G$ is a continuous potential on R^m and

$$U\mu = \hat{R}_p^G = p = h + c \quad \text{on} \quad G .$$

6. Remark. It should be noted that it is an easy consequence of Theorem 5.2.2 (Fortsetzungssatz) in [1] that there are continuous potentials p, q which are harmonic on G and

$$h = p - q \quad \text{on} \quad G .$$

7. Definition. A unit vector Θ is called the exterior normal of a Borel set $M \subset R^m$ at $y \in R^m$ in the sense of Federer provided the symmetric difference of the sets M and the half-space $\{x \in R^m;\ (x - y)\,\Theta < 0\}$ has the m-dimensional density 0 at y. In what follows we shall put $n_M(y) = \Theta$ if Θ is the exterior normal of M in the sense of Federer and we denote by $n_M(y)$ the zero vector if there is no exterior normal Θ at y in the above mentioned sense. (See [8], where relevant references can be found.) If f is of the class C^1 on a neighborhood of $\overline{M}$, we define

$$\frac{\partial f}{\partial n_M}(y) = n_M(y)\operatorname{grad} f(y)\,; \quad y \in \overline{M}\,.$$

Finally, for $M = G_\varrho$ we shall write $n_\varrho(y)$ instead of $n_M(y)$.

8. Remark. The normal in the above mentioned sense is obviously uniquely determined and it is easily seen that

$$n_\varrho(t) = n_G(t/\varrho)\,, \quad t \in R^m\,.$$

Relations between the "classical" normal and the normal in the sense of Federer are studied e.g. in [10].

9. Lemma. *Let h be harmonic on G. For $\varrho \in (0, 1)$ and $y \in \overline{G}$ put $h_\varrho(y) = h(\varrho y)$. Define*

$$\tilde{K} = \sup_{\varrho \in (0,1)} \int_{\partial G} \left|\frac{\partial h_\varrho}{\partial n_G}(x)\right| \mathrm{d}H_{m-1}(x)\,,$$

$$K = \sup_{\varrho \in (0,1)} \int_{\partial G_\varrho} \left|\frac{\partial h}{\partial n_\varrho}(x)\right| \mathrm{d}H_{m-1}(x)\,.$$

Then $\tilde{K} < \infty$ if and only if $K < \infty$.

Proof. Fix $\varrho \in (0, 1)$. Then

$$\int_{\partial G} \left|\frac{\partial h_\varrho}{\partial n_G}(x)\right| \mathrm{d}H_{m-1}(x) = \varrho \int_{\partial G} |\operatorname{grad} h(\varrho x)\, n_G(x)|\, \mathrm{d}H_{m-1}(x) =$$

$$= \frac{1}{\varrho^{m-2}} \int_{\partial G_\varrho} |\operatorname{grad} h(t)\, n_G(t/\varrho)|\, \mathrm{d}H_{m-1}(t) = \frac{1}{\varrho^{m-2}} \int_{\partial G_\varrho} \left|\frac{\partial h}{\partial n_\varrho}(x)\right| \mathrm{d}H_{m-1}(x)\,.$$

Since h is harmonic on G, there is $c > 0$ such that

$$|\operatorname{grad} h_\varrho(x)| = |\varrho \operatorname{grad} h(\varrho x)| < c$$

for each $\varrho \in (0, \frac{1}{2}\rangle$ and each $x \in \partial G$. We see that there is $d > 0$ such that

$$\sup_{\varrho \in (0,1/2\rangle} \sup_{\partial G_\varrho} |\operatorname{grad} h(x)| < d\,.$$

We can therefore limit ourselves to $\varrho \in (\frac{1}{2}, 1)$. The equality proved above now implies easily that $K < \infty$ if and only if $\tilde{K} < \infty$.

10. Theorem. *Let G be an open bounded convex subset of R^m, $m > 2$, $0 \in G$ and let h be a harmonic function on G. Then the following conditions are equivalent:*

(i) *There exists a Radon measure $\mu \in C'(\partial G)$ such that $h = U\mu$ on G.*

(ii)

$$K = \sup_{\varrho \in (0,1)} \int_{\partial G_\varrho} \left| \frac{\partial h}{\partial n_\varrho}(x) \right| \mathrm{d}H_{m-1}(x) < +\infty\,.$$

Proof of implication (i) $\Rightarrow$ (ii). The converse will be proved later (see Sec. 12).

Choose $\varrho \in (0, 1)$. The mapping grad $U\mu$ is continuous on a neighborhood of the compact set ∂G_ϱ so that

$$\operatorname{grad} U\,\mu(x) = -\int_{\partial G} \frac{(x-y)}{|x-y|^m}\,\mathrm{d}\mu(y)\,, \quad x \in \partial G_\varrho\,.$$

Hence for each $x \in \partial G_\varrho$

$$\frac{\partial U\mu}{\partial n_\varrho}(x) = -\int_{\partial G} \frac{n_\varrho(x)\,.\,(x-y)}{|x-y|^m}\,\mathrm{d}\mu(y)\,.$$

Applying Fubini's theorem and substituting $x = \varrho t$, we have

$$K_\varrho := \int_{\partial G_\varrho} \left| \frac{\partial U\,\mu(x)}{\partial n_\varrho} \right| \mathrm{d}H_{m-1}(x) \leqq \int_{\partial G_\varrho} \int_{\partial G} \frac{|n_\varrho(x)\,(x-y)|}{|x-y|^m}\,\mathrm{d}|\mu|\,(y)\,\mathrm{d}H_{m-1}(x) =$$

$$= \int_{\partial G} \left(\int_{\partial G_\varrho} \frac{|n_\varrho(x)\,(x-y)|}{|x-y|^m}\,\mathrm{d}H_{m-1}(x) \right) \mathrm{d}|\mu|\,(y) =$$

$$= \int_{\partial G} \left(\int_{\partial G} \frac{|\eta_\varrho(\varrho t)\,(t - y/\varrho)|}{|t - y/\varrho|^m}\,\mathrm{d}H_{m-1}(t) \right) \mathrm{d}|\mu|\,(y)\,.$$

Here, as usual, $|\mu|$ stands for an indefinite variation of μ. Using the equality $n_\varrho(\varrho t) = n_G(t)$ we get

$$K_\varrho \leqq \int_{\partial G} v_\infty^G(y/\varrho)\,\mathrm{d}|\mu|\,(y) \leqq (\sup_{z \in R^m} v_\infty^G(z))\,\|\mu\| \tag{3}$$

where $v_\infty^M(z)$ is the quantity introduced in [8] as follows: Let $0 \neq M \subset R^m$ be an open set with a compact boundary. We call x a hit of a half-line $S \subset R^m$ on M provided $x \in S$ and each ball $\Omega_r(x)$ meets both $S \cap G$ and $S \setminus G$ in a set of positive linear measure. Given $y \in R^m$, $\Theta \in \Gamma = \partial\Omega_1(0)$, consider the total number $n_\infty^M(\Theta, y)$ $(0 \leqq n_\infty^M(\Theta, y) \leqq \infty)$ of all hits of the half-line $\{y + \varrho\Theta;\, \varrho > 0\}$ on M. For fixed y, $n_\infty^M(\Theta, y)$ is a Baire function of the variable $\Theta \in \Gamma$ and we may put

$$v_\infty^M(y) = \int_\Gamma n_\infty^M(\Theta, y)\,\mathrm{d}H_{m-1}(\Theta)\,.$$

In particular, since G is a convex set, then for each $z \in R^m$ obviously $n_\infty^G(\Theta, z) \leqq 2$, so that

(4) $$\sup_{z \in R^m} v_\infty^G(z) \leqq 2H_{m-1}(\Gamma) .$$

By Proposition 2.10 and Lemma 2.12 in [8]

$$v_\infty^G(z) = \int_{\partial G} \frac{|n_G(y)(y-z)|}{|y-z|^m} \, \mathrm{d}H_{m-1}(y) .$$

According to (3),

$$K = \sup_{\varrho \in (0,1)} K_\varrho \leqq (\sup_{z \in R^m} v_\infty^G(z)) \, \|\mu\| \leqq 2H_{m-1}(\Gamma) \, \|\mu\| < \infty .$$

This completes the proof of (i) $\Rightarrow$ (ii).

11. Notation. For each $\mu \in C'$ we shall define a functional $\mathscr{T}_\mu$ on the space $\mathscr{D}$ of all infinitely differentiable functions φ with compact support in R^m as follows:

$$\langle \varphi, \mathscr{T}_\mu \rangle = \int_G \operatorname{grad} \varphi \, . \, \operatorname{grad} U\mu \, \mathrm{d}H_m .$$

The distribution $\mathscr{T}_\mu$ is a weak characterization of the normal derivative of $U\mu$ (see [8]).

Since for any convex set $\sup_{y \in \partial G} v_\infty^G(y) < \infty$ (see (4)), by Theorem 1.13 in [8] it is possible for each $\mu \in C'$ to identify the functional $\mathscr{T}_\mu$ with a unique Radon measure which will be denoted by $\mathscr{T}\mu$. The mapping $\mathscr{T} : \mu \longmapsto \mathscr{T}\mu$ is a bounded operator on C'. Since G is convex, a result of [14] shows that the hypotheses of Theorem 28 of [13] are fulfilled. Consequently, the range of the operator $\mathscr{T}$ is equal to

$$C_0' := \{\nu \in C';\ \nu(\partial G) = 0\} .$$

It is easily seen that $C_0' \subset C'$ is a Banach space.

Thus we know that for each $\nu \in C_0'$ there is $\mu \in C'$ such that

$$\mathscr{T}\mu = \nu .$$

Denote

$$\tilde{\mu} = \mu - \frac{\mu(\partial G)}{\varkappa(\partial G)} \varkappa$$

where $\varkappa \in C'$ is the capacitary distribution for $\bar{G}$. Note that $\varkappa(\partial G) \neq 0$ and $\mathscr{T}\varkappa = 0$, because $U\varkappa$ is constant on G. Obviously $\tilde{\mu} \in C_0'$ and

$$\mathscr{T}\tilde{\mu} = \mathscr{T}\mu - \frac{\mu(\partial G)}{\varkappa(\partial G)} \mathscr{T}\varkappa = \mathscr{T}\mu .$$

We see that $\mathscr{T}(C'_0) = C'_0$. Now we shall show that the restriction $\mathscr{T}_0$ of the operator $\mathscr{T}$ to C'_0 is injective.

With regard to the fact that $\mathscr{T}_0$ is a linear operator, it is sufficient to prove that $\mathscr{T}_0 \nu = 0$ for a $\nu \in C'_0$ implies $\nu = 0$. By the results of [12]–[14], Theorem 26 of [13] is applicable. Therefore $\mathscr{T}\nu = 0$ implies that there exists $c \in R^1$ such that $U\nu = c$ on G.

We first show that $\nu = c\varkappa$. Indeed, since for $\nu^* = \nu - c\varkappa$ we have $\mathscr{T}\nu^* = 0$ and $U\nu^* = 0$ on G, we conclude again by Theorem 26 in [13] that $\nu^* = \nu - c\varkappa = 0$. Hence $0 = \nu(\partial G) = c\,\varkappa(\partial G)$ and $c = 0$ (recall that $\varkappa(\partial G) \neq 0$). We see that $U\nu = 0$ and again from Theorem 26 of [13] we get $\nu = 0$.

Since $\mathscr{T}_0$ is an injective and continuous linear operator mapping C'_0 onto C'_0 the inverse $\mathscr{T}_0^{-1}$ is a bounded linear operator on C'_0 by the open mapping theorem.

Our next objective is

12. Proof of implication (ii) $\Rightarrow$ (i) **of Theorem 10.** For an arbitrary $\varrho \in (0, 1)$ we define a Radon measure ν_ϱ by

$$\nu_\varrho : f \longmapsto \int_{\partial G} f \frac{\partial h_\varrho}{\partial n_G} \, \mathrm{d}H_{m-1}\,, \quad f \in C\,.$$

By Lemma 9 we have

$$\|\nu_\varrho\| = \int_{\partial G} \left| \frac{\partial h_\varrho}{\partial n_G}(x) \right| \mathrm{d}H_{m-1}(x) \leqq \tilde{K} < \infty\,.$$

Obviously there is a function φ infinitely differentiable with compact support in R^m such that $\varphi = h_\varrho$ on a neighborhood of $\bar{G}$. Applying the Gauss-Green theorem (compare e.g. Remark 2.11 in [8], where the corresponding references can be found), we get

$$\int_{\partial G} n_G(x) \operatorname{grad} \varphi(x) \, \mathrm{d}H_{m-1}(x) = \int_G \Delta\varphi(x) \, \mathrm{d}x = 0$$

so that

$$\nu_\varrho(\partial G) = \int_{\partial G} \frac{\partial h_\varrho}{\partial n_G}(x) \, \mathrm{d}H_{m-1}(x) = 0\,.$$

This shows that $\nu_\varrho \in C'_0$.

Let $\tilde{\mu}_\varrho \in C'_0$ be chosen such that

$$\mathscr{T}_0 \tilde{\mu}_\varrho = \nu_\varrho\,.$$

(We know that $\tilde{\mu}_\varrho$ is uniquely determined.) Then

$$\|\tilde{\mu}_\varrho\| \leqq \tilde{K} \|\mathscr{T}_0^{-1}\|\,.$$

By Alaoglu's theorem it is possible to choose a sequence $\{\varrho_n\}$, $\varrho_n \nearrow 1$ and $\tilde{\mu} \in C'$ such that for $\tilde{\mu}_n = \tilde{\mu}_{\varrho_n}$ we have

$$w^* - \lim \tilde{\mu}_n = \tilde{\mu}$$

where w^* refers to the w^*-topology on C'. Since $0 \in G$ and $x \mapsto 1/(m-2) \,.\, 1/|x|^{m-2}$ is a continuous function on ∂G, $U\, \tilde{\mu}_n(0) \to U\, \tilde{\mu}(0)$.

Put for $\varrho \in (0, 1)$

$$\mu_\varrho = \tilde{\mu}_\varrho + [h(0) - U\, \tilde{\mu}_\varrho(0)]\, \varkappa\,, \quad \mu_n = \mu_{\varrho_n}$$

where $\varkappa$ as above denotes the capacitary distribution for $\bar{G}$. Recalling that $\mathcal{T}\varkappa = 0$, we have

$$\mathcal{T}\mu_\varrho = \mathcal{T}\tilde{\mu}_\varrho = \nu_\varrho\,.$$

For $\varrho \in (0, 1)$ the equalities

$$U\, \mu_\varrho(0) = h(0) = h_\varrho(0) \tag{5}$$

hold. In order to show that $U\mu_\varrho = h_\varrho$ on G we apply Proposition 5. Hence for each $\varrho \in (0, 1)$ there is $c_\varrho > 0$ and a Radon measure $\bar{\mu}_\varrho \in C'$ such that

$$U\bar{\mu}_\varrho = h_\varrho + c_\varrho \quad \text{on} \quad G\,. \tag{6}$$

It is easily seen that $\mathcal{T}\, \bar{\mu}_\varrho(\psi) = \nu_\varrho(\psi)$ for all $\psi \in \mathcal{D}$. It follows

$$\mathcal{T}\bar{\mu}_\varrho = \mathcal{T}\tilde{\mu}_\varrho = \mathcal{T}\mu_\varrho\,.$$

Using Theorem 26 of [13], we establish the existence of $\tilde{c}_\varrho$ such that

$$U\bar{\mu}_\varrho = U\mu_\varrho + \tilde{c}_\varrho \quad \text{on} \quad G$$

and according to (5) and (6),

$$U\mu_\varrho = h_\varrho \quad \text{on} \quad G\,.$$

Setting

$$\mu_n = \tilde{\mu}_n + [h(0) - U\, \tilde{\mu}(0)]\, \varkappa\,,$$

from

$$w^* - \lim \tilde{\mu}_n = \tilde{\mu}\,, \quad h(0) - U\, \tilde{\mu}_n(0) \to h(0) - U\, \tilde{\mu}(0)$$

we obtain

$$w^* - \lim \mu_n = \tilde{\mu} + [h(0) - U\, \tilde{\mu}(0)]\, \varkappa\,.$$

Denote

$$\mu = \tilde{\mu} + [h(0) - U\, \tilde{\mu}(0)]\, \varkappa$$

and fix $y \in G$. Then the function $x \mapsto 1/(m-2) \,.\, 1/|x - y|^{m-2}$ is continuous on ∂G so that

$$h(\varrho_n y) = h_{\varrho_n}(y) = U\, \mu_n(y) \to U\mu(y)\,.$$

This together with the fact that $h(\varrho_n y) \to h(y)$ establishes the equality

$$U\mu = h \quad \text{on} \quad G\,.$$