Časopis pro pěstování matematiky, roč. 102 (1977), Praha

ON DISTRIBUTIONAL SOLUTIONS OF SOME SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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(Received July 17, 1975)

1. INTRODUCTION

Let $A(t) = (a_{ij}(t))$ be a matrix such that $a_{ij}(t)$ is a measure in the interval $(a, b) \subseteq \subseteq R^1$ for $i, j = 1, \ldots, n$, and let $f(t)$ be a vector whose all components $f_i(t)$ are also measures (defined in (a, b)). In this note we consider the system of equations

$$(*) \qquad y'(t) = A(t)\, y(t) + f(t)\,,$$

where y is an unknown vector. The derivative is understood in the distributional sense. Our result generalizes some theorems for linear differential equations (see [1], [2], [3], [6]).

2. THE PRINCIPAL RESULT

First we introduce some notations.

Let $A(t) = (a_{ij}(t))$ be a matrix whose all elements $a_{ij}(t)$ are measures defined in the interval (a, b) $(i, j = 1, \ldots, n)$, and let $\hat{A}(t) = (\hat{a}_{ij}(t))$ be a matrix whose all elements $\hat{a}_{ij}$ are functions such that $a_{ij} = [\hat{a}_{ij}]'$. We put $\Delta\, \hat{A}(t) = \hat{A}(t+) - \hat{A}(t-)$, $\|A\|\,(t) = \sum_{i,j=1}^{n} |a_{ij}|\,(t)$, $y_0 = (y_1^0, \ldots, y_n^0)$, $|y_0| = \sum_{i=1}^{n} |y_i^0|$, $y^*(t) = (y_1^*(t), \ldots, y_n^*(t))$, $|y|^*\,(t) = \sum_{i=1}^{n} |y_i|^*\,(t)$, where $t \in (a, b)$, $\hat{A}(t+) = (\hat{a}_{ij}(t+))$, $\hat{A}(t-) = (\hat{a}_{ij}(t-))$ and $y_i^0 \in R^1$. The remaining notations in this paper are taken from [5].

Theorem 2.1. *We assume that $a_{ij}(t)$ and $f_i(t)$ are measures defined in the interval (a, b) for $i, j = 1, \ldots, n$. Moreover, for every $t \in (a, b)$*

$$(2.1) \qquad \det(2I - \Delta\, \hat{A}(t)) \neq 0 \quad \text{and} \quad \det(2I + \Delta\, \hat{A}(t)) \neq 0\,,$$

where I denotes the identity matrix. Then the problem

$$(2.2) \qquad \begin{cases} y'(t) = A(t)\, y(t) + f(t) \\ y^*(t_0) = y_0 \end{cases}$$

has exactly one solution in the class $V^n_{(a,b)}$ *for every* $t_0 \in (a, b)$.

Remark 1. The assumption 2.1 in Theorem 2.1 is essential. This can be observed from the following

Example .

$$(2.3) \qquad \begin{cases} y'(t) = 2\, \delta(t)\, y(t) \\ y^*(-1) = 0\,, \end{cases}$$

$$(2.4) \qquad \begin{cases} z'(t) = -2\, \delta(t)\, z(t) \\ z^*(1) = 0 \end{cases}$$

(δ denotes Dirac's delta distribution). In fact, let H denote Heaviside's function and let c be a constant. From the equality

$$(2.5) \qquad H\delta = \tfrac{1}{2}\delta \quad \text{(see [4])}$$

it is not difficult to show that the distributions $y = cH$ and $z = c(H - 1)$ are solutions of the problem (2.3) and (2.4), respectively.

3. PROOFS

Before giving the proof of Theorem 2.1 we shall prove some lemmas.

Lemma 3.1. *We assume that* $P_1, P_2 \in V^1_{(a,b)}$ *and* $\lim_{t \to r} P_1^*(t) = \lim_{t \to r} P_2^*(t)$. *Moreover, let*

$$(3.1) \qquad p_1(t) + c_1\, \delta(t - r) = p_2(t) + c_2\, \delta(t - r)\,,$$

where $P_1' = p_1$, $P_2' = p_2$, $r \in (a, b)$ *and* c_1, c_2 *are constants. Then* $c_1 = c_2$.

This fact follows easily from the equality

$$(3.2) \qquad \int_r^t p_1(s)\, ds + c_1 \int_r^t \delta(s - r)\, ds = \int_r^t p_2(s)\, ds + c_2 \int_r^t \delta(s - r)\, ds\,.$$

Lemma 3.2. *Let* $a_{ij}(t)$ *and* $f_i(t)$ *be measures defined in the interval* (a, b) $(i, j = 1, \ldots, n)$, $t_0 \in (a, b)$. *Then there exists a number* $r > 0$ *such that*:

1. $a < t_0 - r < t_0 + r < b$,
2. *the problem* (2.2) *has exactly one solution* $y(t)$ *in the class*

$$V^n_{(t_0 - r, t_0 + r)}\,,$$

3. *there exist finite limits* $\lim_{t \to t_0 - r+} y^*(t)$, $\lim_{t \to t_0 + r-} y^*(t)$.

Proof. If there exists a number $r > 0$ such that

$$\int_{t_0-r}^{t_0+r} \|A\|(t)\,\mathrm{d}t < 1, \tag{3.3}$$

then in view of [5] the problem (2.2) has exactly one solution in the class $V^n_{(t_0-r,t_0+r)}$. In the opposite case we consider matrices $\hat{A}_1(t, t_0)$ and $\hat{A}_2(t, t_0)$ defined as follows:

$$\hat{A}_1(t, t_0) = \begin{cases} \hat{A}(t), & \text{for } a < t < t_0 \\ \hat{A}(t_0-), & \text{for } t_0 \leqq t < b, \end{cases} \tag{3.4}$$

$$\hat{A}_2(t, t_0) = \begin{cases} \hat{A}(t_0+), & \text{for } a < t \leqq t_0 \\ \hat{A}(t), & \text{for } t_0 < t < b. \end{cases} \tag{3.5}$$

Hence, we have

$$\hat{A}(t) = \hat{A}_1(t, t_0) + \hat{A}_2(t, t_0) - H(t - t_0)\,\hat{A}(t_0-) - H(t_0 - t)\,\hat{A}(t_0+) \tag{3.6}$$

and

$$y'(t) = U(t, t_0)\,y(t) + \delta(t - t_0)\,(\Delta\,\hat{A}(t_0))\,y^*(t_0) + f(t), \tag{3.7}$$

where $U(t, t_0) = A_1(t, t_0) + A_2(t, t_0)$, $A_1(t, t_0) = (\hat{A}_1(t, t_0))'$ and $A_2(t, t_0) = (\hat{A}_2(t, t_0))'$. Moreover, there exists a number $r_1 > 0$ such that

$$\int_{t_0-r_1}^{t_0+r_1} \|U\|(t, t_0)\,\mathrm{d}t < 1. \tag{3.8}$$

Taking into account [5], we infer that the system (2.2) has exactly one solution $y(t)$ in the class V^n_I, where $I = (t_0 - r_1,\ t_0 + r_1)$. We claim that $\sup_{t\in I} |y|^*(t) < \infty$. Indeed, let us put

$$\tilde{f} = \delta(t - t_0)\,(\Delta\,\hat{A}(t_0))\,y^*(t_0) + f, \quad K = 1 - \int_{t_0-r_1}^{t_0+r_1} \|U\|(t, t_0)\,\mathrm{d}t, \tag{3.9}$$

$$M_i = \left|\int_{t_0-r_1}^{t_0+r_1} \tilde{f}_i(t)\,\mathrm{d}t\right|, \quad M = \sum_{i=1}^{n} M_i, \quad \varepsilon > 0, \quad J = [t_0 - r_1 + \varepsilon,\ t_0 + r_1 - \varepsilon]$$

$$(a < t_0 - r_1 + \varepsilon < t_0 + r_1 - \varepsilon < b).$$

Then the relation (2.2) and [5] imply

$$\sup_{t\in J} |y|^*(t) \leqq |y_0| + \sup_{t\in J} |y|^*(t) \int_J \|U\|(t, t_0)\,\mathrm{d}t + M \tag{3.10}$$

and

$$\sup_{t\in I} |y|^*(t) \leqq K^{-1}(|y_0| + M). \tag{3.11}$$

Now we consider an arbitrary sequence $\{t_k\}$ such that $t_k \in I$ $(k = 1, 2, \ldots)$ and $t_k \to t_0 + r_1-$. Using (2.2), [5] and (3.11), we have

$$(3.12)\quad |y_i^*(t_k) - y_i^*(t_m)| \leqq \sup_{t\in I} |y|^*(t) \left|\int_{t_m}^{t_k} \|U\|(t, t_0)\,dt\right| + \left|\int_{t_m}^{t_k} \tilde{f}_i(t)\,dt\right| \leqq$$

$$\leqq K^{-1}(|y_0| + M)|\,Z^*(t_k) - Z^*(t_m)| + |G_i^*(t_k) - G_i^*(t_m)|\,,$$

where $Z' = \|U\|$, $G_i' = \tilde{f}_i$ and $i = 1, \ldots, n$. Similarly we prove that there exists a finite limit $\lim\limits_{t\to t_0 - r_1 +} y^*(t)$. Thus our assertion follows.

Lemma 3.3. *Let the assumptions of Theorem* 2.1 *be fulfilled and let* $y(t)$ *be a solution of the problem* (2.2) *in the class* $V^n_{(a,b)}$. *Then for every* $t \in (a, b)$

$$(3.13)\qquad y(t+) = (2I - \Delta\,\hat{A}(t))^{-1}\,[(2I + \Delta\,\hat{A}(t))\,y(t-) + 2\Delta\,F(t)]$$

and

$$(3.14)\qquad y(t-) = (2I + \Delta\,\hat{A}(t))^{-1}\,[(2I - \Delta\,\hat{A}(t))\,y(t+) - 2\Delta\,F(t)]\,,$$

where $F' = f$ and $\Delta\,F(t) = F(t+) - F(t-)$.

Proof. Let $y(t)$ be a solution of the problem (2.2). We consider vectors $\tilde{y}(t)$ and $\bar{y}(t)$ defined as follows:

$$(3.15)\qquad \tilde{y}(t) = \begin{cases} y(t)\,, & \text{for } a < t < t_1 \\ y(t_1-)\,, & \text{for } t_1 \leqq t < b\,, \end{cases}$$

$$(3.16)\qquad \bar{y}(t) = \begin{cases} y(t_1+)\,, & \text{for } a < t \leqq t_1 \\ y(t)\,, & \text{for } t_1 < t < b\,. \end{cases}$$

Then

$$(3.17)\qquad y(t) = \tilde{y}(t) + \bar{y}(t) - H(t - t_1)\,y(t_1-) - H(t_1 - t)\,y(t_1+)$$

and

$$(3.18)\qquad y'(t) = \tilde{y}'(t) + \bar{y}'(t) + \delta(t - t_1)\,(\Delta\,y(t_1))\,.$$

On the other hand,

$$(3.19)\quad y'(t) = U(t, t_1)\,y(t) + \delta(t - t_1)\,(\Delta\,\hat{A}(t_1))\,y^*(t_1) + \tilde{f}(t) + \bar{f}(t) +$$

$$+ \delta(t - t_1)\,(\Delta\,F(t_1))\,,$$

where $\tilde{f} = \tilde{F}'$, $\bar{f} = \bar{F}'$ and

$$(3.20)\qquad \tilde{F}(t) = \begin{cases} F(t)\,, & \text{for } a < t < t_1 \\ F(t_1-)\,, & \text{for } t_1 \leqq t < b\,, \end{cases}$$

$$(3.21)\qquad \bar{F}(t) = \begin{cases} F(t_1+)\,, & \text{for } a < t \leqq t_1 \\ F(t)\,, & \text{for } t_1 < t < b\,. \end{cases}$$