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THE EXISTENCE AND THE UNIQUENESS OF DISTRIBUTIONAL SOLUTIONS OF SOME SYSTEMS OF NON-LINEAR DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Let f_i $(i = 1, \ldots, n)$ be operations defined for every system of real functions $(y_1(t), \ldots, y_n(t))$ of locally bounded variation in the interval $(a, b) \subseteq R^1$. Moreover, let $f_i(y_1(t), \ldots, y_n(t))$ be a measure in (a, b) (i.e. $f_i(y_1(t), \ldots, y_n(t))$ is the first distributional derivative of a real function of locally bounded variation in (a, b)). In this paper we consider the following system of equations

$$(*) \qquad y_i'(t) = f_i(y_1(t), \ldots, y_n(t)) \quad (i = 1, \ldots, n),$$

where the derivative is understood in the distributional sense. By a solution of the system $(*)$ we understand every system of real functions $(y_1(t), \ldots, y_n(t))$ of locally bounded variation in the interval (a, b), which satisfies equation $(*)$. This class will be denoted by $V^n_{(a,b)}$. We prove some theorems on the existence and the uniqueness of solutions of the system $(*)$. Our results generalize some theorems for linear and non-linear differential systems (see [6], [9], [10], [12], [13]). The sequential theory of distributions will be used (see [4]).

2. THE PRINCIPAL RESULTS

First we introduce some notations.

A sequence of smooth, non-negative functions $\{\delta_k(t)\}$ satisfying: $\int_{-\infty}^{\infty} \delta_k(t)\, dt = 1$, $\delta_k(t) = \delta_k(-t)$, $\delta_k(t) = 0$ for $|t| \geqq \alpha_k$, where $\{\alpha_k\}$ is a sequence of positive numbers with $\alpha_k \to 0$ as $k \to \infty$ is called a δ-sequence.

We understand the product, the mean value and the modulus of distributions as generalized operations (see [2], [3], [4]).

One may prove that if P is a function of locally bounded variation in the interval (a, b), then for every $t_0 \in (a, b)$ the mean value $P^*(t_0)$ of P at the point t_0 exists and

$$(2.1) \qquad P^*(t_0) = \frac{P(t_0+) + P(t_0-)}{2},$$

where $P(t_0+)$ ($P(t_0-)$) denote the right (resp. left) hand side limits of the function P at the point t_0 (see [3]).

Let p be a measure defined in the interval (a, b) ($(-\infty, \infty)$). Then we put

$$(2.2) \qquad \int_c^d p(t)\,\mathrm{d}t = P^*(d) - P^*(c)\,, \quad \int_{-\infty}^{\infty} p(t)\,\mathrm{d}t = \lim_{\substack{d\to\infty\\ c\to-\infty}} \int_c^d p(t)\,\mathrm{d}t\,,$$

where $P' = p$ and $c, d \in (a, b)$.

In the case when P is a function of locally bounded variation in the interval (a, b) and q is a measure (in (a, b)), then it has been proved in [11] that

$$(2.3) \qquad \left|\int_c^d P(t)\, q(t)\,\mathrm{d}t\right| \leqq \left|\int_c^d |P|\,(t)\,|q|\,(t)\,\mathrm{d}t\right| \leqq \sup_{t\in[c,d]} |P|^*(t) \left|\int_c^d |q|\,(t)\,\mathrm{d}t\right|.$$

Now we shall introduce two hypotheses.

Hypothesis $\mathbf{H}_1$. 1. *Let f_i $(i = 1, \ldots, n)$ be operations defined for every system of functions $(y_1(t), \ldots, y_n(t))$ of locally bounded variation in R^1. Moreover, let $f_i(y_1(t), \ldots, y_n(t))$ be measures in R^1.*

2. *There exist non-negative measures $L_{ij}(t)$ $(i, j = 1, \ldots, n)$ defined in R^1 such that for two arbitrary systems of functions $(y_1(t), \ldots, y_n(t))$ and $(\bar{y}_1(t), \ldots, \bar{y}_n(t))$ of locally bounded variation in R^1 we have*

$$(2.4) \qquad |f_i(y_1(t), \ldots, y_n(t)) - f_i(\bar{y}_1(t), \ldots, \bar{y}_n(t))| \leqq \sum_{j=1}^{n} L_{ij}(t)\,|y_j(t) - \bar{y}_j(t)|\,,$$

$$(2.5) \qquad \sum_{i,j=1}^{n} \int_{-\infty}^{\infty} L_{ij}(t)\,\mathrm{d}t < 1\,, \quad \int_{-\infty}^{\infty} |f_i(t_1, \ldots, t_n)|\,\mathrm{d}t < \infty\,,$$

where $t_1, \ldots, t_n$ denote constant functions[1]).

Hypothesis $\mathbf{H}_2$. 1. *Let f_i $(i = 1, \ldots, n)$ be operations defined for every system of functions $(y_1(t), \ldots, y_n(t))$ of locally bounded variation in the interval $(a, b) \subseteq R^1$ and such that $f_i(y_1(t), \ldots, y_n(t))$ is a measure.*

2. *There exist non-negative measures $L_{ij}(t)$ defined in the interval (a, b) such that for arbitrary two systems of functions $(y_1(t), \ldots, y_n(t))$ and $(\bar{y}_1(t), \ldots, \bar{y}_n(t))$ of locally bounded variation in the interval (a, b), inequality (2.4) holds.*

Example 1. Let $L(t)$ and $g(t)$ be measures defined in R^1 and such that $\int_{-\infty}^{\infty} |L|\,(t)\,\mathrm{d}t < 1$, $\int_{-\infty}^{\infty} |g|\,(t)\,\mathrm{d}t < \infty$. Moreover, let h be a constant and let $y(t)$ be a function of locally bounded variation in R^1. Then it is not difficult to check that the operation f defined by

$$(2.6) \qquad f(y(t)) = L(t)\frac{1}{1 + y^2(t + h)} + g(t)$$

satisfies the hypotheses H_1 and H_2.

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[1]) The inequality between two distributions is understood as in [2].

Theorem 2.1. *Let hypothesis H_1 be fulfilled. Moreover, let h be a constant. Then the problem*

$$\begin{cases} y_i'(t) = f_i(y_1(t+h), \ldots, y_n(t+h)) \\ y_i^*(t_0) = y_i^0, \quad i = 1, \ldots, n \end{cases} \tag{2.7}$$

has exactly one bounded solution in the class $V^n_{(-\infty,\infty)}$.

Remark 1. We understand that two systems of functions from the class $V^n_{(a,b)}$ are equal, if they are equal in the distributional sense.

Remark 2. The assumptions (2.5) in Theorem 2.1 is essential. This can be observed from the following

Example 2.

$$\begin{cases} y'(t) = 2\,\delta(t)\,y(t) \\ y^*(-1) = 0, \end{cases} \tag{2.8}$$

where δ denotes Dirac's delta distribution. In fact, let H denote Heaviside's function and let c denote a constant. From the equality

$$H\delta = \tfrac{1}{2}\delta \quad \text{(see [14])} \tag{2.9}$$

it is not difficult to show that the distribution $y = cH$ is a solution of the problem (2.8).

Let all elements of the matrix $L = (L_{ij})$ $(i, j = 1, \ldots, n)$ be measures defined in the interval $(a, b) \subseteq R^1$. We say that the matrix L has the property (P) in the interval (a, b) if for every $t_0 \in (a, b)$ there exists a number $\varepsilon > 0$ such that

$$[t_0 - \varepsilon, t_0 + \varepsilon] \subset (a, b) \quad \text{and} \quad \sum_{i,j=1}^{n} \int_{t_0-\varepsilon}^{t_0+\varepsilon} |L_{ij}|\,(t)\,\mathrm{d}t < 1\,. \tag{2.10}$$

It is easy to verify that every locally integrable function in the interval (a, b) has the property (P). There exists a matrix of measures, which has not the property (P). In fact, let us put $L(t) = 2\,\delta(t)$, $a = -\infty$, $b = \infty$ and $t_0 = 0$.

Theorem 2.2. *Let hypothesis H_2 be satisfied. Moreover, let the matrix $L = (L_{ij})$ have the property (P) in the interval (a, b). Then the problem*

$$\begin{cases} y_i'(t) = f_i(y_1(t), \ldots, y_n(t)) \\ y_i^*(t_0) = y_i^0, \quad t_0 \in (a, b), \quad i = 1, \ldots, n \end{cases} \tag{2.11}$$

has exactly one solution in the class $V^n_{(a,b)}$.

Remark 3. Let $\bar{f}_i(t, v_1, \ldots, v_n)$ $(i = 1, \ldots, n)$ be real functions defined in the set

$$D\colon a < t < b\,, \quad -\infty < v_1, \ldots, v_n < \infty\,.$$

Moreover, let us assume that:

1. The functions $\bar{f}_i(t, v_1, \ldots, v_n)$ are measurable with respect to t for every system $(v_1, \ldots, v_n)$.

2. The functions $\bar{f}_i(t, v_1, \ldots, v_n)$ are continuous with respect to $(v_1, \ldots, v_n)$ for every $t \in (a, b)$.
3. There exist non-negative, locally integrable functions (in the interval (a, b)) $L_{ij}(t)$ $(i, j = 1, \ldots, n)$ and $u(t)$ such that

$$(2.12) \qquad |\bar{f}_i(t, v_1, \ldots, v_n) - \bar{f}_i(t, \bar{v}_1, \ldots, \bar{v}_n)| \leqq \sum_{j=1}^{n} L_{ij}(t) \, |v_j - \bar{v}_j| \,,$$

$$(2.13) \qquad |\bar{f}_i(t, 0, \ldots, 0)| \leqq u(t) \,.$$

Then the problem

$$(2.14) \qquad \begin{cases} y_i'(t) = \bar{f}_i(t, y_1(t), \ldots, y_n(t)) \\ y_i(t_0) = y_i^0 \,, \quad t_0 \in (a, b) \,, \quad i = 1, \ldots, n \end{cases}$$

has exactly one solution in the Carathéodory sense in the interval (a, b) (see [5]). It is easy to verify that the right-hand side of the system (2.14) satisfies hypothesis H_2, too. Thus in this case Theorem 2.2 generalizes the classical Carathéodory's result.

Remark 4. Non-continuous solutions of ordinary differential equations have been considered either by means of integral equations with generalized Stieltjes integral (see [7], [8], [15], [16]) or by means of theory of distributions (see [6], [9], [10], [12], [13]). The distributional solutions of non-linear differential equations have not been sufficiently studied. In [6], [9] and [10] the authors give theorems on the distributional solutions of some linear differential equations, but the product of two distributions is understood more generally in our paper than by those authors. More precisely, the existence of the product of a measure and a continuous function or a function of locally bounded variation does not result in general from the definition given in [6]. Hence our results may be applied even to some types of linear differential equations in the case when the theorems from [6], [9] and [10] cannot be used.

3. PROOFS

Proof of Theorem 2.1. We shall apply the method of successive approximations. Thus we consider the sequence of functions $\{g_{iv}\}$ defined as follows

$$(3.1) \qquad g_{i0}(t) = y_i^0 \,, \quad g_{iv}(t) = y_i^0 + \int_{t_0}^{t} f_i(g_{1v-1}(s + h), \ldots, g_{nv-1}(s + h)) \, ds \,,$$

$$i = 1, \ldots, n \,, \quad v = 1, 2, \ldots \,, \quad t \in R^1 \,.$$

We put

$$(3.2) \qquad \tilde{L}_i = \sum_{j=1}^{n} \int_{-\infty}^{\infty} L_{ij}(t) \, dt \,, \quad M_i = \int_{-\infty}^{\infty} |f_i(y_1^0, \ldots, y_n^0)| \, dt \,.$$

In view of (2.3) and (2.4), we have

$$(3.3) \qquad |g^*_{iv}(t) - g^*_{iv-1}(t)| \leqq M_i \mathcal{L}_i^{v-1} \quad \text{for every} \quad t \in (-\infty, \infty)\,.$$

Hence we infer that the sequence of functions $\{g^*_{iv}(t)\}$ is uniformly convergent to a function $g_i \in V^1_{(-\infty,\infty)}$ as $v \to \infty$. In fact, the inequality (3.3) implies

$$(3.4) \qquad |g_{iv}|^* (t) \leqq |y_i^0| + \frac{M_i}{1 - \mathcal{L}_i}\,.$$

Thus g_i is a bounded function in R^1. We consider a finite sequence of numbers $\{t_i\}$ such that $t_1 \leqq t_2 \leqq \ldots \leqq t_\iota$. Since

$$(3.5) \qquad \sum_{r=1}^{\iota} |g^*_{iv}(t_r) - g^*_{iv}(t_{r-1})| \leqq \int_{-\infty}^{\infty} |f_i(y_1^0, \ldots, y_n^0)|\, dt +$$

$$+ \sum_{j=1}^{n} \int_{-\infty}^{\infty} L_{ij}(t)\, |g_{jv-1}(t + h) - y_j^0|\, dt \leqq M_i +$$

$$+ \sum_{j=1}^{n} \int_{-\infty}^{\infty} L_{ij}(t) \left(|y_j^0| + \frac{M_j}{1 - \mathcal{L}_j} \right) dt + \sum_{j=1}^{n} \int_{-\infty}^{\infty} |y_j^0|\, L_{ij}(t)\, dt\,,$$

g_i is a function of locally bounded variation in R^1. Taking into account that by (3.3) the sequence of functions $\{g^*_{iv}(t)\}$ is uniformly convergent, we obtain

$$(3.6) \qquad g_i(t_0+) = y_i^0 + \lim_{t \to t_0^+} \lim_{v \to \infty} (F^*_{iv}(t) - F^*_{iv}(t_0)) =$$

$$= y_i^0 + \lim_{v \to \infty} \lim_{t \to t_0^+} (F^*_{iv}(t) - F^*_{iv}(t_0)) = y_i^0 + \tfrac{1}{2} \lim_{v \to \infty} (F_{iv}(t_0+) - F_{iv}(t_0-))\,,$$

where $F'_{iv}(t) = f_i(g_{1v}(t + h), \ldots, g_{nv}(t + h))$. Similarly

$$(3.7) \qquad g_i(t_0-) = y_i^0 + \lim_{t \to t_0^-} \lim_{v \to \infty} (F^*_{iv}(t) - F^*_{iv}(t_0)) =$$

$$= y_i^0 + \lim_{v \to \infty} \lim_{t \to t_0^-} (F^*_{iv}(t) - F^*_{iv}(t_0)) = y_i^0 - \tfrac{1}{2} \lim_{v \to \infty} (F_{iv}(t_0+) - F_{iv}(t_0-))\,.$$

Hence $g_i^*(t_0) = y_i^0$. Next, by (2.3) and (2.4) we conclude that

$$(3.8) \qquad \left| \int_{t_0}^{t} [f_i(g_1(s + h), \ldots, g_n(s + h)) - f_i(g_{1v}(s + h), \ldots, g_{nv}(s + h))]\, ds \right| \leqq$$

$$\leqq \mathcal{L}_i \Big(\sum_{j=1}^{n} \sup_{t \in R^1} |g_j - g_{jv}|^* (t) \Big)\,.$$

Thus the system of the functions $(g_1(t), \ldots, g_n(t))$ is a solution of the problem (2.7). It remains to prove the uniqueness of the solution. Let $\bar{g}_1, \ldots, \bar{g}_n$ be bounded functions of locally bounded variation in R^1 such that $g_i^*(t_0) = y_i^0$, $(g_1(t), \ldots, g_n(t)) \neq (\bar{g}_1(t), \ldots, \bar{g}_n(t))$ $(i = 1, \ldots, n)$. Moreover, let the system of the functions $(\bar{g}_1(t), \ldots, \bar{g}_n(t))$ satisfy the system (2.7). Then the inequality (2.4) yields

$$(3.9) \qquad K \leqq K \Big(\sum_{i,j=1} \int_{-\infty}^{\infty} L_{ij}(t)\, dt \Big)\,,$$