Časopis pro pěstování matematiky, roč. 102 (1977), Praha

EXCEPTIONAL VALUES OF LINEAR COMBINATIONS OF THE DERIVATIVES OF A MEROMORPHIC FUNCTION

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(Received April 16, 1975)

We denote by C the set of all finite complex numbers and by $\bar{C}$ the extended complex plane consisting of all (finite) complex numbers and ∞. By a meromorphic function we shall always mean a transcendental meromorphic function in the plane. We use the usual notations of the Nevanlinna theory of meromorphic functions as explained in [2] and [4].

If f is a meromorphic function we denote by $S(r, f)$ any quantity satisfying

$$\int_{r_0}^{r} \frac{S(x,f)}{x^{1+\lambda}} \, dx = O\left(\int_{r_0}^{r} \frac{\log T(x,f)}{x^{1+\lambda}}\right) \tag{1}$$

as $r \to \infty$, whenever $\lambda > 0$ and

$$S(r,f) = o(T(r,f)) \tag{2}$$

as $r \to \infty$, through all values if f is of finite order and outside a set of finite linear measure if f is of infinite order.

If f is a meromorphic function, then we have the following fundamental results of NEVANLINNA [3, page 63].

$$m(r, f'/f) = S(r,f)$$

and

$$(q-2)\, T(r,f) \leqq \sum_{i=1}^{q} N(r, a_i, f) - N_1(r) + S(r,f)$$

whenever $a_1, \ldots, a_q$ are distinct elements of $\bar{C}$, where

$$N_1(r) = 2\, N(r,f) - N(r,f') + N(r, 1/f') .$$

Generalisations and extensions of these results have been obtained by MILLOUX, HAYMAN and others and most of them are found in [2]. In [2], Hayman denotes

*) Research of the second author is supported by the Department of Atomic Energy, Bombay.

by $S(r, f)$ any quantity satisfying (2) above. However, since all the results are obtained from the fundamental results of Nevanlinna it is easy to see that the theorems in [2] are valid with $S(r, f)$ satisfying (1) and (2) also.

In particular, we have [2, Theorem 3.1], for a meromorphic function f,

$$m(r, f^{(k)}/f) \doteq S(r, f) \tag{3}$$

for each integer $k \geqq 1$.

If f is a meromorphic function of order ϱ, $0 \leqq \varrho \leqq \infty$ and $a \in \bar{C}$, we define

$$\varrho(a, f) = \limsup_{r \to \infty} \frac{\log^+ n(r, a, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ N(r, a, f)}{\log r},$$

$$\bar{\varrho}(a, f) = \limsup_{r \to \infty} \frac{\log^+ \bar{n}(r, a, f)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ \bar{N}(r, a, f)}{\log r}$$

and we call a

(i) an evB (exceptional value in the sense of Borel) for f if $\varrho(a, f) < \varrho$,

(ii) an evB for f for distinct zeros if $\bar{\varrho}(a, f) < \varrho$, and

(iii) an evP (exceptional value in the sense of Picard) for f if f assumes the value a only a finite number of times or, equivalently, if $n(r, a, f) = O(1)$.

If $\varrho > 0$ and a is an evP for f then a is clearly an evB for f whereas if $\varrho = 0$ then, trivially, f has no evB in $\bar{C}$.

In [1] Hayman proved the following theorem [2, Theorem 3.5, Corollary].

Theorem A. *If f is a meromorphic function and m is a positive integer, then either f has no evP in C or $f^{(m)}$ has no evP in C except possibly zero.*

In this paper we extend this theorem to certain linear combinations in the successive derivatives of f.

We first prove the following lemma.

Lemma 1. *Let f be a meromorphic function and $\psi_f = a_1 f^{(1)} + \ldots + a_{k-2} f^{(k-2)} + a_k f^{(k)}$ with $k \geqq 3$, where $a_1, \ldots, a_{k-2}, a_k \in C$ and $a_k \neq 0$. If ψ_f is not a constant, then*

$$2 N_1(r, f) \leqq \bar{N}(r, f) + \bar{N}(r, 1/(\psi_f - 1)) + \bar{N}_0(r, 1/\psi'_f) + S(r, f), \tag{4}$$

where $N_1(r, f)$ is obtained by considering only the simple poles of f and in $\bar{N}_0(r, 1/\psi'_f)$ only distinct zeros of ψ'_f which are not zeros of $\psi_f - 1$ are to be considered.

Proof. Let

$$g(z) = \frac{\{\psi'_f(z)\}^{k+1}}{\{1 - \psi_f(z)\}^{k+2}}.$$

Let a be a simple pole of f. Then in a neighbourhood of a we have

$$f(z) = \frac{b}{z - a} + h(z)$$

where $b \in C$, $b \neq 0$ and $h(z)$ is analytic.

Thus,

$$1 - \psi_f(z) = 1 + \frac{(-1)^{k+1}\, k!\, a_k b}{(z - a)^{k+1}} - \sum_{i=1}^{k-2} \frac{(-1)^i\, i!\, a_i b}{(z - a)^{i+1}} - \phi(z)$$

where

$$\phi(z) = \sum_{i=1}^{k-2} a_i\, h^{(i)}(z) + a_k\, h^{(k)}(z)\,.$$

Hence,

$$1 - \psi_f(z) = \frac{1}{(z - a)^{k+1}} \{(-1)^{k+1}\, k!\, a_k b + (z - a)^2\, u(z)\}\,,$$

where

$$u(z) = (z - a)^{k-1}\,(1 - \phi(z)) - \sum_{i=1}^{k-2} (-1)^i\, i!\, a_i b (z - a)^{k-2-i}$$

is analytic.

Also,

$$\psi'_f(z) = \frac{1}{(z - a)^{k+2}} \{(-1)^{k+1}\,(k + 1)!\, a_k b + (z - a)^2\, v(z)\}$$

where

$$v(z) = (z - a)^k\, \phi'(z) + \sum_{i=1}^{k-2} (-1)^{i+1}\,(i + 1)!\, a_i b (z - a)^{k-2-i}$$

is analytic.

Therefore, in a neighbourhood of a,

(5) $$g(z) = \frac{[(-1)^{k+1}\,(k + 1)!\, a_k b + (z - a)^2\, v(z)]^{k+1}}{[(-1)^{k+1}\, k!\, a_k b + (z - a)^2\, u(z)]^{k+2}}\,.$$

Hence

$$g(a) = \frac{(-1)^{k+1}\,(k + 1)^{k+1}}{k!\, a_k b} \neq 0\,, \quad \neq \infty\,.$$

Thus, a is neither a zero nor a pole of g.

On the other hand, it is easily verified from (5) that a is a zero of g'.

Hence $N_1(r, f) \leqq \bar{N}_0(r, 1/g')$, where, in $\bar{N}_0(r, 1/g')$ only distinct zeros of g' which are not zeros of g are to be considered.

Thus,

$$N_1(r, f) \leqq \bar{N}_0(r, 1/g') = \bar{N}(r, g/g') \leqq T(r, g/g') =$$
$$= T(r, g'/g) + O(1) = N(r, g'/g) + S(r, g)$$

Hence,

(6) $$N_1(r, f) \leqq \bar{N}(r, g) + \bar{N}(r, 1/g) + S(r, g)\,.$$

Clearly zeros and poles of g can occur only at multiple poles of f or zeros of $\psi_f - 1$ or zeros of ψ'_f other than the zeros of $\psi_f - 1$.

Thus,

$$(7) \qquad \bar{N}(r, g) + \bar{N}(r, 1/g) \leqq \bar{N}(r, f) - N_1(r, f) + \\ + \bar{N}(r, 1/(\psi_f - 1)) + \bar{N}_0(r, 1/\psi'_f).$$

From (6) and (7) we obtain (4), since it is easy to see that $S(r, g) = S(r, \psi)$ and $S(r, \psi) = S(r, f)$.

Theorem 1. *Let f be a meromorphic function and ψ_f be as in Lemma* 1. *If ψ_f is not a constant, then*

$$(8) \qquad T(r, f) < 3\, N(r, 1/f) + 4\, \bar{N}(r, 1/(\psi_f - 1)) + S(r, f).$$

Proof. By [2, Theorem 3.2] we have

$$(9) \qquad T(r, f) < \bar{N}(r, f) + N(r, 1/f) + \bar{N}(r, 1/(\psi_f - 1)) - \\ - N_0(r, 1/\psi'_f) + S(r, f),$$

where in $N_0(r, 1/\psi'_f)$ only zeros of ψ'_f which are not zeros of $\psi_f - 1$ are to be considered.

Now

$$2\, \bar{N}(r, f) \leqq N(r, f) + N_1(r, f) \leqq T(r, f) + N_1(r, f)$$

Hence, from (4) and (9),

$$\bar{N}(r, f) < 2\, N(r, 1/f) + 3\, \bar{N}(r, 1/(\psi_f - 1)) - 2\, N_0(r, 1/\psi'_f) + \\ + \bar{N}_0(r, 1/\psi'_f) + S(r, f).$$

Using this in (9) we obtain

$$T(r, f) < 3\, N(r, 1/f) + 4\, \bar{N}(r, 1/(\psi_f - 1)) - 3\, N_0(r, 1/\psi'_f) + \\ + \bar{N}_0(r, 1/\psi'_f) + S(r, f)$$

which yields (8) since $\bar{N}_0(r, 1/\psi'_f) \leqq N_0(r, 1/\psi'_f)$.

The following theorem is an extension of Theorem A of Hayman mentioned earlier.

Theorem 2. *Let f be a meromorphic function and $\psi_f = a_1 f^{(1)} + \ldots + a_{k-2} f^{(k-2)} + a_k f^{(k)}$ with $k \geqq 3$, where $a_1, \ldots, a_{k-2}$, $a_k \in C$ and $a_k \neq 0$. If ψ_f is not a constant then*

(i) *either f has no evP in C or ψ_f has no evP in C except possibly zero, and*

(ii) *either f has no evB in C or ψ_f has no evB for distinct zeros in C except possibly zero.*

Note. It is easy to see that the order of $\psi_f \leqq$ the order of f. When the order of ψ_f is positive, (ii) implies (i).

Proof. Let $w_1, w_2 \in C$ and $w_2 \neq 0$. Define F by

$$F(z) = \frac{f(z) - w_1}{w_2}.$$

Then $T(r, F) = T(r, f) + O(1)$ and $S(r, F) = S(r, f)$.

If ψ_F denotes $a_1 F^{(1)} + \ldots + a_{k-2} F^{(k-2)} + a_k F^{(k)}$, then $\psi_F = \psi_f / w_2$.