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LATTICES OF TOLERANCES

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In the paper [12] the concept of a tolerance is introduced. The tolerance relation (more briefly tolerance) is a reflexive and symmetric binary relation on a given set. Compatible tolerances are defined on algebras; they are a generalization of congruences. The concept of the compatible tolerance was introduced first for graphs in [13], later it was defined for arbitrary algebraic structures in [14] and [15]. The papers [7], [9], [10], [13], [14], [15] concern the investigation of the existence of compatible tolerances on various algebras. Although the conditions of the existence of compatible relations on algebras were investigated in some papers, there are still very few results on the set of all compatible tolerances on a given algebra. Only in the paper [7] it was proved that the set of all compatible tolerances on a given algebra forms a lattice with respect to the set inclusion. The aim of this paper is to find further properties which characterize this lattice.

1. LATTICE OPERATIONS IN $LT(\mathfrak{A})$

By the symbol $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ we denote an algebra $\mathfrak{A}$ with the support A and with the set $\mathscr{F}$ of fundamental operations. A tolerance on a non-empty set M is a reflexive and symmetric binary relation on M. A binary relation R on the set A is called compatible with $\mathfrak{A} = \langle A, \mathscr{F} \rangle$, if for any n-ary operation $f \in \mathscr{F}$, where n is a positive integer, and for arbitrary elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of A fulfilling $a_i \mathrel{R} b_i$ for $i = 1, \ldots, n$ we have $f(a_1, \ldots, a_n) \mathrel{R} f(b_1, \ldots, b_n)$. If R is moreover a tolerance on A, we say that it is a compatible tolerance on $\mathfrak{A}$. By the symbol $LT(\mathfrak{A})$ we denote the set of all compatible tolerances on $\mathfrak{A}$. Evidently $LT(\mathfrak{A}) \neq \emptyset$ for every $\mathfrak{A}$, because the identity relation I (such that $x \mathrel{I} y \Leftrightarrow x = y$) and the universal relation U (such that $x \mathrel{U} y$ for each x and each y) are compatible tolerances on $\mathfrak{A}$. Further, each congruence on $\mathfrak{A}$ is a compatible tolerance on $\mathfrak{A}$.

Theorem 1. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra. Then $LT(\mathfrak{A})$ is a complete lattice with the least element I and the greatest element U with respect to the set inclusion. The meet in $LT(\mathfrak{A})$ is equal to the set intersection.*

Proof follows immediately from Theorem 17 in [11].

It is evident (see [7]) that in general $LT(\mathfrak{A})$ is not a sublattice of the lattice of all tolerances on the support A. The join in the lattice $LT(\mathfrak{A})$ need not be equal to the set union, because the set union of some compatible tolerances on A need not be compatible with $\mathfrak{A}$. Nevertheless, $\bigcup_{\gamma\in\Gamma} T_\gamma \subseteq \bigvee_{\gamma\in\Gamma} T_\gamma$.

In the sequel we shall use the concept of a polynomial on an algebra, as was defined by GRÄTZER [5]. If a polynomial p on an algebra $\mathfrak{A}$ contains variables $x_1, \ldots, x_n$ and no other variables, then it is denoted by $p(x_1, \ldots, x_n)$. Let $a_1, \ldots, a_n$ be elements of A. If we substitute for each x_i the element a_i (for $i = 1, \ldots, n$) wherever x_i occurs in the polynomial $p(x_1, \ldots, x_n)$, then we obtain an element of A which will be denoted by $p(a_1, \ldots, a_n)$.

Theorem 2. *Let $\mathfrak{A} = \langle A, \mathscr{F}\rangle$ be an algebra and let $T_\gamma \in LT(\mathfrak{A})$ for each γ from a subscript set Γ. Let T be a binary relation on A defined so that $a\,T\,b$ if and only if there exist elements $\gamma_1, \ldots, \gamma_m$ of Γ and elements $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ of A, where m is a positive integer, and there exists a polynomial $p(x_1, \ldots, x_m)$ on $\mathfrak{A}$ such that $a_i\,T_{\gamma_i}\,b_i$ for $i = 1, \ldots, m$ and $p(a_1, \ldots, a_m) = a$, $p(b_1, \ldots, b_m) = b$. Then $T = \bigvee_{\gamma\in\Gamma} T_\gamma$.*

Proof. Evidently $T_\gamma \subseteq T$ for each $\gamma \in \Gamma$; it suffices to choose $m = 1$, $a_1 = a$, $b_1 = b$, $p(x_1) = x_1$. This implies the reflexivity of T. The symmetry of T is evident from its definition, therefore T is a tolerance on A. Now let $c_1, \ldots, c_n$, $d_1, \ldots, d_n$ be elements of A such that $c_i\,T\,d_i$ for $i = 1, \ldots, n$ and let $f \in \mathscr{F}$ be an n-ary operation. Then there exist elements $c_{i1}, \ldots, c_{im_i}$, $d_{i1}, \ldots, d_{im_i}$ of A and elements $\gamma_{i1}, \ldots, \gamma_{im_i}$ of Γ such that $c_{ij}\,T_{\gamma_{ij}}\,d_{ij}$ for each $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$ (where m_i is a positive integer dependent on i). Further, there exist polynomials $p_i(x_1, \ldots, x_{m_i})$ of $\mathfrak{A}$ such that $c_i = p_i(c_{i1}, \ldots, c_{im_i})$, $d_i = p_i(d_{i1}, \ldots, d_{im_i})$ for $i = 1, \ldots, n$. According to the definition of a polynomial, $f(p_1, \ldots, p_n)$ is again a polynomial on $\mathfrak{A}$ and

$$f(c_1, \ldots, c_n) = f(p_1(c_{11}, \ldots, c_{1m_1}), \ldots, p_n(c_{n1}, \ldots, c_{nm_n}))\,,$$
$$f(d_1, \ldots, d_n) = f(p_1(d_{11}, \ldots, d_{1m_1}), \ldots, p_n(d_{n1}, \ldots, d_{nm_n}))\,,$$

therefore $c_{ij}\,T_{\gamma_{ij}}\,d_{ij}$ implies $f(c_1, \ldots, c_n)\,T f(d_1, \ldots, d_n)$, which means that $T \in LT(\mathfrak{A})$. The tolerance T is compatible with $\mathfrak{A}$ and contains T_γ for all $\gamma \in \Gamma$, therefore $\bigvee_{\gamma\in\Gamma} T_\gamma \subseteq$ $\subseteq T$. Let $a \in A$, $b \in A$ and $a\,T\,b$. Then there exist elements $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ of A and elements $\gamma_1, \ldots, \gamma_n$ of Γ and a polynomial $p(x_1, \ldots, x_n)$ on $\mathfrak{A}$ so that $a_i\,T_{\gamma_i}\,b_i$ for $i = 1, \ldots, n$, $p(a_1, \ldots, a_n) = a$, $p(b_1, \ldots, b_n) = b$. Then $a_i\,(\bigcup_{\gamma\in\Gamma} T_\gamma)\,b_i$ and also $a_i\,(\bigvee_{\gamma\in\Gamma} T_\gamma)\,b_i$. But $\bigvee_{\gamma\in\Gamma} T_\gamma \in LT(\mathfrak{A})$, therefore for each n-ary operation $f \in \mathscr{F}$ we have $f(a_1, \ldots, a_n)\,(\bigvee_{\gamma\in\Gamma} T_\gamma)\,f(b_1, \ldots, b_n)$ and the compatibility of $\bigvee_{\gamma\in\Gamma} T_\gamma$ implies

$$a = p(a_1, \ldots, a_n)\,(\bigvee_{\gamma\in\Gamma} T_\gamma)\,p(b_1, \ldots, b_n) = b\,.$$

As a and b were chosen arbitrarily, we have $T \subseteq \bigvee_{\gamma\in\Gamma} T_\gamma$ and thus $T = \bigvee_{\gamma\in\Gamma} T_\gamma$.

Now we shall study some interrelations between the lattice $LT(\mathfrak{A})$ and congruences on the algebra $\mathfrak{A}$.

Definition 1. Let T_γ be a tolerance on a set M for each $\gamma \in \Gamma$, where Γ is a subscript set. The least (with respect to the set inclusion) equivalence E on M such that $T_\gamma \subseteq E$ for each $\gamma \in \Gamma$ will be called the transitive hull of the tolerances T_γ for $\gamma \in \Gamma$.

Thus for $|\Gamma| = 1$ we obtain that the transitive hull of a tolerance T on M is the least equivalence E on M such that $T \subseteq E$. If T is an equivalence, then $T = E$.

The following three propositions are evidently true.

Proposition 1. *If $R_1, \ldots, R_n$ are binary relations compatible with $\mathfrak{A}$ (where n is a positive integer), then also $R_1 R_2 \ldots R_n$ is a binary relation compatible with $\mathfrak{A}$.*

Proposition 2. *If $\{R_i\}_{i \in I}$ (where $I \neq \emptyset$) is a system of binary relations compatible with $\mathfrak{A}$, which is directed upwards with respect to the set inclusion, then also $\bigcup_{i \in I} R_i$ is a relation compatible with $\mathfrak{A}$.*

Proposition 3. *If $T_i \in LT(\mathfrak{A})$ for each i from a non-empty subcsript set I, then the transitive hull C of the tolerances T_i is the relation $\bigcup T_{i_1} T_{i_2} \ldots T_{i_n}$, where the union is taken over all positive integers n and all subscripts $i_1, i_2, \ldots, i_n$ from I. C is thus expressed as a union of a system of relations which is directed upwards.*

Now we shall present some theorems.

Theorem 3. *Let A be a set, let T_γ be a tolerance on A for each $\gamma \in \Gamma$ (where Γ is a subscript set). Let C_γ be the transitive hull of T_γ and let C be the transitive hull of all C_γ for $\gamma \in \Gamma$. Then C is the transitive hull of all T_γ for $\gamma \in \Gamma$.*

Proof follows from Propositions 1, 2, 3.

Theorem 4. *Let $T_\gamma \in LT(\mathfrak{A})$ for $\gamma \in \Gamma$ and let C be the transitive hull of the tolerances T_γ for $\gamma \in \Gamma$. Then C is a congruence on $\mathfrak{A}$.*

Proof. By Proposition 3 the relation C is the union of a system of products of elements of $\{T\}_{\gamma \in \Gamma}$ which is directed upwards. By Proposition 1 each of these products is compatible with $\mathfrak{A}$, by Proposition 2 the union of this system is compatible with $\mathfrak{A}$. Thus C is compatible with $\mathfrak{A}$. As C is an equivalence on the support of $\mathfrak{A}$, it is a congruence on $\mathfrak{A}$.

Corollary 1. *Let $T_\gamma \in LT(\mathfrak{A})$ for $\gamma \in \Gamma$ and let C be the transitive hull of T_γ for $\gamma \in \Gamma$. Then $C \in LT(\mathfrak{A})$ and $\bigvee_{\gamma \in \Gamma} T_\gamma \subseteq C$.*

Definition 2. Let L be a lattice. A mapping t of L into itself is called a closure operation on L, if for any $a \in L$, $b \in L$ the following three conditions are satisfied:

(i) $a \leqq t(a)$;

(ii) $t(t(a)) = t(a)$;

(iii) $t(a) \vee t(b) \leqq t(a \vee b)$.

Theorem 5. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let $LT(\mathfrak{A})$ be the lattice of all compatible tolerances on $\mathfrak{A}$. For each $T \in LT(\mathfrak{A})$ let $t(T)$ be the transitive hull of T. Then t is a closure operation on $LT(\mathfrak{A})$.*

Proof. The conditions (i) and (ii) follow from the definition of the transitive hull. Further, the same definition implies also the implication $T_1 \subseteq T_2 \Rightarrow t(T_1) \subseteq t(T_2)$ and we have $t(T') \subseteq t(T' \vee T'')$, $t(T'') \subseteq t(T' \vee T'')$, which means $t(T') \vee t(T'') \subseteq \subseteq t(T' \vee T'')$ for any $T' \in LT(A)$, $T'' \in LT(\mathfrak{A})$, which means (iii).

Now we shall add some remarks concerning graphs of tolerances. If T is a tolerance on a set M, then the undirected graph $G(T)$ whose vertex set is M and in which two vertices x, y are adjacent if and only if $x\ T\ y$ is called the graph of T.

Theorem 6. *Let T be a tolerance on a set M, let $G(T)$ be the graph of T. Let E be the transitive hull of T. Then each equivalence class of E is the vertex set of a connected component of $G(T)$ and vice versa.*

This assertion is evident.

Theorem 7. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra. For each tolerance $T \in LT(\mathfrak{A})$ which is not a congruence choose a partition $P(T)$ of A with these properties:*

(i) *$P(T)$ is a refinement of the partition of A into equivalence classes of the transitive hull $C(T)$ of T;*

(ii) *if K_1, K_2 are two distinct classes of $P(T)$ which are subsets of the same equivalence class of $C(T)$, then there exist elements $k_1 \in K_1$, $k_2 \in K_2$ such that $k_1\ T k_2$.*

Let $E(T)$ be the equivalence on A whose equivalence classes form the partition $P(T)$, let C be a congruence on $\mathfrak{A}$ which contains all $E(T)$ for all tolerances $T \in \in LT(\mathfrak{A})$ which are not congruences. Then each tolerance compatible with the factor-algebra $\mathfrak{A}/C$ is a congruence.

Proof. Let T_1 be a tolerance compatible with $\mathfrak{A}/C$. Let χ be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/C$. Let T_2 be a tolerance on A defined so that $a\ T_2\ b$, if and only if $\chi(a)\ T_1\ \chi(b)$. It is easy to prove that T_2 is a tolerance compatible with $\mathfrak{A}$. Suppose that T_1 is not a congruence. Then neither T_2 is a congruence. Thus the partition $P(T_2)$ was chosen in accordance with the assumptions of the theorem and there exists an equivalence $E(T_2)$ corresponding to it. We have $E(T_2) \subseteq C$. Let x, y be two elements of A for which $x\ C(T_2)\ y$, where $C(T_2)$ is the transitive hull of T, but not $x\ T_2\ y$; as T_2 is not a congruence, such a pair of elements must exist. If $x\ C\ y$, then $\chi(x) = \chi(y)$ and $\chi(x)\ T_1\ \chi(y)$, because T_1 is reflexive. But from the definition of T_2 we have $x\ T_2\ y$, which is a contradiction. If x and y are not in C, then neither x nor y are in $E(T_2)$. But, as they are in $C(T_2)$, there exist elements x', y' such that $x\ E(T_2)\ x'$, $y\ E(T_2)\ y'$, $x'\ T_2\ y'$. As $x\ E(T_2)\ x'$ and $E(T_2) \subseteq C$, we have $x\ C\ x'$ and analogously $y\ C\ y'$. This means $\chi(x) = \chi(x')$, $\chi(y) = \chi(y')$. But by the definition of T_2, from $x'\ T_2\ y'$ we have $\chi(x')\ T_1\ \chi(y')$, which means $\chi(x)\ T_1\ \chi(y)$ and this implies $x\ T_2\ y$, which is again a contradiction.

An admissible colouring of a graph G is a partition of the vertex set of G such that no two distinct vertices of the same class of this partition are joined by an edge. If it has the minimal possible number of classes (colours), it is called a minimal admissible colouring of G. If K_1, K_2 are two distinct classes of a minimal admissible colouring of a connected graph G, then there exists a vertex $k_1 \in K_1$ and a vertex $k_2 \in K_2$ such that k_1 and k_2 are joined by an edge (otherwise we could substitute K_1 and K_2 by their union and we should obtain in this way an admissible colouring with a smaller number of classes). Thus we have a corollary.

Corollary 2. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra. For each tolerance T compatible with $\mathfrak{A}$ choose a partition $P(T)$ such that the restriction of $P(T)$ onto any equivalence class of $C(T)$, where $C(T)$ is the transitive hull of T, is a minimal admissible colouring of the corresponding connected component of $G(T)$. Let $E(T)$ be the equivalence on A whose equivalence classes form the partition $P(T)$, let C be a congruence on $\mathfrak{A}$ which contains all $E(T)$ for all tolerances $T \in LT(\mathfrak{A})$ which are not congruences. Then each tolerance compatible with the factor-algebra $\mathfrak{A}/C$ is a congruence.*

We have still another corollary.

Corollary 3. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra. For each tolerance T compatible with $\mathfrak{A}$ which is not a congruence let $E(T)$ be the least equivalence containing $C(T) - T$, where $C(T)$ is the transitive hull of T. Then each tolerance compatible with the factor-algebra $\mathfrak{A}/C$ is a congruence.*

If T_1 and T_2 are two tolerances compatible with $\mathfrak{A}$ with the same transitive hull, then evidently also their join has the same transitive hull. For their meet this need not hold. The subset of $LT(\mathfrak{A})$ consisting of all tolerances with the given transitive hull is an upper subsemilattice of $LT(\mathfrak{A})$, but it may have more than one minimal element.

If T_0 is a tolerance on A and T is the least tolerance from $LT(\mathfrak{A})$ which contains T_0, we say that T is generated by T_0.

Theorem 8. *Let C be a congruence on an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$, let $S(C)$ be the set of all tolerances from $LT(\mathfrak{A})$ whose transitive hull is C. Let T be a minimal element from $S(C)$. Then T is generated by a tolerance T_0 on A such that its graph $G(T_0)$ is a forest and the vertex set of each connected component of $G(T_0)$ is an equivalence class of C.*

Proof. By Theorem 7 the vertex set of each connected component of $G(T)$ is an equivalence class of C. In each of these connected components we choose a spanning tree; these spanning trees of all connected components of $G(T)$ form a forest F. Define a tolerance T_0 on A so that $x\ T_0\ y$ if and only if $x = y$ or x and y are joined by an edge in F. Then $F = G(T_0)$ and $T_0 \subseteq T$. Suppose that there exists $T_1 \in LT(\mathfrak{A})$ such that $T_0 \subseteq T_1 \subset T$ (therefore $T_1 \neq T$). Let $C(T)$, $C(T_0)$, $C(T_1)$ denote the transitive hulls of T, T_0, T_1 respectively. As $T_0 \subseteq T$, any connected component of T_0 must be contained in a connected component of T_1 and thus $C = C(T_0) \subseteq C(T_1)$. As $T_1 \subset T$, we obtain analogously $C(T_1) \subseteq C(T) = C$, thus $C \subseteq C(T_1) \subseteq C$ and $C(T_1) = C$. But this is a contradiction with the minimality of T.

Theorem 9. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let C be a congruence of $\mathfrak{A}$. Let $F(C)$ be the filter of $LT(\mathfrak{A})$ consisting of all elements of $LT(\mathfrak{A})$ which are greater than or equal to C. Let $\mathfrak{A}/C$ be the factor-algebra of $\mathfrak{A}$ by C. Then there exists a join-homomorphism of $F(C)$ onto $LT(\mathfrak{A}/C)$.*

Proof. Let $T \in F(C)$, this means $C \subseteq T$. Let χ be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/C$. Define $\varphi(T)$ as a tolerance on $\mathfrak{A}/C$ such that $x\ \varphi(T)\ y$ if and only if there exist elements x', y' of A such that $\chi(x') = x$, $\chi(y') = y$, $x'\ T\ y'$. The tolerance $\varphi(T)$ is evidently compatible with $\mathfrak{A}/C$. Now let T_0 be a tolerance compatible with $\mathfrak{A}/C$. Let T_1 be a tolerance on A such that $a\ T_1\ b$ if and only if $\chi(a)\ T_0\ \chi(b)$. Again T_1 is evidently compatible with $\mathfrak{A}$. If $c\ C\ d$ for some elements c, d of A, then $\chi(c) = \chi(d)$, thus $\chi(c)\ T_1\ \chi(d)$ and this implies $c\ T_1\ d$; we have $C \subseteq T_1$ and thus $T_1 \in F(C)$. Evidently $\varphi(T_1) = T_0$. We see that φ is a surjection. Now let $T \in F(C)$, $T' \in F(C)$. Let $a\ (T \vee T')\ b$. Then there exist elements $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ of A and a polynomial $p(x_1, \ldots, x_n)$ of A such that $p(a_1, \ldots, a_n) = a$, $p(b_1, \ldots, b_n) = b$ and for each $i = 1, \ldots, n$ either $a_i\ T\ b_i$, or $a_i\ T'\ b_i$. In the polynomial $p(x_1, \ldots, x_n)$ substitute each element $c \in A$ by the element $\chi(c)$ and each operation $f \in \mathscr{F}$ by the operation on $\mathfrak{A}/C$ corresponding to f in the homomorphism χ; we obtain a polynomial $p^*(x_1, \ldots, x_n)$ of $\mathfrak{A}/C$. We have $p^*(\chi(a_1), \ldots, \chi(a_n)) = \chi(a)$, $p^*(\chi(b_1), \ldots, \chi(b_n)) = \chi(b)$ and for $i = 1, \ldots, n$ we have either $\chi(a_i)\ \varphi(T)\ \chi(b_i)$ or $\chi(a_i)\ \varphi(T')\ \chi(b_i)$. Thus $\chi(a)\ (\varphi(T) \vee \varphi(T'))\ \chi(b)$. We have proved $\varphi(T \vee T') \subseteq \varphi(T) \vee \varphi(T')$. It remains to prove that $\varphi(T) \subseteq \varphi(T \vee T')$, $\varphi(T') \subseteq \varphi(T \vee T')$. If $c\ \varphi(T)\ d$ for some elements c, d of $\mathfrak{A}/C$, then there exist elements c', d' of A such that $\chi(c') = c$, $\chi(d') = d$, $c'\ T\ d'$. Then $c'\ (T \vee T')\ d'$ and $c = \chi(c')\ \varphi(T \vee T')\ \chi(d') = d$, thus $\varphi(T) \subseteq \varphi(T \vee T')$. Analogously $\varphi(T') \subseteq \varphi(T \vee T')$. The relation $\varphi(T \vee T')$ contains both $\varphi(T)$ and $\varphi(T')$ and is contained in $\varphi(T) \vee \varphi(T')$, therefore $\varphi(T \vee T') = \varphi(T) \vee \varphi(T')$ and φ is a join-homomorphism.

We shall give an example showing that this theorem cannot be strengthened by substituting the word "join-homomorphism" by "homomorphism".

Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be a groupoid with the support $A = \{a_1, a_2, a_3, b\}$ and with a binary operation given by the following Cayley table:

	a_1	a_2	a_3	b
a_1	a_1	a_3	a_2	a_1
a_2	a_3	a_2	a_1	a_2
a_3	a_2	a_1	a_3	a_3
b	a_1	a_2	a_3	b

Let C be an equivalence on A whose classes are $\{a_1, a_2, a_3\}$, $\{b\}$. It is evidently a congruence on $\mathfrak{A}$. The set $F(C)$ consists of the tolerance C, T_1, T_2, T_3, U, where T_i for $i = 1, 2, 3$ is obtained from C by adding the pairs $(a_i, b), (b, a_i)$, U is the universal relation on A. Evidently $T_1 \wedge T_2 = T_1 \wedge T_3 = T_2 \wedge T_3 = C$, $T_1 \vee T_2 = T_1 \vee T_3 = T_2 \vee T_3 = U$. The factor-algebra $\mathfrak{A}/C$ consists only of two elements, therefore there are only two tolerances on it, the identity relation I_0 and the universal relation U_0. Suppose that there exists a homomorphism φ of $F(C)$ onto $LT(\mathfrak{A}/C)$. If $\varphi(C) = U_0$, then $\varphi(T) = \varphi(T \vee C) = \varphi(T) \vee \varphi(C) = \varphi(T) \vee U_0 = U_0$ for each $T \in F(C)$ and φ is not a surjection, which is a contradiction. Thus $\varphi(C) = I_0$. Dually, if $\varphi(U) = I_0$, then $\varphi(T) = I_0$ for each $T \in F(C)$ and this is also a contradiction. Thus $\varphi(C) = I_0$, $\varphi(U) = U_0$. From the elements $\varphi(T_1), \varphi(T_2), \varphi(T_3)$ at least two must be equal, without loss of generality let $\varphi(T_1) = \varphi(T_2)$. Then $U_0 = \varphi(U) = \varphi(T_1 \vee T_2) = \varphi(T_1) \vee \varphi(T_2) = \varphi(T_1) = \varphi(T_1) \wedge \varphi(T_2) = \varphi(T_1 \wedge T_2) = \varphi(C) = I_0$, which is a contradiction.

Theorem 10. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let C be a congruence of $\mathfrak{A}$. Let $F(C)$ be the filter of $LT(\mathfrak{A})$ consisting of all elements of $LT(\mathfrak{A})$ which are greater than or equal to C. Let $\mathfrak{A}/C$ be the factor-algebra of $\mathfrak{A}$ by C. Then there exists a meet-isomorphism of $LT(\mathfrak{A}/C)$ into $F(C)$.*

Proof. Let χ be the natural homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/C$. Let $T \in LT(\mathfrak{A}/C)$. By $\psi(T)$ we denote the binary relation on A defined so that $x\, \psi(T)\, y$ for $x \in A$, $y \in A$ if and only if $\chi(x)\, T\, \chi(y)$. We have $\psi(T) \in F(C)$; the proof is left to the reader. Now let $F_0(C)$ be the set of all $\psi(T)$ for $T \in LT(\mathfrak{A}/C)$. Evidently each tolerance $T_0 \in F(C)$ has the property that for any two congruence classes K_1, K_2 of C the relation $k_1\, T_0\, k_2$ holds either for each $k_1 \in K_1$ and each $k_2 \in K_2$, or for no $k_1 \in K_1$ and no $k_2 \in K_2$; this property will be denoted by **P**. It is easy to prove that $T_1 = \psi(\varphi(T_1))$, where φ is the homomorphism defined in the proof of Theorem 9; thus $T_1 \in F_0(C)$. Thus the subset of $F(C)$ consisting of all tolerances with the property **P** coincides with $F_0(C)$. The set $F_0(C)$ is closed under meet; if $T_1 \in F_0(C)$, $T_2 \in F_0(C)$ and K_1, K_2 are two congruence classes of C, then either $k_1\, T_1\, k_2$, $k_1\, T_2\, k_2$ for each $k_1 \in K_1$ and each $k_2 \in K_2$, then $k_1\, (T_1 \wedge T_2)\, k_2$ for each $k_1 \in K_1$ and each $k_2 \in K_2$, or $k_1\, T_i\, k_2$ for some i from the numbers 1, 2 holds for no $k_1 \in K_1$, $k_2 \in K_2$, then $k_1\, (T_1 \wedge T_2)\, k_2$ holds for no $k_1 \in K_2$ and no $k_2 \in K_2$. We see that $F_0(C)$ is a lower subsemilattice of $F(C)$. If $T \in LT(\mathfrak{A}/C)$, then evidently $\varphi(\psi(T)) = T$. Since we have $\psi(\varphi(T_0)) = T_0$ for each $T_0 \in F_0(C)$, we see that ψ is a bijective mapping and so is the restriction of φ onto $F_0(C)$; this restriction is equal to ψ^{-1}. As φ is an order-preserving mapping, so is ψ. As in $F_0(C)$ to any two elements their meet exists and is equal to their meet in $LT(\mathfrak{A})$, ψ must be a meet-isomorphism of $LT(\mathfrak{A}/C)$ onto $F_0(C)$, this means into $F(C)$.

We shall give an example showing that ψ need not be an isomorphism.

Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be the semigroup with the support $A = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$, given by the following Cayley table:

	a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2
a_1	a_1	a_2	b_2	b_2	c_2	c_2	d_2	d_2
a_2	a_2	a_2	b_2	b_2	c_2	c_2	d_2	d_2
b_1	b_2	b_2	b_1	b_2	d_2	d_2	d_2	d_2
b_2	b_2	b_2	b_2	b_2	d_2	d_2	d_2	d_2
c_1	c_2	c_2	d_2	d_2	c_1	c_2	d_2	d_2
c_2	c_2	c_2	d_2	d_2	c_2	c_2	d_2	d_2
d_1	d_2	d_2	d_2	d_2	d_2	d_2	d_1	d_2
d_2	d_2	d_2	d_2	d_2	d_2	d_2	d_2	d_2

Let C be the congruence on $\mathfrak{A}$ whose classes are $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{c_1, c_2\}$, $\{d_1, d_2\}$. Let T_1 be the congruence on $\mathfrak{A}$ whose classes are $\{a_1, a_2, b_1, b_2\}$, $\{c_1, c_2, d_1, d_2\}$, let T_2 be the congruence on $\mathfrak{A}$ whose classes are $\{a_1, a_2, c_1, c_2\}$, $\{b_1, b_2, d_1, d_2\}$. We have $T_1 \in F_0(C)$, $T_2 \in F_0(C)$. Let $T = T_1 \vee T_2$. Then $a_2\, T d_2$, because $a_2\, T_1\, b_2$, $a_2\, T_2\, c_2$, thus $a_2 = a_2 a_2\, T b_2\, c_2 = d_2$, but the elements a_1, d_1 are not in T. As $a_1\, C\, a_2$, $d_1\, C\, d_2$, this means that $T = T_1 \vee T_2 \notin F_0(C)$ and $F_0(C)$ is not closed under join, thus ψ is not an isomorphism of $LT(\mathfrak{A}/C)$ onto $F_0(C)$.

Theorem 11. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let C be a congruence on $\mathfrak{A}$. Let $J(C)$ be the ideal of $LT(\mathfrak{A})$ consisting of all elements of $LT(\mathfrak{A})$ which are less or equal to C. Let there exist a subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ whose support A_0 is one of the congruence classes of C. Then there exists a join-homomorphism of $LT(\mathfrak{A}_0)$ into $J(C)$.*

Proof. Let $T \in LT(\mathfrak{A}_0)$, let $\alpha(T)$ be the least compatible tolerance on $\mathfrak{A}$ which contains T. Then $\alpha(T) \subseteq C$; as C obviously contains T, this means $\alpha(T) \in J(C)$. Now if $T' \in J(C)$ and T'' is the restriction of T' onto A_0, then evidently $T'' \in LT(\mathfrak{A}_0)$ and $\alpha(T'') \subseteq T'$. Thus α is a mapping of $LT(\mathfrak{A}_0)$ into $J(C)$. Now let $T_1 \in LT(\mathfrak{A}_0)$, $T_2 \in LT(\mathfrak{A}_0)$. Then $\alpha(T_1) \vee \alpha(T_2)$ is a tolerance from $J(C)$ which contains both T_1 and T_2. As it contains T_1 and T_2 and is compatible with $\mathfrak{A}$, it must contain also $T_1 \vee T_2$, which means $\alpha(T_1 \vee T_2) \subseteq \alpha(T_1) \vee \alpha(T_2)$. But evidently $\alpha(T_1) \subseteq \alpha(T_1 \vee T_2)$, $\alpha(T_2) \subseteq \alpha(T_1 \vee T_2)$, thus $\alpha(T_1) \vee \alpha(T_2) \subseteq \alpha(T_1 \vee T_2)$ and $\alpha(T_1 \vee T_2) = \alpha(T_1) \vee \alpha(T_2)$. We see that α is a join-homomorphism of $LT(\mathfrak{A}_0)$ into $J(C)$.

Theorem 12. *Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, let C be a congruence on $\mathfrak{A}$. Let $J(C)$ be the ideal of $LT(\mathfrak{A})$ consisting of all elements of $LT(\mathfrak{A})$ which are less than or equal to C. Let there exist a subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ whose support A_0 is one of the congruence classes of C. Then there exists a meet-homomorphism of $J(C)$ into $LT(\mathfrak{A}_0)$.*

Proof. Let $T \in J(C)$, let $\beta(T)$ be the restriction of T onto A_0. Then evidently $\beta(T) \in LT(\mathfrak{A}_0)$ and $\alpha(\beta(T)) \subseteq T$, where α is the join-homomorphism from the proof of Theorem 12. Let L_0 be the set of all elements of $LT(\mathfrak{A}_0)$ which are images of elements of $J(C)$ in the mapping β. Evidently an element $T \in LT(\mathfrak{A}_0)$ is in L_0 if and only if the restriction of $\alpha(T)$ onto A_0 is equal to T. Let $T_1 \in L_0$, $T_2 \in L_0$ and consider $T_1 \wedge T_2$.

For each $T \in J(C)$ the relation $\beta(T)$ is the intersection of T with the universal relation U_0 on A_0. Therefore $\beta(T_1 \wedge T_2) = T_1 \cap T_2 \cap U_0 = (T_1 \cap U_0) \cap (T_2 \cap \cap U_0) = \beta(T_1) \wedge \beta(T_2)$. Thus the mapping β is a meet-homomorphism of $J(C)$ onto L_0, this means into $LT(\mathfrak{A}_0)$.

3. COMPACTNESS ON THE LATTICE $LT(\mathfrak{A})$

Definition 3. An element c of a complete lattice L is called compact, if for each subset M of L such that $c \leqq \bigvee_{x \in M} x$ there exists a finite subset N of M such that $c \leqq \leqq \bigvee_{x \in N} x$. A lattice L is called compactly generated, if each element of the lattice L is a join of compact elements.

This definition is from [11], p. 65, § 25.

Definition 4. Let $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ be an algebra, $a \in A$, $b \in A$. By the symbol T_{ab} we denote the least (with respect to the lattice ordering) tolerance from $LT(\mathfrak{A})$ for which $a\, T_{ab}\, b$ holds.

The correctness of this definition follows from Theorem 5.

Theorem 13. *The tolerance T_{ab} is a compact element of the lattice $LT(\mathfrak{A})$ for each $a \in A$, $b \in A$, where A is the support of the algebra $\mathfrak{A}$.*

Proof. Let $T_\gamma \in LT(\mathfrak{A})$ for $\gamma \in \Gamma$, where Γ is a subscript set and $T_{ab} \subseteq \bigvee_{\gamma \in \Gamma} T_\gamma$. Then $a\, T_{ab}\, b$ implies $a \left(\bigvee_{\gamma \in \Gamma} T_\gamma \right) b$. By Theorem 2 there exist elements $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ of A, elements $\gamma_1, \ldots, \gamma_n$ of Γ and a polynomial $p(x_1, \ldots, x_n)$ on $\mathfrak{A}$ such that $p(a_1, \ldots, a_n) = a$, $p(b_1, \ldots, b_n) = b$ and $a_i\, T_{\gamma_i}\, b_i$ for $i = 1, \ldots, n$. Therefore also $a \left(\bigvee_{i=1}^{n} T_{\gamma_i} \right) b$. As T_{ab} is the least tolerance in $LT(\mathfrak{A})$ fulfilling $a\, T_{ab}\, b$, we have $T_{ab} \subseteq \bigvee_{i=1}^{n} T_{\gamma_i}$.

Instead of $a\, T\, b$ we shall sometimes write $(a, b) \in T$.

Theorem 14. *Let $T \in LT(\mathfrak{A})$ for an algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$. Then $T = \bigvee_{(a,b) \in T} T_{ab}$.*

Proof. Let $T \in LT(\mathfrak{A})$, let $c \in A$, $d \in A$, $c\, T d$. By Theorem 14 we have $T_{cd} \in$

$\in LT(\mathfrak{A})$ and thus $c\, T_{cd}\, d$, which implies $c\,(\bigvee_{(a,b)\in T} T_{ab})\, d$; this means $T \subseteq \bigvee_{(a,b)\in T} T_{ab}$. Conversely, let $c\,(\bigvee_{(a,b)\in T} T_{ab})\, d$; then by Theorem 2 there exist elements $c_1, \ldots, c_n$, $d_1, \ldots, d_n$ of A and tolerances $T_i \in \{T_{ab}; (a, b) \in T\}$ for $i = 1, \ldots, n$ so that $c_i\, T_i\, d_i$ and there exists a polynomial $p(x_1, \ldots, x_n)$ of the algebra $\mathfrak{A}$ so that $p(c_1, \ldots, c_n) = c$, $p(d_1, \ldots, d_n) = d$. As $T_i \in \{T_{ab}; (a, b) \in T\}$, there exist elements $a(i)$, $b(i)$ of A such that $a(i)\, T\, b(i)$ and $T_i = T_{a(i)b(i)}$. Evidently $T_{ab} \subseteq T$ for each $a \in A$, $b \in A$, $a\, T\, b$, therefore $c_i\, T_i\, d_i$ implies $c_i\, T\, d_i$. The compatibility of T and Definition 1 imply $c = p(c_1, \ldots, c_n)\, T\, p(d_1, \ldots, d_n) = d$, therefore $\bigvee_{(a,b)\in T} T_{ab} \subseteq T$, which completes the proof.

Corollary 4. *The lattice $LT(\mathfrak{A})$ for every algebra $\mathfrak{A} = \langle A, \mathscr{F}\rangle$ is compactly generated.*

For an arbitrary $T \in LT(\mathfrak{A})$, Theorem 54 from [11] implies that the equality $T = \bigvee_{(a,b)\in T} T_{ab}$ holds. By Theorem 14 each tolerance T_{ab} is a compact element of $LT(\mathfrak{A})$, which implies the assertion.

Remark. This corollary is a generalization of an analogous theorem for lattices of congruences (see [3], Theorem 6).

4. ATOMICITY OF THE LATTICE $LT(\mathfrak{A})$

In [1] the following concept is introduced:

Definition 5. A lattice L is relatively atomic, if each interval of this lattice is an atomic sublattice of L.

In [1] it is proved that the lattice $K(\mathfrak{A})$ of all congruences on an algebra $\mathfrak{A} = \langle A, \mathscr{F}\rangle$ is relatively atomic (Theorem 67). The proof of this property is based on the fact that each congruence on $\mathfrak{A}$ determines a partition of A, which does not hold for tolerances in general. But for an idempotent algebra $\mathfrak{A}$, an analogous assertion holds on classes on A which need not be pairwise disjoint (thus they need not form a partition).

The following theorem shows that this property is sufficient for the validity of an analogous theorem for $LT(\mathfrak{A})$, if also the lattice of subalgebras of $\mathfrak{A}$ is relatively atomic.

Let $\mathfrak{A} = \langle A, \mathscr{F}\rangle$ be an algebra. By $\mathscr{P}(\mathfrak{A})$ we denote the set of all subalgebras of the algebra $\mathfrak{A}$. If $\bigcap_{\mathfrak{A}'\in\mathscr{P}(\mathfrak{A})} \mathfrak{A}' \neq \emptyset$, denote $\mathscr{S}(\mathfrak{A}) = \mathscr{P}(\mathfrak{A})$, in the opposite case denote $\mathscr{S}(\mathfrak{A}) = \mathscr{P}(\mathfrak{A}) \cup \{\emptyset\}$. Then we define the ordering on $\mathscr{S}(\mathfrak{A})$ in the following way: If $\emptyset \in \mathscr{S}(\mathfrak{A})$, then $\emptyset \leqq \mathfrak{A}'$ for each $\mathfrak{A}' \in \mathscr{S}(\mathfrak{A})$. For each $\mathfrak{A}' \in \mathscr{S}(\mathfrak{A})$ and $\mathfrak{A}'' \in \mathscr{S}(\mathfrak{A})$ we have $\mathfrak{A}' \leqq \mathfrak{A}''$ if and only if $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}''$. In [2] it is proved that $\mathscr{S}(\mathfrak{A})$ is a complete lattice with the least and the greatest element with respect to the ordering $\leqq$.

An algebra $\mathfrak{A} = \langle A, \mathscr{F} \rangle$ will be called idempotent, if for each element $a \in A$ and each n-ary operation $f \in \mathscr{F}$ we have $f(a, \ldots, a) = a$.

Theorem 15. *Let $\mathfrak{A}$ be an idempotent algebra such that $\mathscr{S}(\mathfrak{A})$ is a relatively atomic lattice. Then $LT(\mathfrak{A})$ is a relatively atomic (thus also atomic) lattice.*

Proof. Let $T_1 \in LT(\mathfrak{A})$, $T_2 \in LT(\mathfrak{A})$, $T_1 \subset T_2$, $T_1 \neq T_2$. For the sake of simplicity of this proof we shall not distinguish an algebra from its support. According to Theorems 3 and 4 in [10] there exists a system $\{\mathfrak{A}_\gamma, \gamma \in \Gamma\}$ of subalgebras of the algebra $\mathfrak{A}$ such that $\bigcup_{\gamma \in \Gamma} \mathfrak{A}_\gamma = \mathfrak{A}$ and $x\, T_1\, y$ if and only if there exists $\gamma \in \Gamma$ such that $x \in \mathfrak{A}_\gamma$, $y \in \mathfrak{A}_\gamma$. This system has the property that if f is an n-ary operation on $\mathfrak{A}$ and $\gamma_1, \ldots, \gamma_n$ are elements of Γ, then there exists $\gamma_0 \in \Gamma$ such that $x_i \in \mathfrak{A}_{\gamma_i}$ for $i = 1, \ldots, n$ implies $f(x_1, \ldots, x_n) \in \mathfrak{A}_{\gamma_0}$. Analogously there exists a system $\{\mathfrak{A}_\mu, \mu \in M\}$ of subalgebras of $\mathfrak{A}$ such that $\bigcup_{\mu \in M} \mathfrak{A}_\mu = \mathfrak{A}$ and $x\, T_2\, y$ if and only if there exists $\mu \in M$ such that $x \in \mathfrak{A}_\mu$, $y \in \mathfrak{A}_\mu$. This system has a quite analogous property to the above mentioned property of $\{\mathfrak{A}_\gamma, \gamma \in \Gamma\}$. As $T_1 \subset T_2$, to each $\gamma \in \Gamma$ there exists $\mu \in M$ such that $\mathfrak{A}_\gamma$ is a subalgebra of $\mathfrak{A}_\mu$. As $T_1 \neq T_2$, there exists at least one algebra $\mathfrak{A}_{\gamma_1}$ for $\gamma_1 \in \Gamma$ and at least one algebra $\mathfrak{A}_{\mu_1}$ for $\mu_1 \in M$ such that $\mathfrak{A}_{\gamma_1}$ is a proper subalgebra of $\mathfrak{A}_{\mu_1}$. As $\mathscr{S}(\mathfrak{A})$ is relatively atomic, there exists a subalgebra $\mathfrak{B}$ of the algebra $\mathfrak{A}$ which covers A_{γ_1} in the interval $\langle \mathfrak{A}_{\gamma_1}, \mathfrak{A}_{\mu_1} \rangle$ of the lattice $\mathscr{S}(\mathfrak{A})$. Now let us choose a system of algebras $\mathscr{X} = \{\mathfrak{A}_\gamma, \gamma \in \Gamma - \{\gamma_0\}\} \vee \{\mathfrak{B}\}$. This system is a covering of the algebra $\mathfrak{A}$ by subalgebras, therefore it induces a tolerance T_0 (this means $x\, T_0\, y$ if and only if either $x \in \mathfrak{B}$, $y \in \mathfrak{B}$, or there exists $\gamma \in \Gamma - \{\gamma_0\}$ such that $x \in \mathfrak{A}_\gamma$, $y \in \mathfrak{A}_\gamma$). Evidently T_0 covers T_1 in the interval $\langle T_1, T_2 \rangle$ in the lattice of all tolerances on $\mathfrak{A}$. If T_0 is a tolerance compatible with $\mathfrak{A}$, then $T_0 \in LT(\mathfrak{A})$ and T_0 covers T_1 also in the interval $\langle T_1, T_2 \rangle$ of the lattice $LT(\mathfrak{A})$. If T_0 is not compatible with $\mathfrak{A}$, then let $\mathscr{C}$ be the set of all tolerances T from $LT(\mathfrak{A})$ for which $T_0 \subseteq T \subseteq T_2$ holds. Evidently $\mathscr{C} \neq \emptyset$, because $T_2 \in \mathscr{C}$. As $LT(\mathfrak{A})$ is a complete lattice, $T' = \bigwedge_{T \in \mathscr{C}} T$ is again a tolerance compatible with $\mathfrak{A}$ and $T_0 \subseteq T'$, therefore $T_1 \neq T_2$ implies $T_1 \neq T'$ and there does not exist any compatible tolerance between T_1 and T'. Therefore T' covers T_1 in the lattice $LT(\mathfrak{A})$. We have proved that $LT(\mathfrak{A})$ is relatively atomic.

5. DISTRIBUTIVITY OF THE LATTICE $LT(\mathfrak{A})$

In the paper [4] it is proved that the lattice $\mathscr{K}(\mathfrak{L})$ of all congruences on a lattice $\mathfrak{L}$ is distributive and even that $\mathscr{K}(\mathfrak{L})$ is infinitely meet-distributive. For $LT(\mathfrak{L})$ an analogous assertion can be established in the case when $\mathfrak{L}$ is distributive. In this item we shall not distinguish an algebra from its support, as is usual in the lattice theory.

Lemma 1. *Let $\mathfrak{L}$ be a lattice, let $p(x_1, \ldots, x_n)$ be a lattice polynomial. Then for each $a \in \mathfrak{L}$ the equality $p(a, \ldots, a) = a$ holds.*

This follows from Lemma 6 in [6], page 33.

Lemma 2. *Let $\mathfrak{L}$ be a distributive lattice, let $p(x_1, \ldots, x_n)$ be a lattice polynomial. Then for arbitrary elements $a, a_1, \ldots, a_n$ of $\mathfrak{L}$ the following equalities hold:*

$$p(a_1, \ldots, a_n) \wedge a = p(a_1 \wedge a, \ldots, a_n \wedge a),$$
$$p(a_1, \ldots, a_n) \vee a = p(a_1 \vee a, \ldots, a_n \vee a).$$

This follows from Lemma 8 in [6], page 44.

Theorem 16. *Let $\mathfrak{L}$ be a distributive lattice. Then $LT(\mathfrak{L})$ is distributive and infinitely meet-distributive.*

Proof. By Theorem 28 in [11] it is sufficient to prove that $LT(\mathfrak{L})$ is infinitely meet-distributive. In every lattice the so-called distributive inequalities (see [1]) hold, by Theorem 1 the lattice $LT(\mathfrak{L})$ is complete, therefore the inclusion

$$T \wedge \bigvee_{\gamma \in \Gamma} T_\gamma \supseteq \bigvee_{\gamma \in \Gamma} (T \wedge T_\gamma)$$

holds for arbitrary T_γ and T from $LT(\mathfrak{L})$. We shall prove the converse inclusion. Let $a \in \mathfrak{L}$, $b \in \mathfrak{L}$ and $a\,(T \wedge \bigvee_{\gamma \in \Gamma} T_\gamma)\,b$. Then by Theorem 1 we have $a\,T\,b$, $a\,(\bigvee_{\gamma \in \Gamma} T_\gamma)\,b$ and by Theorem 2 there exist elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of $\mathfrak{L}$, elements $\gamma_1, \ldots, \gamma_n$ of Γ and a lattice polynomial $p(x_1, \ldots, x_n)$ such that $p(a_1, \ldots, a_n) = a$, $p(b_1, \ldots, b_n) = b$ and $a_i\,T_{\gamma_i}\,b_i$ for $i = 1, \ldots, n$. Consider the elements y_i, z_i such that

(1) $$y_i = ((a \wedge b) \vee a_i) \wedge (a \vee b),$$
$$z_i = ((a \wedge b) \vee b_i) \wedge (a \vee b).$$

Evidently

(2) $$a \wedge b \leqq y_i \leqq a \vee b, \quad a \wedge b \leqq z_i \leqq a \vee b$$

for $i = 1, \ldots, n$. Further $T \in LT(\mathfrak{L})$, thus

(3) $$a\,T\,b \Rightarrow a \wedge b\,T\,a \vee b.$$

This follows for example from Theorem 1 in [9]. From the quoted theorem and from (3) we obtain

(4) $$y_i\,T\,z_i$$

for $i = 1, \ldots, n$. By Lemma 1 we have $p(y_1, \ldots, y_n) = ((p(a, \ldots, a) \wedge p(b, \ldots, b)) \vee p(a_1, \ldots, a_n)) \wedge (p(a, \ldots, a) \vee p(b, \ldots, b))$ which is equal (by Lemma 1) to

(5) $$p(y_1, \ldots, y_n) = ((a \wedge b) \vee a) \wedge (a \vee b) = a.$$

Analogously we can prove

(6) $$p(z_1, \ldots, z_n) = b.$$

The compatibility of the relations T_γ and (1) implies (by Lemma 2)

(7) $$y_i \, T_{\gamma_i} \, z_i$$

for $i = 1, \ldots, n$, because $a_i \, T_{\gamma_i} \, b_i$. From (4) and (7) it follows that $y_i \, (\bigvee_{\gamma \in \Gamma} (T \wedge T_\gamma)) \, z_i$ and the property $\bigvee_{\gamma \in \Gamma} (T \wedge T_\gamma) \in LT(\mathfrak{L})$ implies also

$$a = p(y_1, \ldots, y_n) \, (\bigvee_{\gamma \in \Gamma} (T \wedge T_\gamma)) \, p(z_1, \ldots, z_n) = b \, .$$

Therefore we have

$$T \wedge \bigvee_{\gamma \in \Gamma} T_\gamma \subseteq \bigvee_{\gamma \in \Gamma} (T \wedge T_\gamma) \, ,$$

which completes the proof.

6. TOLERANCES ON A SET

Let A be a non-empty set. In the paper [11] it is proved that the set of all equivalences on A is a complete relatively complemented semimodular atomic lattice. Let $\mathcal{T}(A)$ denote the set of all tolerances on the set A and let us investigate the structural properties of $\mathcal{T}(A)$. The following theorem shows that the structure of the lattice $\mathcal{T}(A)$ is much simpler than that of the lattice of all equivalences on A. This is a difference in comparison with the case of the lattices $LT(\mathfrak{A})$ and $\mathcal{K}(\mathfrak{A})$ of an algebra $\mathfrak{A}$.

Theorem 17. *Let A be a non-empty set. Then $\mathcal{T}(A)$ is a complete atomic Boolean algebra, the operation $\wedge$ is equal to the set intersection, the operation $\vee$ is equal to the set union.*

Proof. It is evident that if $T_\gamma \in \mathcal{T}(A)$ for each γ from a subscript set Γ, then $\bigcap_{\gamma \in \Gamma} T_\gamma$ and $\bigcup_{\gamma \in \Gamma} T_\gamma$ are again tolerances on A; thus $\vee$ is equal to the set union and $\wedge$ is equal to the set intersection. Further, the identity relation I (or the universal relation U) is the least (or greatest, respectively) element of $\mathcal{T}(A)$. The lattice $\mathcal{T}(A)$ is a sublattice of the lattice of all binary relations on the set A, i.e. of the lattice $\mathcal{P}(A \times A)$ of all subsets of the Cartesian product $A \times A$. But $\mathcal{P}(A \times A)$ is a complete Boolean algebra, the distributivity is a hereditary property, therefore $\mathcal{T}(A)$ is a complete distributive lattice with both the least and the greatest element. Let a, b be two distinct elements of A. By T_{ab} we denote the tolerance on A such that $x \, T y$ if and only if $x = y$, or $x = a$, $y = b$, or $x = b$, $y = a$. The tolerances T_{ab} for all pairs of distinct elements a, b of A are evidently atoms of $\mathcal{T}(A)$. If a tolerance on A is different from the identity relation, there exist at least two distinct elements a, b of A which are in this tolerance and thus this tolerance contains the atom T_{ab} of $\mathcal{T}(A)$. Therefore $\mathcal{T}(A)$ is atomic. It remains to prove the complementarity. Let $T \in \mathcal{T}(A)$. Then $T = I \cup S$, where S is a symmetric irreflexive binary relation. Evidently $U = I \cup S_U$, where S_U is the relation of inequality (this means that $a \, S_U \, b$ if and only if $a \in A$, $b \in A$, $a \neq b$). Denote $S' = S_U - S$. Evidently S' is again a symmetric irreflexive relation and thus $T' = I \cup S'$ is a tolerance on A. Then

$$T \cap T' = (I \cup S) \cap (I \cup S') = I \cup (S \cap S') = I \cup I = I\,,$$
$$T \cup T' = (I \cup S) \cup (I \cup S') = I \cup S \cup S' = I \cup S_U = U\,.$$

This means that T' is a complement to T, which completes the proof.

In [10] a τ-covering of a set M was defined as a covering $\mathfrak{M} = \{M_\gamma, \gamma \in \Gamma\}$ of M by subsets for which the following conditions hold:

(i) if $\gamma_0 \in \Gamma$, $\Gamma_0 \subseteq \Gamma$, then

$$M_{\gamma_0} \subseteq \bigcup_{\gamma \in \Gamma_0} M_\gamma \Rightarrow \bigcap_{\gamma \in \Gamma_0} M_\gamma \subseteq M_{\gamma_0}\,,$$

(ii) if $N \subseteq M$ and N is contained in no set from $\mathfrak{M}$, then N contains a two-element subset with the same property.

In [10] it was proved that there is a one-to-one correspondence between tolerances on M and τ-coverings of M such that if a τ-covering $\mathfrak{M}_T$ corresponds to a tolerance T, then $\mathfrak{M}_T$ consists of the maximal subsets of M with the property that any two elements of such a subset are in the relation T. If T is an equivalence, then $\mathfrak{M}_T$ is the partition of M into equivalence classes of T.

Now let $\mathfrak{M}$ be a τ-covering of a set M, let $\mathfrak{P}$ be a partition of M. The partition $\mathfrak{P}$ will be called the partition hull of $\mathfrak{M}$, if $\mathfrak{P}$ is the least partition of M such that each set of $\mathfrak{M}$ is contained in a certain class of $\mathfrak{P}$.

If two sets of $\mathfrak{M}$ have a non-empty intersection, then they must be contained in the same class of $\mathfrak{P}(\mathfrak{M})$, where $\mathfrak{P}(\mathfrak{M})$ is the partition hull of $\mathfrak{M}$; otherwise $\mathfrak{P}(\mathfrak{M})$ would contain two distinct classes having a non-empty intersection, which is impossible. If $M_1, \ldots, M_n$ are sets of $\mathfrak{M}$ such that $M_i \cap M_{i+1} \neq \emptyset$ for $i = 1, \ldots, n-1$, then M_i and M_{i+1} are contained in the same class of $\mathfrak{P}(\mathfrak{M})$; as to each set of $\mathfrak{M}$ exactly one class of $\mathfrak{P}(\mathfrak{M})$ containing it can exist, all sets $M_1, \ldots, M_n$ are contained in the same class of $\mathfrak{P}(\mathfrak{M})$. Thus we may consider a tolerance $\mathcal{N}$ on $\mathfrak{M}$ such that $(M, M') \in \mathcal{N}$ if and only if $M \cap M' \neq \emptyset$. Let $\mathscr{C}(\mathcal{N})$ be the transitive hull of $\mathcal{N}$. Then the union of all sets belonging to a class of $\mathscr{C}(\mathcal{N})$ must be contained in a class of $\mathfrak{P}(\mathfrak{M})$. On the other hand, these unions form a partition of M with the property that each set of $\mathfrak{M}$ is contained in a certain class of this partition. Thus $\mathfrak{P}(\mathfrak{M})$ is a partition, each of whose classes is the union of all sets belonging to a certain equivalence class of $\mathscr{C}(\mathcal{N})$.

Theorem 18. *Let T be a tolerance on a set M, let $C(T)$ be its transitive hull. Let $\mathfrak{M}_T$ be the τ-covering of M corresponding to T, let $\mathfrak{P}$ be the partition of M into the equivalence classes of $C(T)$. Then $\mathfrak{P}$ is the partition hull of $\mathfrak{M}_T$.*

Proof. Let $a \in M$, $b \in M$, $a\ C(T)\ b$. This means that there exist elements $x_1, \ldots, x_n$ of M such that $x_1 = a$, $x_n = b$, $x_i\,T\,x_{i+1}$ for $i = 1, \ldots, n$. Then for each i from the numbers $1, \ldots, n-1$ there exists a set $M_i \in \mathfrak{M}_T$ such that $x_i \in M_i$, $x_{i+1} \in M_i$. We