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SETS OF σ-POROSITY AND SETS OF σ-POROSITY (q)

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1. INTRODUCTION

The notion of a set of σ-porosity was defined by E. P. DOLŽENKO [1]. There exists a number of theorems in the theory of cluster sets which use this notion. (See [1], [2], [3], [4], [5].) It is easy to see that any set of σ-porosity is of the first category and of measure zero. The existence of a set of the first category and of measure zero which is not of σ-porosity is claimed without a proof in [1]. In the present article we shall prove this result.

N. YANAGIHARA [2] defined and used the notion of a set of σ-porosity (q), $0 < q \leqq 1$, which coincides with the notion of a set of σ-porosity for $q = 1$. We shall prove that the notions of a set of σ-porosity (q) and σ-porosity (p) coincide for any p, q, $0 < p, q < 1$.

The main aim of the present article is to prove the results mentioned above and some other results on the sets of σ-porosity (q) (in our notation on sets of (x^q)-σ-porosity) in Euclidean spaces.

We shall generalize the notion of a set of σ-porosity (q) and we shall formulate some results in a general metric space in order to clarify the proofs.

2. DEFINITIONS

Let (P, ϱ) be a metric space. Then we define:

2.1. The open sphere with the centre $x \in P$ and the radius $r > 0$ is denoted by $K(x, r)$.

2.2. Let $M \subset P$, $x \in P$, $R > 0$. Then we denote the supremum of the set $\{r > 0;$ for some $z \in P$, $K(z, r) \subset K(x, R)$ and $K(z, r) \cap M = \emptyset\}$ by $\gamma(x, R, M)$.

2.3. Let $K(x, r) \subset P$. Let f be an arbitrary function. Then we put $f * K(x, r) = K(x, f(r))$ if $f(r) > 0$.

2.4. Let $M \subset P$. Let f be an arbitrary function. Then we put $S(f, r, M) = \bigcup\{f * K;\ K \cap M = \emptyset,\ K = K(x, \sigma),\ \sigma < r \text{ and } f(\sigma) > 0\}$.

2.5. We denote by G (resp. G_1, resp. G_2) the system of all real functions which are increasing and continuous (resp. for which $\infty > g'(x) \geqq 1$, resp. for which $\infty > g'(x) \geqq 1$ and $g(x) > x$) on $(0, \delta)$ for some $\delta > 0$.

2.6. We denote by G_3 the system of all functions $g \in G$ such that for any $A > 0$ and $e > 1$ there exists an integer r and $\delta > 0$ such that

$$\underbrace{(eg) \circ \ldots \circ (eg)}_{r\text{-times}}(x) \geqq A\, g(x) \quad \text{for} \quad 0 < x < \delta .$$

2.7. Let $f \in G$, $M \subset P$, $x \in P$. Then we say that x is a point of (f)-porosity of M if

$$\limsup_{R \to 0+} \frac{1}{R} f(\gamma(x, R, M)) > 0 .$$

2.8. Let $f \in G$, $M \subset P$, $x \in P$, $c > 0$. Then we say that x is a point of (f, c)-porosity of M if

$$\limsup_{R \to 0+} \frac{1}{R} f(\gamma(x, R, M)) \geqq c .$$

2.9. Let $g \in G$, $H \subset G$, $M \subset P$, $x \in P$. Then we say that x is a point of $\langle g \rangle$-porosity of M if $x \in \bigcap\{S(g, r, M);\ r > 0\}$. We say that x is a point of $\langle H \rangle$-porosity of M if it is a point of $\langle h \rangle$-porosity of M for any $h \in H$.

2.10. Let V be one of the symbols (f), (f, c), $\langle h \rangle$, $\langle H \rangle$. Let $M \subset P$, $N \subset P$. Then we say that M is of V-porosity if any point $x \in M$ is a point of V-porosity of M. We say that N is a set of V-σ-porosity if it is the union of a sequence of sets of V-porosity.

2.11. We shall write "porosity" instead of "(x)-porosity" and "σ-porosity" instead of "(x)-σ-porosity".

Let us note:

2.12. The notions of a set of (x^q)-porosity and of a set of (x^q)-σ-porosity coincide with the notions of N. Yanagihara of a set of porosity (q) and of a set of σ-porosity (q).

2.13. Let V be one of the symbols (f), (f, c), $\langle h \rangle$, $\langle H \rangle$. Then the system of all sets of V-porosity is an ideal of sets and the system of all sets of V-σ-porosity is a σ-ideal of sets.

2.14. A point $x \in R^k$ is a point of $(x, 1/2)$-porosity of a set $M \subset R^k$ iff there exists a sequence of spheres $\{K(s_n, r_n)\}$ such that $\lim_{n \to \infty} s_n = x$, $\lim_{n \to \infty} \varrho(x, s_n)/r_n = 1$ and $K(s_n, r_n) \cap M = \emptyset$ for $n = 1, 2, \ldots$.

2.15. Evidently, we may always write (af, ac) instead of (f, c) if $a > 0$.

2.16. A point $x \in P$ is a point of $\langle h \rangle$-porosity of a set $M \subset P$ iff there exists a sequence of spheres $\{K(s_n, r_n)\}$ such that $\lim_{n \to \infty} s_n = x$, $K(s_n, r_n) \cap M = \emptyset$ and $h(r_n) > \varrho(x, s_n)$ for $n = 1, 2, \ldots$.

2.17. Let V be one of the symbols (f), (f, c), $\langle h \rangle$, $\langle H \rangle$. Let $x \in P$, $M \subset P$. Then the point x is a point of V-porosity of the set M iff it is a point of V-porosity of the set $\overline{M}$.

3. SEVERAL LEMMAS

3.1. Lemma. *Let (P, ϱ) be a metric space, $M \subset P, f \in G$. Then:*

(i) *If $x \in P$ is a point of $(f, 2)$-porosity of M then it is a point of $\langle f \rangle$-porosity of M.*

(ii) *If $x \in M$ is a point of $\langle f \rangle$-porosity of M then it is a point of $(2f, 1)$-porosity of M.*

Proof. The assertion (i) immediately follows from the continuity of f on $(0, \delta)$ and from the definitions. The assertion (ii) follows from the fact that if $K(y, h) \subset \subset P - M$, $x \in f * K(y, h)$ and h is sufficiently small then $f(\gamma(x, R, M)) > R/2$ where $R = 2\varrho(x, y)$.

3.2. Lemma. *Let $g \in G_1$. Then if $d > 0$ is a sufficiently small number, the relations $z \in g * I, I = (t - r, t + r)$ and $I \subset J = (u - d, u + d)$ imply $z \in g * J$.*

Proof. Let $d < \delta$, where δ is the number from the definition of the system G_1 (see 2.5). Then

$$\varrho(z, u) \leqq \varrho(z, t) + d - r < g(r) + d - r \leqq g(d)$$

and therefore $z \in g * J$.

3.3. Lemma. *Let $M \subset (a, b)$ be a nowhere dense set. Let $g \in G_2$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint open intervals such that $(a, b) - \overline{M} = \bigcup_{n=1}^{\infty} I_n$. Let H be the set of all endpoints of intervals I_n. Let P be the set of all points of $\langle g \rangle$-porosity of M which lie in $\overline{M} \cap (a, b)$. Then*

$$P = (H \cup \limsup_{n \to \infty} g * I_n) \cap (a, b).$$

Proof. Let $z \in (H \cup \limsup_{n \to \infty} g * I_n) \cap (a, b)$. If $z \in H$, then $z \in P$, since $g(x) > x$ for sufficiently small x. If $z \in \limsup_{n \to \infty} g * I_n$, then evidently $z \in P$. Let $y \in P - H$. Then 2.16 and 3.2 clearly imply $z \in \limsup_{n \to \infty} g * I_n$.

3.4. Lemma. *Let $H \subset G$, $f \in G$, $c > 0$. Let n be an integer and $M \subset R^1$. Put $N = M \times R^n$. Then N is a set of (f, c)-σ-porosity or of (f, c)-porosity, or of (f)-σ-porosity, or of (f)-porosity, or of $\langle H \rangle$-σ-porosity, or of $\langle H \rangle$-porosity in the space R^{n+1} iff M is of the same type as a subset of R^1.*

Proof. We shall prove only the part concerning (f, c)-σ-porosity, the proofs of the other parts being quite similar. The implication "if" follows from the fact that $\gamma((x, y), r, A \times R^n) \geqq \gamma(x, r, A)$ for any $x \in R^1$, $y \in R^n$, $A \subset R^1$ and $r > 0$. Now we shall prove the implication "only if". Let $N = \bigcup_{k=1}^{\infty} N_k$ where any N_k is a set of (f, c)-porosity. Let $\{B_t\}_{t=1}^{\infty}$ be a basis of open sets in R^n. Denote by $A_{k,t}$ the set of all points $x \in M$ for which the set $\{z; (x, z) \in N_k\}$ is dense in B_t. Clearly $M = \bigcup_{k,t} A_{k,t}$ and therefore it is sufficient to prove that each set $A_{k,t}$ is of (f, c)-porosity. Let $x \in A_{k,t}$ and $z \in B_t$ be such that $(x, z) \in N_k$. Clearly for any $r > 0$ such that $K(z, r) \subset B_t$, the inequality $\gamma((x, z), r, N_k) \leqq \gamma(x, r, A_{k,t})$ holds. Since N_k is a set of (f, c)-porosity in R^{n+1}, the set $A_{k,t}$ is a set of (f, c)-porosity in R^1.

3.5. Lemma. *Let P be a metric space and $f \in G$. Let $A \subset P$ be a set of (f)-σ-porosity. Then $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is a set of (f, c_n)-porosity for some $c_n > 0$, $n = 1, 2 \dots$.*

Proof. Let $A = \bigcup_{i=1}^{\infty} B_i$ where each set B_i is a set of (f)-porosity. For any i let $B_{i,k}$ be the set of all points $x \in B_i$ which are points of $(f, 1/k)$-porosity of the set B_i. Clearly $B_i = \bigcup_{k=1}^{\infty} B_{i,k}$ and each set $B_{i,k}$ is a set of $(f, 1/k)$-porosity. Now it is sufficient to order the sets $B_{i,k}$ in a sequence $\{A_n\}_{n=1}^{\infty}$.

4. SOME AFFIRMATIVE RESULTS

In the present part we shall prove that some properties like σ-porosity are equivalent with other, seemingly weaker properties of this type. We use only one method which is contained in the following basic proposition.

4.1. Proposition. *Let $h \in G$, $f \in G$. Let there exist an integer n and $\delta > 0$ such that*

(1) $$h^{(n)}(x) = \underbrace{h \circ \dots \circ h}_{n\text{-times}}(x) \geqq f(x) \quad \textit{for} \quad 0 < x < \delta .$$

Let P be a metric space and let $M \subset P$ be a set of $\langle f \rangle$-σ-porosity. Then M is a set of $\langle h \rangle$-σ-porosity.

Proof. It is clearly sufficient to prove that if A is a set of $\langle f \rangle$-porosity then it is a set of $\langle h \rangle$-σ-porosity. Put $C_k = A \cap \bigcap_{r>0} S(h^{(k)}, r, A)$, (see 2.4). Then $A \subset C_n$ and therefore $A \subset \bigcup_{k=2}^{n} (C_k - C_{k-1}) \cup C_1$. Since obviously C_1 is a set of $\langle h \rangle$-porosity it is sufficient to prove that $C_k - C_{k-1}$ is a set of $\langle h \rangle$-σ-porosity for $k = 2, \ldots, n$. Put $T_{k,m} = C_k - S(h^{(k-1)}, 1/m, A)$ for $k = 2, \ldots, n$ and $m = 1, 2, \ldots$. Since clearly $C_k - C_{k-1} = \bigcup_{m=1}^{\infty} T_{k,m}$, it is sufficient to prove that any set $T_{k,m}$ is a set of $\langle h \rangle$-porosity. Let $z \in T_{k,m}$, $r > 0$. Then there exists an open sphere $K(y, t)$ such that $t < \min(1/m, r)$, $K(y, t) \cap A = \emptyset$ and $z \in h^{(k)} * K(y, t)$. Put $K = h^{(k-1)} * K(y, t)$. Then $z \in h * K$ and $K \cap T_{k,m} = \emptyset$ since $K \subset S(h^{(k-1)}, 1/m, A)$. Since the radius of the sphere K is arbitrarily small provided r is sufficiently small, z is a point of $\langle h \rangle$-porosity of the set $T_{k,m}$. Therefore $T_{k,m}$ is a set of $\langle h \rangle$-porosity. Thus the proof is complete.

4.2. Proposition. *Let $h \in G$, $f \in G$. For any $B > 0$, let there exist $A > 0$, $\delta > 0$ and an integer r such that*

$$(2) \qquad \underbrace{(Ah) \circ \ldots \circ (Ah)}_{r\text{-times}}(x) \geqq B f(x) \quad \textit{for} \quad 0 < x < \delta .$$

Let P be a metric space and let $M \subset P$ be a set of (f)-σ-porosity. Then M is a set of (h)-σ-porosity.

Proof. By 3.5, $M = \bigcup_{m=1}^{\infty} M_m$ where M_m is a set of (f, c_m)-porosity, $c_m > 0$. By 2.15 and 3.1 M_m is a set of $\langle 2f/c_m \rangle$-porosity. By 4.1 and (2) it is a set of $\langle Ah \rangle$-σ-porosity for some $A > 0$. Therefore by 2.15 and 4.1 it is a set of (h)-σ-porosity. Consequently M is of (h)-σ-porosity.

4.3. Theorem. *Let $0 < q < p < 1$ and let M be a subset of a metric space. Then M is a set of (x^q)-σ-porosity iff it is a set of (x^p)-σ-porosity.*

Proof. Let $B > 0$. Then the inequality (2) from Proposition 4.2 holds for $A = 1$, $h = x^p$, $f = x^q$, an integer r such that $p^r < q$ and for a sufficiently small $\delta > 0$. Therefore the statement of the theorem follows from 4.2.

4.4. Proposition. *Let P be a metric space and $g \in G_3$ (see 2.6). Let $M \subset P$ be a set of (g)-σ-porosity and $0 < c < \frac{1}{2}$. Then M is a set of (g, c)-σ-porosity.*

Proof. By 3.5 it is sufficient to prove that any set N of (g, a)-porosity is a set of (g, c)-σ-porosity. By 3.1, N is a set of $\langle 2g/a \rangle$-porosity. Put $A = 2/a$ and $e = 1/2c$. Let r be the integer from 2.6. Then the inequality (1) from 4.1 holds for $f = 2g/a$, $h = g/2c$ and for sufficiently small $\delta > 0$. Therefore by 4.1, N is a set of $\langle g/2c \rangle$-σ-porosity and consequently it is a set of (g, c)-σ-porosity.

Since obviously $x^q \in G_3$ for $0 < q \leqq 1$, we have

4.5. Theorem. *Let P be a metric space, $0 < q \leqq 1$, $0 < c < \frac{1}{2}$. Then a subset of P is a set of (x^q)-σ-porosity iff it is a set of (x^q, c)-σ-porosity.*

5. SOME NEGATIVE RESULTS

In the present part we shall prove that some properties like σ-porosity are not equivalent with the others. We use only one method which is contained in the following basic proposition.

5.1. Proposition. *Let $f \in G$ and $H \subset G_2$ (see 2.5). Let there exist a sequence $\{h_i\}_{n=1}^{\infty}$ of functions from H and a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ such that*

$$h_n \circ \ldots \circ h_1(x) < f(x) \quad \text{for} \quad 0 < x < \varepsilon_n \,. \tag{3}$$

Then in any Euclidean space there exists a perfect set F of $\langle f \rangle$-porosity which is not a set of $\langle H \rangle$-σ-porosity.

Along with 5.1, we shall prove the following proposition.

5.2. Proposition. *Let $g \in G$ and $\lim_{x \to 0+} x/g(x) = 0$. Then in any Euclidean space there exists a perfect set F of $(g, 1)$-porosity and of measure zero which is not of σ-porosity.*

Proof. 3.4 implies that it is sufficient to construct a set F on the line. Let $\{k_i\}_{i=1}^{\infty}$ be an increasing sequence of integers such that $k_1 = 1$. Our construction depends on this sequence. For a proof of 5.1, the sequence $\{k_i\}$ may be chosen in an arbitrary way but for a proof of 5.2 we must choose it in a special way. Given $\{k_i\}$ define a sequence $\{s_p\}_{p=1}^{\infty}$ by the relations $k_{s_p} \leqq p \leqq k_{s_p+1}$. We may and will assume that $\lim_{n\to\infty} \varepsilon_n = 0$.

From the segment $\langle 0, 1 \rangle$ we shall delete in the k-th step a finite number of pairwise disjoint intervals, D-intervals of the order k. The points from $\langle 0, 1 \rangle$ not contained in any D-interval will form the set F. For any integer k we shall define a system of remaining intervals (R-intervals) of the order k. Any R-interval will be closed. The system of all R-intervals of the order k and of all D-intervals of orders $j \leqq k$ will form a covering of $\langle 0, 1 \rangle$ and any two members of this system will have disjoint interiors.

Define the D-intervals and the R-intervals by induction:

1. A D-interval of the order 1 does not exist. As the system of all R-intervals of the order 1, let us choose any covering of $\langle 0, 1 \rangle$ by closed intervals of a length smaller than ε_2 such that any two its members have disjoint interiors.

2. Let k be an integer. Let D-intervals and R-intervals of all orders smaller than $k + 1$ be defined. Let $R_1, \ldots, R_{i_k}$ be all R-intervals of the order k. For $j = 1, \ldots, i_k$

define an open interval $D_j \subset R_j$ by the relation $(h_{s_k+1} \circ \ldots \circ h_1) * \bar{D}_j = R_j$. Define the system of all D-intervals of the order $k + 1$ as the system $D_1, \ldots, D_{i_k}$. The endpoints of the intervals D_j and $(h_t \circ \ldots \circ h_1) * D_j$, $j = 1, \ldots, i_k$, $t = 1, \ldots, s_k + 1$ divide $\langle 0, 1\rangle$ to a finite number of closed intervals. Let $A_1, \ldots, A_{b_k}$ be all of these intervals which are disjoint with each D-interval of the order $k + 1$.

For $1 \leqq r \leqq b_k$ let C_r be a system of closed intervals of a length smaller than $\varepsilon_{s_{k+1}+1}$ such that $\bigcup\{X; X \in C_r\} = A_r$ and any two members of C_r have disjoint interiors. Define the system of all R-intervals of the order $k + 1$ as the system $\bigcup_{r=1}^{b_k} C_r$.

The following assertions are easily verified:

(i) F is a perfect set of $\langle f\rangle$-porosity.

(ii) Let R be an R-interval of an order k and let $m \leqq s_k$ be an integer. Then the set $R - \bigcup\{(h_m \circ \ldots \circ h_1) * D;\ D \subset R \text{ is a } D\text{-interval}\}$ is a nonempty perfect set. If a contiguous interval of this set lies in R then it is of the form $(h_m \circ \ldots \circ h_1) * D$, where $D \subset R$ is a D-interval.

(iii) Let D be a D-interval of an order k and let $m \leqq s_{k-1} + 1$ be an integer. Let R be a R-interval such that $\operatorname{Int} R \cap D = \emptyset$. Then either $\operatorname{Int} R \subset (h_m \circ \ldots \circ h_1) * D$ or $R \cap (h_m \circ \ldots \circ h_1) * D = \emptyset$.

Now suppose that F is a set of $\langle H\rangle$-σ-porosity. Then $F = \bigcup_{i=1}^{\infty} P_i$ where each P_i is a set of $\langle H\rangle$-porosity. We shall define a sequence $\{F_i\}_{i=0}^{\infty}$ of nonempty perfect sets such that $F \supset F_{i-1} \supset F_i$ and $F_i \cap P_i = \emptyset$ for $i = 1, 2, \ldots$. The existence of such a sequence yields a contradiction since it implies that there exists a point $x \in \bigcap_{i=0}^{\infty} F_i \subset F$ which does not lie in $\bigcup_{i=1}^{\infty} P_i = F$. Each set F_i will have the form

$$F_i = R_i - \bigcup\{(h_i \circ \ldots \circ h_1) * D;\ D \subset R_i \text{ is a } D\text{-interval}\}, \tag{4}$$

where R_i is an R-interval of an order $j \geqq k_{i+1}$ and $(h_0 \circ \ldots \circ h_1) * D = D$. By (ii) any set of the form (4) is a nonempty perfect set.

Define the sets F_i by induction:

A. Put $F_0 = R_0 \cap F$ where R_0 is an R-interval of the order 1.

B. Suppose that we have defined the set F_i. We shall distinguish two cases:

B 1. $F_i \not\subset \bar{P}_{i+1}$. Then define R_{i+1} as an R-interval of an order $j \geqq k_{i+2}$ such that $R_{i+1} \cap \bar{P}_{i+1} = \emptyset$ and $R_{i+1} \cap F_i$ is an infinite set. Define the set F_{i+1} by (4).

B 2. $F_i \subset \bar{P}_{i+1}$. Then any point of P_{i+1} is a point of $\langle h_{i+1}\rangle$-porosity of F_i. Therefore by 3.3 and (ii) any point $x \in \operatorname{Int} R_i \cap P_{i+1}$ lies in an interval of the form

$$h_{i+1} * ((h_i \circ \ldots \circ h_1) * D) = (h_{i+1} \circ \ldots \circ h_1) * D,$$

where $D \subset R_i$ is a D-interval. Therefore the nonempty perfect set $A = R_i - \bigcup\{(h_{i+1} \circ \ldots \circ h_1) * D;\ D \subset R_i\}$ and the set $\operatorname{Int} R_i \cap P_{i+1}$ are disjoint. Define R_{i+1}

as an R-interval of an order $j \geqq k_{i+2}$ such that $R_{i+1} \subset \operatorname{Int} R_i$ and $R_{i+1} \cap A$ is an infinite set. Then define the set F_{i+1} by (4). Since (iii) implies $F_{i+1} = R_{i+1} \cap A$ we have $F_{i+1} \cap P_{i+1} = \emptyset$. Thus the proof of 5.1 is complete.

To prove 5.2 put $f = g/2$ and $H = \{6x\}$. Then the assumptions of 5.1 are obviously fulfilled. If we denote by m_i the measure of the union of all R-intervals of the order k_i, then evidently

$$m_{i+1} = m_i(1 - 1/6^{i+1})^{k_{i+1}-k_i}$$

Therefore there exists a sequence $\{k_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} m_i = 0$ and consequently $\mu F = 0$. The set F is a set of $\langle g/2\rangle$-porosity and therefore it is of $(g, 1)$-porosity. On the other hand, F is not a set of $\langle 6x\rangle$-σ-porosity and therefore it is not a set of $(3x, 1)$-σ-porosity. Now 4.5 implies that F is not a set of σ-porosity.

5.3. Proposition. *Let $h \in G_3$, $f \in G_1$. Let there exist $B > 0$ such that for any $A > 0$ and any integer r there exists $\delta > 0$ such that*

$$\underbrace{(Ah) \circ \ldots \circ (Ah)}_{r\text{-times}}(x) < B f(x) \quad \text{for} \quad 0 < x < \delta\,.$$

Then in any Euclidean space there exists a perfect set of (f)-porosity which is not of (h)-σ-porosity.

Proof. By 5.1, in any Euclidean space there exists a perfect set F of $\langle Bf\rangle$-porosity which is not of $\langle 6h\rangle$-σ-porosity. Thus F is a set of (f)-porosity but not of $(3h, 1)$-σ-porosity and by 4.4 it is not of (h)-σ-porosity.

The following theorem is a consequence of 5.2.

5.4. Theorem. *Let $0 < q < 1$. Then in any Euclidean space there exists a perfect set F of $(x^q, 1)$-porosity and of measure zero which is not of σ-porosity.*

The existence of a perfect set of (x^q)-porosity which is not of σ-porosity follows also from the following easy theorem.

5.5. Theorem. *Let $0 < q < 1$. Then in any Euclidean space there exists a perfect set D of (x^q)-porosity and of positive Lebesgue measure.*

Proof. 3.4 implies that it is sufficient to construct the set D on the line. We shall define a sequence of sets such that S_k contains 2^k disjoint closed intervals:

1. $S_0 = \{\langle 0, 1/2\rangle\}$.

2. Suppose that we have defined $S_k = \{I_1, \ldots, I_{2^k}\}$. For $j = 1, \ldots, 2^k$ define closed disjoint intervals I'_j, I''_j such that

$$x^q * (I_j - (I'_j \cup I''_j)) = \operatorname{Int} I_j\,.$$

Put $S_{k+1} = \{I'_1, I''_1, \ldots, I'_{2^k}, I''_{2^k}\}$. Put $D_k = \bigcup\{I; I \in S_k\}$ and $D = \bigcap_{k=0}^{\infty} D_k$. The set D is clearly a perfect set of (x^q)-porosity. We have

$$\mu(I'_j \cup I''_j) = \mu I_j(1 - 2^{1-1/q}(\mu I_k)^{1/q-1}).$$

Since $\mu I_j < 1/2^{n+1}$, we have

$$\mu(I'_j \cup I''_j) > \mu I_j(1 - 2^{(1-1/q)(n+2)}).$$

If we denote $\mu D_k = m_k$, we have

$$m_{n+1} > m_n(1 - 2^{(1-1/q)(n+2)})$$

and therefore

$$m_{n+1} > \tfrac{1}{2}\prod_{k=0}^{n}(1 - 2^{(1-1/q)(k-2)})$$

and

$$\mu D \geqq \tfrac{1}{2}\prod_{k=0}^{\infty}(1 - 2^{(1-1/q)(k+2)}) > 0.$$

Thus the proof is complete.

The following theorem justifies the complicated form of 5.1.

5.6. Theorem. *In any Euclidean space there exists a perfect set F of porosity which is not a set of $(x, 1/2)$-σ-porosity.*

Proof. Let $H = \{ax; a > 1\}$. For an integer n put $h_n = (1 + 1/n^2)\,x$. Put $c = \prod_{k=1}^{\infty}(1 + 1/k^2)$ and $f(x) = 2cx$. Then the assumption (3) from 5.1 is obviously fulfilled and therefore in any Euclidean space there exists a perfect set F of $\langle 2cx\rangle$-porosity which is not a set of $\langle H\rangle$-σ-porosity. The set F is clearly a set of porosity but not of $(x, 1/2)$-σ-porosity since a set is of $(x, 1/2)$-porosity iff it is of $\langle H\rangle$-porosity.

5.7. Theorem. *Let $0 < q < 1$. Then in any Euclidean space there exists a perfect set D which is not a set of (x^q)-σ-porosity.*

Proof. The theorem immediately follows from 5.3 if we put $h = x^q$, $f = (\log(1/x))^{-1}$, $B = 1$.

6. SOME OPEN PROBLEMS

6.1. Problem. *Does there exist a (perfect) set on the line of the first category and of measure zero which is not a set of (x^q)-σ-porosity for $0 < q < 1$?*

6.2. Problem. *Does there exist $f \in G$ such that any (perfect) set on the line of measure zero and of the first category is a set of (f)-σ-porosity?*