ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
Vydává Matematický ústav ČSAV, Praha
SVAZEK 101 * PRAHA 24. 11. 1976 * ČÍSLO 4

SOME FORMULAS THAT ENUMERATE CERTAIN PARTITIONS AND GRAPHS

ROGER C. GRIMSON, Chapel Hill
(Received May 16, 1975)

Let $p(2n + 1, k)$ denote the number of partitions of $2n + 1$ into k parts, each part not exceeding n. Letting $[x]$ be the greatest integer $\leqq x$, MICHAL BUČKO [1] showed that

$$p(2n + 1, 3) = \sum_{k=0}^{[(n-1)/3]} \left[\frac{n + 1 - 3k}{2}\right],$$

$$p(2n + 1, 4) = \sum_{\substack{i\geqq 0 \\ 0\leqq 3i+j\leqq n-2}} \sum_{j\geqq 0} \left[\frac{n - 3i - j}{2}\right],$$

$$\frac{1}{6}\binom{n+2}{2} \leqq p(2n + 1, 3) \leqq \frac{1}{6}\binom{n+3}{2},$$

$$\frac{1}{6}\left\{\binom{n+2}{3} - 1\right\} \leqq p(2n + 1, 4) \leqq \frac{1}{6}\left\{\binom{n+3}{3} - 4\right\}.$$

Bučko also showed how this function, p, is useful in counting cycles in certain graphs; this is described in the second section of [1].

The purpose of this paper is to exhibit further properties of p; in particular, we evaluate the two above sums and some more general similar sums.

Let b and m be positive integers such that $b < m$ and let q be defined so that qb is as large as possible and yet $qb - 1 \leqq m$. Then $\sum_{k=0}^{m} f(k) = \sum_{k=0}^{qb-1} f(k) + s$ where $s = f(qb) + f(qb + 1) + \ldots + f(m)$ if $qb - 1 < m$ and $s = 0$ if $qb - 1 = m$. Thus, s is a "residual" sum containing only a few terms. Also, we observe that $\sum_{k=0}^{qb-1} f(k) = \sum_{j=0}^{q-1} \sum_{k=0}^{b-1} f(k + jb)$. Hence, if x is a positive real number, if a is a positive integer and if $f(k) = [(x + ak)/b]$ then

$$(1) \qquad \sum_{k=0}^{m} \left[\frac{x + ak}{b}\right] = \sum_{j=0}^{q-1} \sum_{k=0}^{b-1} \left[\frac{x + ajb + ak}{b}\right] + s\,.$$

But is is known [2] that if $(a, b) = 1$, then $\sum_{k=0}^{b-1} [(x + ak)/b] = \frac{1}{2}(a - 1)(b - 1) + [x]$. Substituting for the inner sum of the right side of (1) and simplifying, (1) becomes

$$(2) \qquad \sum_{k=0}^{m} \left[\frac{x + ak}{b}\right] = \tfrac{1}{2}(abq^2 - (a + b - 1)q) + q[x] + s, \quad (a, b) = 1.$$

Note that $p(2n + 1, 3)$ may be given by

$$(3) \qquad p(2n + 1, 3) = \sum_{k=0}^{[(n-1)/3]} \left[\frac{n + 1 - 3[(n - 1)/3] + 3k}{2}\right].$$

Since $n + 1 - 3[(n - 1)/3] \geqq 0$, we may apply (2). In this case, we want to select an integer q which makes $2q - 1$ as large as possible, constrained by $2q - 1 \leqq [(n - 1)/3]$. That is, we want the largest value of q for which $q \leqq \frac{1}{2}[(n - 1)/3] + \frac{1}{2} = \frac{1}{2}[(n + 2)/3]$. Hence,

$$(4) \qquad q = \left[\frac{1}{2}\left[\frac{n + 2}{3}\right]\right].$$

Now, according to the definition of the residual sum s, we have in our case the following two possibilities: $s = f(2q) + \ldots + f([(n - 1)/3])$ if $2q - 1 < [(n - 1)/3]$; $s = 0$ if $2q - 1 = [(n - 1)/3]$. Note that q can be selected so that s contains no terms or only one term, depending on whether $[(n - 1)/3]$ is odd or even, respectively. $[(n - 1)/3]$ is odd if and only if $n \equiv 0$ or -1 or $-2 \pmod 6$. If $[(n - 1)/3]$ is even, then s is simply the last term in the sum (3). Therefore,

$$(5) \qquad s = \begin{cases} 0 & \text{if} \quad n \equiv 0 \quad \text{or} \quad -1 \quad \text{or} \quad -2 \pmod 6 \\ f\left(\left[\frac{n - 1}{3}\right]\right) = \left[\frac{n + 1}{2}\right] & \text{otherwise}. \end{cases}$$

Applying (2) to (3), using (4) and (5), and simplifying, we have the formula

$$p(2n + 1, 3) = 3\left[\frac{1}{2}\left[\frac{n + 2}{3}\right]\right]^2 + \left[\frac{1}{2}\left[\frac{n + 2}{3}\right]\right]\left(n - 1 - 3\left[\frac{n - 1}{3}\right]\right) + \begin{cases} 0 & \text{if} \quad n \equiv 0, \quad -1 \quad \text{or} \quad -2 \pmod 6 \\ \left[\frac{n + 1}{2}\right] & \text{otherwise}. \end{cases}$$

Next we find some formulas for $p(2n + 1, 4)$.

In general,

$$\sum_{\substack{0 \leqq a_1 i + \ldots + a_t i_t \leqq m \\ i_1 \geqq 0, \ldots, i_t \geqq 0}} \left[\frac{n - a_1 i_1 - \ldots - a_t i_t}{b}\right] = \sum_{k=0}^{m} \sum_{a_1 i_1 + \ldots + a_t i_t = k} \left[\frac{n - k}{b}\right] =$$

$$= \sum_{k=0}^{m} \left[\frac{n-k}{b}\right] D(k; a_1, \dots, a_t),$$

where $D(k; a_1, \dots, a_t)$ is a denumerant, i.e., the number of partitions of k into the parts $a_1, \dots, a_t$. As a special case we have

$$(6)\qquad \sum_{\substack{0 \le ai+j \le n-c \\ i,j \ge 0}} \left[\frac{n-ai-j}{b}\right] = \sum_{k=0}^{n-c} \left[\frac{n-k}{b}\right] \sum_{i=0}^{[k/a]} 1 = \sum_{k=0}^{n-c} \left[\frac{n-k}{b}\right] \left(\left[\frac{k}{a}\right] + 1\right) =$$

$$= \sum_{k=0}^{n-c} \left[\frac{n-k}{b}\right]\left[\frac{k}{a}\right] + \sum_{k=0}^{n-c} \left[\frac{n-k}{b}\right] = \sum_{k=0}^{n-c} \left[\frac{n-k}{b}\right]\left[\frac{k}{a}\right] + \sum_{k=0}^{n} \left[\frac{k}{b}\right] - \sum_{k=0}^{c-1} \left[\frac{k}{b}\right].$$

It can be shown (see [3], for example) that

$$(7)\qquad \sum_{k=0}^{n} \left[\frac{k}{b}\right] = \frac{b}{2}\left[\frac{n}{b}\right]\left[\frac{n+b}{b}\right] - \left(b\left[\frac{n}{b}\right] + b - n - 1\right)\left[\frac{n}{b}\right].$$

Now we study the sum

$$T = \sum_{k=0}^{n} \left[\frac{n-k}{b}\right]\left[\frac{k}{a}\right].$$

LIPSCHITZ [4] gave the useful definition

$$(8)\qquad \Phi(x) = \begin{cases} 1\,; & x \ge 1 \\ 0\,; & 0 \le x < 1\,, \end{cases}$$

which allows for the following representation of the greatest integer function:

$$\left[\frac{k}{a}\right] = \sum_{i=1}^{\infty} \Phi\left(\frac{k}{ia}\right).$$

Then

$$T = \sum_{k=1}^{n} \sum_{i=1}^{\infty} \Phi\left(\frac{k}{ia}\right) \sum_{j=1}^{\infty} \Phi\left(\frac{n-k}{jb}\right).$$

Now, $\Phi(k/ia) = 1$ whenever $k/ia \ge 1$, i.e., when $i \le [k/a]$. Since $k \le n$, an upper bound for i is $[n/a]$. Similarly, $\Phi[(n-k)/jb] = 1$ when $j \le [(n-k)/b]$ so that an upper bound for j is $[(n-1)/b]$. Therefore,

$$T = \sum_{k=1}^{n} \sum_{i=1}^{[n/a]} \sum_{j=1}^{[(n-1)/b]} \Phi\left(\frac{k}{ia}\right) \Phi\left(\frac{n-k}{jb}\right) = \sum_{i=1}^{[n/a]} \sum_{j=1}^{[(n-1)/b]} \sum_{k=1}^{n} \Phi\left(\frac{k}{ia}\right) \Phi\left(\frac{n-k}{jb}\right).$$

But $k/ia \ge 1$ if $k \ge ia$; also $(n-k)/jb \ge 1$ if $k \le n - jb$. Using (8) we find that

$$(9)\qquad T = \sum_{i=1}^{[n/a]} \sum_{j=1}^{[(n-1)/b]} \sum_{ia \le k \le n-jb} 1\,.$$

The inner sum exists if and only if $ia \leqq n - jb$, i.e., if and only if

$$j \leqq [(n - ia)/b] \qquad \text{(Case 1)}$$

or, the same thing,

$$i \leqq [(n - jb)/a] \qquad \text{(Case 2)}.$$

Regarding case 1, (9) may be expressed by

$$\sum_{i=1}^{[n/a]} \sum_{j=1}^{\min\{[(n-1)/b],[(n-ia)/b]\}} (n + 1 - jb - ia) =$$

$$= \sum_{i=1}^{[n/a]} \sum_{j=1}^{[(n-ia)/b]} (n + 1 - jb - ia) =$$

$$= \sum_{i=1}^{[n/a]} \left[\frac{n - ia}{b}\right]\left(n + 1 - ia - \frac{b}{2}\left[\frac{n - ia + b}{b}\right]\right).$$

In the second case, we have a similar argument; thus we have a pair of formulas for our sum:

$$T = \sum_{i=1}^{[n/a]} \left[\frac{n - ia}{b}\right]\left(n + 1 - ia - \frac{b}{2}\left[\frac{n - ia + b}{b}\right]\right), \tag{10}$$

$$T = \sum_{j=1}^{[(n-1)/b]} \left[\frac{n - jb}{a}\right]\left(n + 1 - jb - \frac{a}{2}\left[\frac{n - jb + a}{a}\right]\right).$$

Using (6) and (7) with $a = 3$, $b = c = 2$, we get

$$\sum_{\substack{0 \leqq 3i+j \leqq n-2 \\ i,j \geqq 0}} \left[\frac{n - 3i - j}{2}\right] = \underbrace{\sum_{k=0}^{n-2} \left[\frac{n - k}{2}\right]\left[\frac{k}{3}\right]}_{C} + \tag{11}$$

$$+ \underbrace{\left[\frac{n}{2}\right]\left[\frac{n + 2}{2}\right] - \left(2\left[\frac{n}{2}\right] + 1 - n\right)\left[\frac{n}{2}\right]}_{D}.$$

With a little manipulation a simpler expression can be found for D:

$$D = \left[\frac{n}{2}\right]\left(\left[\frac{3n}{2}\right] - 2\left[\frac{n}{2}\right]\right). \tag{12}$$

We may replace the upper bound in C, $n - 2$, by n because the two additional terms are zero. Then applying (10) and simplifying, we obtain the curious sum

$$C = \sum_{i=1}^{[n/3]} \left[\frac{n - 3i}{2}\right]\left(n - 3i - \left[\frac{n - 3i}{2}\right]\right) = -\sum_{i=1}^{[n/3]} \left[\frac{n - 3i}{2}\right]\left[\frac{3i - n}{2}\right].$$

Now we use the following property of the greatest integer function: If a and b are positive integers then

$$-\left[-\frac{a}{b}\right]=\begin{cases}[a/b] & \text{if } a\mid b,\\ [a/b]+1 & \text{if } a\nmid b.\end{cases}$$

Then, on cosidering two cases depending on whether $3i-n$ is even or odd we find

$$C=\sum_{\substack{3i\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq n/3}}\left[\frac{n-3i}{2}\right]^2+\sum_{\substack{3i-1\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq n/3}}\left[\frac{n-3i}{2}\right]\left(\left[\frac{n-3i}{2}\right]+1\right)=$$

$$=\sum_{\substack{3i\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq n/3}}\left(\frac{n-3i}{2}\right)^2+\sum_{\substack{3i-1\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq n/3}}\left(\frac{n-3i-1}{2}\right)\left(\left(\frac{n-3i-1}{2}\right)+1\right).$$

Using the following four formulas,

$$(n\ \text{even})\qquad \sum_{\substack{3i-1\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq m}}f(i)=\sum_{i=1}^{[(m+1)/2]}f(2i-1);\qquad \sum_{\substack{3i\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq m}}=\sum_{i=1}^{[m/2]}f(2i),$$

$$(n\ \text{odd})\qquad \sum_{\substack{3i-1\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq m}}f(i)=\sum_{i=1}^{[m/2]}f(2i);\qquad \sum_{\substack{3i\equiv n(\mathrm{mod}\,2)\\ 1\leqq i\leqq m}}f(i)=\sum_{i=1}^{[(n+1)/2]}f(2i-1),$$

and applying some elementary summation methods we arrive at an evaluation of C. We shall leave out these remaining details but the result is as follows:
If we substitute (12) into (11) and define

$$A=\left[\frac{1}{2}\left[\frac{n}{3}\right]\right],\quad B=\left[\frac{1}{2}\left[\frac{n+3}{3}\right]\right],$$

$$X=\tfrac{1}{4}n^2A-\tfrac{3}{4}nA(A+1)+\tfrac{3}{2}A(A+1)(2A+1)+$$
$$+\tfrac{1}{4}(n+2)(n+4)B-\tfrac{3}{4}(n+3)B(B+1)+\tfrac{3}{2}B(B+1)(2B+1),$$

$$Y=\tfrac{1}{4}(n+3)^2B-\tfrac{3}{4}(n+3)B(B+1)+\tfrac{3}{2}B(B+1)(2B+1)+$$
$$+\tfrac{1}{4}(n+1)(n-1)A-\tfrac{3}{4}nA(A+1)+\tfrac{3}{2}A(A+1)(2A+1),$$

then

$$p(2n+1,4)=\sum_{\substack{i\geqq 0,j\geqq 0\\ 0\leqq 3i+j\leqq n-2}}\left[\frac{n-3i-j}{2}\right]=$$

(13) $$=\left[\frac{n}{2}\right]\left(\left[\frac{3n}{2}\right]-2\left[\frac{n}{2}\right]\right)-\sum_{i=1}^{[n/3]}\left[\frac{n-3i}{2}\right]\left[\frac{3i-n}{2}\right]$$

(14) $$=\left[\frac{n}{2}\right]\left(\left[\frac{3n}{2}\right]-2\left[\frac{n}{2}\right]\right)+\begin{cases}X & \text{if } n \text{ is even},\\ Y & \text{if } n \text{ is odd}.\end{cases}$$