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GROUPS AND POLAR GRAPHS

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In this paper the results of [2] will be transferred to polar graphs. A polar graph was defined by F. ZÍTEK [1] at the Czechoslovak Conference on Graph Theory at Štiřín in May 1972. Their properties are studied in the papers [3]–[9].

A polar graph is an ordered quintuple $\langle V, E, P, \varkappa, \lambda \rangle$, where V, E, P are sets, $\varkappa$ and λ are mappings of the set V and E respectively into the set of unordered pairs of distinct elements of P and the following conditions are satisfied:

(1) For each $u \in V$, $v \in V$, $u \neq v$, we have $\varkappa(u) \cap \varkappa(v) = \emptyset$.

(2) For each $e \in E, f \in E$, $e \neq f$, we have $\lambda(e) \neq \lambda(f)$.

(3) For each $p \in P$ there exists $v \in V$ so that $p \in \varkappa(v)$.

The elements of the sets V, E, P are called respectively vertices, edges and poles. If $p \in P$, $v \in V$, $p \in \varkappa(v)$, we say that the pole p belongs to the vertex v. If $p \in P$, $e \in E$ and $p \in \lambda(e)$, we say that the edge e is incident with the pole p. If an edge e is incident with a pole p which belongs to a vertex v, we say that e is incident with v.

Let $\mathfrak{G}$ be a group, A its subset. The polar graph $PG(\mathfrak{G}, A)$ is defined as follows: Its vertex set V is the support of $\mathfrak{G}$, its pole set P is the disjoint union of two sets P_1, P_2 such that there exist bijections $p_1 : \mathfrak{G} \to P_1$ and $p_2 : \mathfrak{G} \to P_2$. The edge set E of $PG(\mathfrak{G}, A)$ consists of the edges joining $p_1(x)$ with $p_2(y)$ for such x and y of $\mathfrak{G}$ that $x^{-1}y \in A$. (An edge e joins two poles p_1, p_2 of a polar graph, if it is incident with both of them.)

This is an analogue of a directed graph studied in [2]. In that graph there was a directed edge from x into y if and only if $x^{-1}y \in A$.

A polar graph is called vertex-transitive, if and only if to any two vertices u, v of this graph there exists an automorphism φ of this graph such that $\varphi(u) = v$.

An isomorphism of a polar graph $G_1 = \langle V_1, E_1, P_1, \varkappa_1, \lambda_1 \rangle$ onto a polar graph $G_2 = \langle V_2, E_2, P_2, \varkappa_2, \lambda_2 \rangle$ is a one-to-one mapping $\varphi : V_1 \cup E_1 \cup P_1 \to V_2 \cup E_2 \cup \cup P_2$ such that $\varphi(V_1) = V_2$, $\varphi(E_1) = E_2$, $\varphi(P_1) = P_2$, $\varkappa_2 \varphi(v) = \varphi \varkappa_1(v)$ for each $v \in V_1$, $\lambda_2 \varphi(e) = \varphi \lambda_1(e)$ for each $e \in E_1$. An isomorphism of a polar graph G onto itself is called an automorphism of G.

(For the vertex-transitive graph – in the non-polar case – in [2] we have used the term "symmetric". Here we prefer the term "vertex-transitive", because the term "symmetric graph" is used by other authors in different senses.)

Now we shall define a homogeneous polar graph in accordance with the similar concept for non-polar graphs. A polar graph G is called homogeneous if and only if the following conditions are satisfied:

(α) To any two poles p_1, p_2 of G there exists an automorphism φ of G such that $\varphi(p_1) = p_2$.

(β) For any pole p of G and any permutation π of the set of edges incident with p there exists an automorphism ψ_π of G such that $\psi_\pi(p) = p$ and the permutation π is induced by ψ_π.

It is easy to see that every homogeneous polar graph is also vertex-transitive.

Now we shall prove some theorems analogous to those of [2].

Theorem 1. *For every group $\mathfrak{G}$ and any one of its subsets A the polar graph $PG(\mathfrak{G}, A)$ is vertex-transitive.*

Proof. If u, v are two vertices of $PG(\mathfrak{G}, A)$, we take a mapping $\varphi_{vu^{-1}}$ such that $\varphi_{vu^{-1}}(a) = vu^{-1}a$ for any $a \in \mathfrak{G}$; this is a one-to-one mapping, because $\mathfrak{G}$ is a group. For the poles $p_1(a)$, $p_2(a)$ of the vertex a we put $\varphi_{vu^{-1}}(p_1(a)) = p_1(vu^{-1}a)$, $\varphi_{vu^{-1}}(p_2(a)) = p_2(vu^{-1}a)$. Now the mapping $\varphi_{vu^{-1}}$ can be naturally extended also to the edges of $PG(\mathfrak{G}, A)$. If x, y are two vertices of $PG(\mathfrak{G}, A)$, then $p_1(x)$ and $p_2(y)$ are joined by an edge if and only if $x^{-1}y \in A$. The images of the poles $p_1(x)$, $p_2(y)$ in $\varphi_{vu^{-1}}$ are $p_1(vu^{-1}x)$, $p_2(vu^{-1}y)$. We have

$$(vu^{-1}x)^{-1}\,(vu^{-1}x) = x^{-1}uv^{-1}vu^{-1}y = x^{-1}y\,.$$

Thus the poles $\varphi_{vu^{-1}}(p_1(x))$, $\varphi_{vu^{-1}}(p_2(y))$ are joined by an edge if and only if $p_1(x)$, $p_2(y)$ are joined by an edge. The pairs $p_1(x)$, $p_1(y)$ or $p_2(x)$, $p_2(y)$ are never joined by an edge. Therefore $\varphi_{uv^{-1}}$ is an automorphism of $PG(\mathfrak{G}, A)$. Further we have $\varphi_{uv^{-1}}(u) = v$. Therefore $PG(\mathfrak{G}, A)$ is vertex-transitive.

Theorem 2. *Let $\mathfrak{G}$ be a group, A its subset. Let φ be an automorphism of the group $\mathfrak{G}$ such that either $\varphi(A) = A$ or $\varphi(A) = \bar{A}$, where $\bar{A} = \{y \in \mathfrak{G} \mid y = x^{-1}, x \in A\}$. Then φ is induced on the vertex set of $PG(\mathfrak{G}, A)$ by an automorphism of $PG(\mathfrak{G}, A)$.*

Proof. Let $\varphi(A) = A$. Let x, y be two vertices of $PG(\mathfrak{G}, A)$. The poles $p_1(x)$, $p_2(y)$ are joined by an edge if and only if $x^{-1}y \in A$. Let φ^* be a mapping such that $\varphi^*(v) = \varphi(v)$ for each $v \in V$, $\varphi^*(p_1(v)) = p_1(\varphi(v))$, $\varphi^*(p_2(v)) = p_2(\varphi(v))$. We have $[\varphi(x)]^{-1}\,\varphi(y) = \varphi(x^{-1}y)$, because φ is an automorphism of $\mathfrak{G}$. Thus the poles $p_1(\varphi(x)) = \varphi^*(p_1(x))$, $p_2(\varphi(y)) = \varphi^*(p_2(y))$ are joined by an edge if and only if $\varphi(x^{-1}y) \in A$. However, as $\varphi(A) = A$ and φ is one-to-one, this is so if and only if

$x^{-1}y \in A$, i.e., if $p_1(x)$ and $p_2(y)$ are joined by an edge in $PG(\mathfrak{G}, A)$. Therefore φ^* is an automorphism of $PG(\mathfrak{G}, A)$. Let $\varphi(A) = \bar{A}$. We have again $[\varphi(x)]^{-1}\varphi(y) = \varphi(x^{-1}y)$. Let φ^{**} be a mapping such that $\varphi^*(v) = \varphi(v)$ for each $v \in V$, $\varphi^{**}(p_1(v)) = p_2(\varphi(v))$, $\varphi^{**}(p_2(v)) = p_1(\varphi(v))$. The poles $\varphi^{**}(p_1(x)) = p_2(\varphi(x))$, $\varphi^{**}(p_2(y)) = p_1(\varphi(y))$ are joined by an edge if and only if $[\varphi(y)]^{-1}\varphi(x) \in A$. But $[\varphi(y)]^{-1} . \varphi(x) = \varphi(y^{-1}x)$; this is in A if and only if $x^{-1}y \in \bar{A}$. Thus φ^{**} is an automorphism of $PG(\mathfrak{G}, A)$. Both φ^* and φ^{**} induce φ on the vertex set of $PG(\mathfrak{G}, A)$. (We have tacitly assumed that these mappings are naturally extended also onto the edge set.)

Theorem 3. *Let $\mathfrak{G}$ be a group, A a system of its generators, $\bar{A} = \{y \in \mathfrak{G} \mid y = x^{-1}, x \in A\}$. Let any permutation of A be induced by an automorphism of $\mathfrak{G}$ and let there exist an automorphism α of $\mathfrak{G}$ such that $\alpha(A) = \bar{A}$. Then $PG(\mathfrak{G}, A)$ is a homogeneous polar graph.*

Proof. According to Theorem 1, to any two vertices x, y of $PG(\mathfrak{G}, A)$ there exists an automorphism φ of this graph such that $\varphi(x) = y$. In the proof of Theorem 1 we have constructed an automorphism such that $\varphi(p_1(x)) = p_1(y)$, $\varphi(p_2(x)) = p_2(y)$. Now let e be the unit element of $\mathfrak{G}$. The pole $p_1(e)$ is joined with the poles $p_2(a)$, where $a \in A$, and with no other poles, the pole $p_2(e)$ is joined with the poles $p_1(b)$, where $b \in \bar{A}$, and with no other poles. According to Theorem 2 the automorphism α of $\mathfrak{G}$ is induced by the automorphism α^{**} of $PG(\mathfrak{G}, A)$ which is defined so that $\alpha^{**}(x) = \alpha(x)$, $\alpha^{**}(p_1(x)) = p_2(\alpha(x))$, $\alpha^{**}(p_2(x)) = p_1(\alpha(x))$ for each $x \in \mathfrak{G}$. We see that $\alpha^{**}(p_1(e)) = p_2(e)$, $\alpha^{**}(p_2(e)) = p_1(e)$. Now if we have two poles $p_1(x)$, $p_2(y)$, the former is mapped onto the latter by the automorphism $\varphi_y^*\alpha^{**}\varphi_{x^{-1}}^*$, where $\varphi_y^*(p_i(u)) = p_i(yu)$, $\varphi_{x^{-1}}^*(p_i(u)) = p_i(x^{-1}u)$ for each $u \in \mathfrak{G}$ and i equal to 1 or 2. Thus the condition (α) is proved. To any permutation π of the set of edges incident with $p_1(e)$ there corresponds in a one-to-one manner a permutation π' of A; for any $a \in A$ the element $\pi'(a)$ is the end vertex of the edge $\pi(h)$ which is in A, where h joins $p_1(e)$ and $p_2(a)$. Each π' is induced by an automorphism ψ_π of $\mathfrak{G}$ (according to the assumption) and this automorphism is induced by an automorphism ψ_π^* of $PG(\mathfrak{G}, A)$ (according to Theorem 2). Thus (β) holds for $p_1(e)$. Now let $x \in \mathfrak{G}$, let $p_i(x)$ be a pole of x, where $i = 1$ or $i = 2$. Let β be an automorphism of $PG(\mathfrak{G}, A)$ which maps $p_i(x)$ onto $p_1(e)$; its existence was proved above. Let ϱ be a permutation of the set of edges incident with $p_i(x)$. The mapping $\beta\varrho\beta^{-1}$ is a permutation of the set of edges incident with $p_1(e)$. To this permutation there exists an automorphism γ of $PG(\mathfrak{G}, A)$ inducing it. Then $\beta^{-1}\gamma\beta$ is the required automorphism for ϱ.

Theorem 4. *Let $\mathfrak{G}$ be an Abelian group, A a system of its generators. Let any permutation of A be induced by an automorphism of $\mathfrak{G}$. Then $PG(\mathfrak{G}, A)$ is a homogeneous polar graph.*

Proof. As $\mathfrak{G}$ is Abelian, there exists an automorphism α of $\mathfrak{G}$ such that $\alpha(x) = x^{-1}$ for any $x \in \mathfrak{G}$. This automorphism maps A onto $\bar{A}$. Therefore according to Theorem 3 the graph $PG(\mathfrak{G}, A)$ is a homogeneous polar graph.

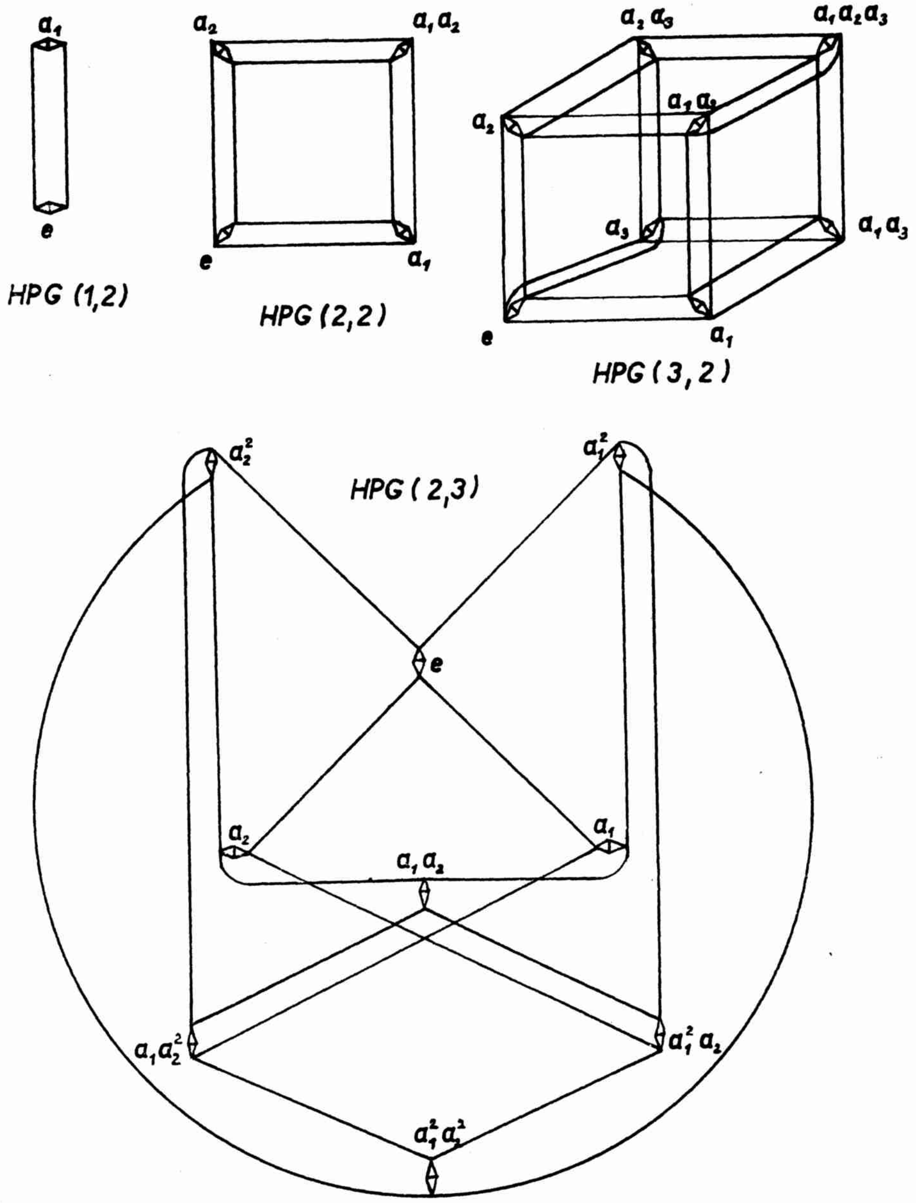

Fig. 1.

Analogously as in [2] we shall construct a certain class of homogeneous polar graphs. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ be cyclic groups of the same order r, let a_i be the generator of $\mathfrak{A}_i$ for $i = 1, \ldots, k$. Let $\mathfrak{G}$ be the direct product of $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$, let $A = \{a_1, \ldots, a_k\}$. The graph $PG(\mathfrak{G}, A)$ is evidently homogeneous and we denote it by $HPG(k, r)$. We have obviously $r \geqq 2$. Some of these graphs are in Fig. 1. They can be generalized